BIOMETRIKA

A JOURNAL FOR THE STATISTICAL STUDY OF BIOLOGICAL PROBLEMS

FOUNDED BY
W. F. R. WELDON, FRANCIS GALTON AND KARL PEARSON

EDITED BY
KARL PEARSON
ASSISTED BY
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CONTENTS OF VOLUME XXII

Memoirs

Miscellanea

List of Plates, etc.

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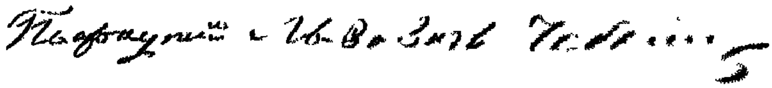

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BIOMETRIKA

ON THE REMAINING TABLES FOR DETERMINING THE VOLUMES OF A BI-VARIATE NORMAL SURFACE.

EDITORIAL.

WE start with the fundamental tetrachoric table

a	b	$a+b$
c	d	$c+d$
$a+c$	$b+d$	N

and assume the frequency distribution to be normal; we suppose

$$\left.\begin{aligned}(b+d)/N &= \int_h^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \tfrac{1}{2}(1-\alpha_h)\\ (c+d)/N &= \int_k^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy = \tfrac{1}{2}(1-\alpha_k)\end{aligned}\right\} \quad \text{.................(i)},$$

and we take as our standard case h and k both positive. We can always arrange our table so that this shall be so. But having done this the correlation will sometimes be positive and sometimes negative.

The equation for r is known to be

$$\frac{d}{N} = \tau_0(h)\,\tau_0(k) + \tau_1(h)\,\tau_1(k)\,r + \tau_2(h)\,\tau_2(k)\,r^2 + \ldots + \tau_n(h)\,\tau_n(k)\,r^n + \ldots \quad \text{(ii)},$$

where τ_n is the tetrachoric function of the nth order, and $\tau_0(h) = \frac{1}{2}(1-\alpha_h)$, $\tau_0(k) = \frac{1}{2}(1-\alpha_k)$.

Tables of $\frac{d}{N}$ for the triple entry h, k, r, r being positive, have been published in *Biometrika**, and for $r = -\cdot80$ to $-1\cdot00$ in the same Journal†. Both these tables were computed by Dr Alice Lee. The present tables complete the whole series by providing the values of d/N from $r = \cdot00$ to $-\cdot75$. They have been computed by Margaret Moul, Ethel M. Elderton, E. C. Fieller, J. Pretorius and A. E. R. Church, all members of the Galton Laboratory. Up to $r = -\cdot60$ the values of d/N were obtained by aid of Dr Lee's table of the first twenty tetrachoric functions‡. After $r = -\cdot60$ it was found that twenty tetrachoric functions to only seven figures were not adequate and the integral value of d/N was obtained by quadrature, Weddle's formula being used, in the manner indicated in *Biometrika*, Vol. VIII.

* Vol. XIX. (1927), pp. 354—404. † Vol. XI. pp. 284—291.
‡ *Biometrika*, Vol. XVII. pp. 343—354.

p. 886. The difference between the table there provided for $r = \cdot 80$ to $1 \cdot 00$ being that the present table is worked to more decimal places for the high values of h and k, those of the 1917 table having for certain cases been found inadequate.

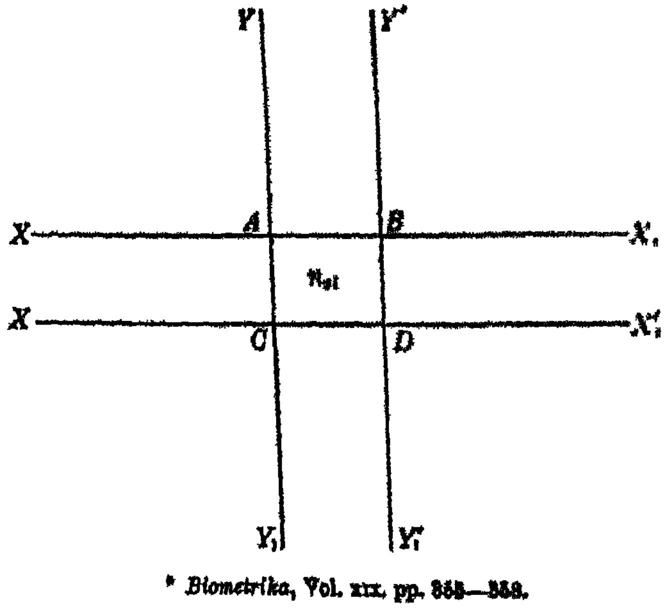
The complete tables thus furnished will serve three fundamental purposes: (i) to find r from any fourfold table, (ii) to find r from any cell of a table when the table is known or assumed to be normal in character, and (iii) to find when r has been ascertained for a table, for example by the product-moment method, what should be the theoretical contents of a given cell.

The general method of interpolating into tables of triple entry like the present has been discussed at adequate length in the paper of 1927*. Examples of the use of these methods were provided, but it appeared that to work out effectively by these methods the contents of all the cells of a normal table we required the d/N tables for r negative. These are now supplied in association with the tables published in this Journal, Vol. XI. pp. 284—291.

It must be remembered that in our standard table we suppose d to be the contents of the quadrant for which the limits of integration are $x = h$ to ∞, $y = k$ to ∞, h and k being positive. It may be needful at times to find a, b or c from d, or on the contrary d from a, b or c. Since h and k are supposed known the connecting equations clearly are:

$$\left.\begin{aligned}\frac{a}{N} &= \tfrac{1}{2}(1+\alpha_h) - \tfrac{1}{2}(1-\alpha_k) + \frac{d}{N}\\ \frac{b}{N} &= \tfrac{1}{2}(1-\alpha_h) - \frac{d}{N}\\ \frac{c}{N} &= \tfrac{1}{2}(1-\alpha_k) - \frac{d}{N}\end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots \text{(iii)}$$

Now let n_{st} be the contents of the cell in the sth row and tth column of a correlation table.



* *Biometrika*, Vol. XIX. pp. 355—358.

Let n_u equal the total frequency or volume of the normal surface in the quadrant standing on YAX_1; n_u' that in the quadrant $Y'BX_1$; n_v that in the quadrant YCX_1'; and n_v' that in $Y'DX_1'$. Then $n_v - n_u = n_{st} + V$, where V is the volume standing on X_1BDX_1'. But $V = n_v' - n_u'$.

Accordingly $$n_{st} = n_v - n_u - n_v' + n_u' \quad\text{.........................(iv).}$$

Now it is clear that the h_1, h_2 giving the lines YY_1 and $Y'Y_1'$, and the k_1, k_2 giving the lines XX_1 and $X'X_1'$, will be known; also r, the correlation coefficient, will be known. Thus either n_v, n_u, n_v', n_u' form the d's of four tetrachoric tables and are known, or, if they be the a, b or c's, the corresponding d can be obtained from the tables and their values found from Equations (iii) above.

Thus we deduce the "normal value" of n_{st}. We propose first to illustrate this process.

Illustration I. In a table for the correlation of Father and Son for stature we find, for the heights of Fathers 68″·875—69″·875, twelve Sons of the heights 66″·875—67″·875. This is a perfectly arbitrary cell taken out of a table of 20 × 17 cells*. The correlation coefficient of this table worked by the product-moment method is ·5189. The problem we put before ourselves is this: Supposing the table corresponds to a normal surface, are twelve individuals a reasonable frequency for this cell? As much of the table as concerns our present purpose can be written as follows:

Sons' Stature	Fathers' Stature			Totals
	Below 68″·875	68″·875—69″·875	Above 69″·875	
Below 66″·875	206	9	10	225
66″·875—67″·875	106	$\boxed{12}$	12	129
Above 67″·875	326	104	216	646
Totals	637	125	238	1000

Clearly $$n_u = 19, \quad n_u' = 10, \quad n_v = 43, \quad n_v' = 22,$$
and $$n_{st} = 12 = n_v - n_u - n_v' + n_u' = 43 - 19 - 22 + 10.$$

We can now examine the requisite four tables which have to be solved to obtain n_{st} for the normal surface. They are:

(i)			(ii)			(iii)			(iv)		
206	(n_u) 19	225	215	(n_u') 10	225	311	(n_v) 43	354	332	(n_v') 22	354
431	344	775	547	228	775	326	320	646	430	216	646
637	363	1000	762	238	1000	637	363	1000	762	238	1000

* See *Biometrika*, Vol. xiv. p. 151, Table XV.

If we re-arrange these tables in standard form, we have

(i)			(ii)			(iii)			(iv)		
(a_1) 431	(b_1) 344	775	(a_2) 547	(b_2) 228	775	(a_3) 326	(b_3) 320	646	(a_4) 430	(b_4) 216	646
(c_1) 206	$(d_1=n_u)$ 19	225	(c_2) 215	$(d_2=n_u')$ 10	225	(c_3) 311	$(d_3=n_v)$ 43	354	(c_4) 332	$(d_4=n_v')$ 22	354
637	363	1000	762	238	1000	637	363	1000	762	238	1000

and we see at once that $ad-bc$ is *negative* for all of them, or the d/N is to be found from the present issue of tables; i.e. $r=-\cdot 5189$. In the next place in every case the n_u, n_u', n_v, n_v' of the quadrant to be found is the d of the standard form. Hence we have, by Equation (iv),

$$n_{st}=N\left(\frac{d_3}{N}-\frac{d_1}{N}-\frac{d_4}{N}+\frac{d_2}{N}\right),$$

where the d/N's are to be found from our present table. To use these, however, we require to ascertain the h and k corresponding to the above four tables. This is most easily done by the use of the first and last columns in Table XXIX of the *Tables for Statisticians*, Part I, which give h (or k) for $\frac{1}{2}(1-\alpha)$. In the present case we have:

$$\tfrac{1}{2}(1-\alpha_{h_1})=\cdot 363,\quad \tfrac{1}{2}(1-\alpha_{k_1})=\cdot 225,\quad \text{or: } h_1=\cdot 35045,\quad k_1=\cdot 75541,$$
$$\tfrac{1}{2}(1-\alpha_{h_2})=\cdot 238,\quad \tfrac{1}{2}(1-\alpha_{k_2})=\cdot 225,\quad \text{or: } h_2=\cdot 71275,\quad k_2=\cdot 75541,$$
$$\tfrac{1}{2}(1-\alpha_{h_3})=\cdot 363,\quad \tfrac{1}{2}(1-\alpha_{k_3})=\cdot 354,\quad \text{or: } h_3=\cdot 35045,\quad k_3=\cdot 37454,$$
$$\tfrac{1}{2}(1-\alpha_{h_4})=\cdot 238,\quad \tfrac{1}{2}(1-\alpha_{k_4})=\cdot 354,\quad \text{or: } h_4=\cdot 71275,\quad k_4=\cdot 37454.$$

For most cases h and k to five decimal figures are fully adequate*.

If the four tables be now worked out by the interpolation formula for (i) use of four entries, and (ii) for twelve entries (i.e. formulae (α) and (β) of *Biometrika*, Vol. XIX. p. 356), we find:

	d_1	d_2	d_3	d_4
(β)	27·118	12·752	56·437	28·348
(α)	27·318	12·847	56·674	28·467
Observed values:	19	10	43	22

In both cases linear interpolation alone has been used to deduce $r=-\cdot 5189$ from the tables for $r=-\cdot 50$ and $r=-\cdot 55$, after the readings have been obtained for these from the corresponding h's and k's either by (α) or (β). It will be seen at once that (α) and (β) are in very close agreement, and that, at any rate in this portion of the tables, the hyperbolic formula (α) is fully adequate for most practical statistical purposes.

* A table of h to $\frac{1}{2}(1-\alpha_h)$ with far more figures will shortly be published.

But the deviations from the observed values of d in the four cases are very considerable. Notwithstanding, if we proceed to determine n_{st} we have:

$$\text{from } (\beta):\ n_{st} = d_s - d_1 - d_4 + d_2$$

$$= 56{\cdot}437 - 27{\cdot}113 - 28{\cdot}348 + 12{\cdot}752 = 13{\cdot}728,$$

$$\text{from } (\alpha):\ n_{st} = 56{\cdot}674 - 27{\cdot}313 - 28{\cdot}467 + 12{\cdot}847 = 13{\cdot}741,$$

or (β) only improves on (α) by ·013, a quantity of no practical importance.

Now the standard error of 13·74 in $1000 = \sqrt{\dfrac{13{\cdot}74 \times 986{\cdot}27}{1000}} = 3{\cdot}68$ nearly, corresponding to a probable error of 2·48.

Clearly $13{\cdot}74 \pm 2{\cdot}48$ easily covers the probability of 12 arising in a random sample. Or, the observed cell content of 12 is quite consistent with the table for the correlation of father's and son's statures being of a normal type.

Illustration II. We will take another example from the same correlation table, which indicates a greater variety in the methods of treatment; namely, the cell for fathers of stature 67″·875—68″·875 and for sons 67″·875—68″·875. It contains 27 cases, and the full table condensed for our purposes is as follows:

Sons' Stature	Fathers' Stature			Totals
	Below 67″·875	67″·875—68″·875	Above 68″·875	
Below 67″·875	277	34	43	354
67″·875—68″·875	89	$\boxed{27}$	65	181
Above 68″·875	132	78	255	465
Totals	498	139	363	1000

We have at once:

$$n_u = 77, \quad n_u' = 43, \quad n_v = 169, \quad n_v' = 108.$$

Thus our four tables take the forms:

(i)			(ii)			(iii)			(iv)		
277	(n_u) 77	354	311	(n_u') 43	354	366	(n_v) 169	535	427	(n_v') 108	535
221	425	646	326	320	646	132	333	465	210	255	465
498	502	1000	637	363	1000	498	502	1000	637	363	1000

Or, arranged in standard form :

(i)			(ii)		
$a_1=425$	$b_1=221$	646	$a_2=326$	$b_2=320$	646
(n_u) $c_1=77$	$d_1=277$	354	$c_2=311$	(n_u') $d_2=43$	354
502	498	1000	637	363	1000

(iii)			(iv)		
(n_v) $a_3=169$	$b_3=366$	535	$a_4=427$	(n_v') $b_4=108$	535
$c_3=333$	$d_3=132$	465	$c_4=210$	$d_4=255$	465
502	498	1000	637	363	1000

We see at once that:

$n_u = c_1$, and r is positive in Table (i) $= + \cdot 5189$,

$n_u' = d_2$, and r is negative in Table (ii) $= - \cdot 5189$,

$n_v = a_3$, and r is negative in Table (iii) $= - \cdot 5189$,

$n_v' = b_4$, and r is positive in Table (iv) $= + \cdot 5189$.

We have thus: $$n_{uv} = N\left(\frac{a_3}{N} - \frac{c_1}{N} - \frac{b_4}{N} + \frac{d_2}{N}\right).$$

There are thus two tables to be worked from the tables in *Biometrika*, Vol. XIX. pp. 378—404, i.e. (i) and (iv), and two tables from the present tables, i.e. (ii) and (iii).

Accordingly, in only one case is the n_u or n_v equal to d, namely $n_u' = d_2$. For the other cases we require to use the formulae given in Equations (iii).

Further, we shall need to use special interpolation formulae for three of the cases, as we are at the edges of our tables for d/N. We may arrange our work as shown on the following page, where α, β, γ, γ *bis*, and δ refer to the formulae in *Biometrika*, Vol. XIX. pp. 356—358.

It is, we think, clear that a difference of the order 0·228 is not of much statistical importance in a cell containing 28, and thus the formula α might have been used throughout. We give the work up to third differences, which much increases the labour, in order to show the reasonable effectiveness of the shorter hyperbolic formula.

(i)

$\tfrac{1}{2}(1-\alpha_h)={\cdot}498,\quad \tfrac{1}{2}(1-\alpha_k)={\cdot}354$

$h={\cdot}00501,\quad k={\cdot}37454$

Finial panel in h

Use Formula γ *bis*.

$\theta={\cdot}0501,\quad \chi={\cdot}7454$
$\phi={\cdot}9499,\quad \psi={\cdot}2546$
$\phi\psi={\cdot}24184,\quad \phi\chi={\cdot}70806$
$\theta\psi={\cdot}01276,\quad \theta\chi={\cdot}03734$
$\tfrac{1}{6}\theta\phi={\cdot}00793,\quad \tfrac{1}{6}\chi\psi={\cdot}03163$

$r={\cdot}50$

$z_{00}={\cdot}270{,}344,\quad z_{01}={\cdot}248{,}589$
$z_{10}={\cdot}255{,}392,\quad z_{11}={\cdot}235{,}345$
$\delta^2 z_{10}=-722,\quad \delta'^2 z_{00}=-213$
$\delta^2 z_{11}=-704,\quad \delta'^2 z_{10}=-278$
$\delta^2 z_{20}=-556,\quad \delta'^2 z_{01}=+\ 44$
$\delta^2 z_{21}=-561,\quad \delta'^2 z_{11}=-\ 38$

$$\begin{aligned}
&\phi\psi z_{00}+\phi\chi z_{01}+\theta\psi z_{10}+\theta\chi z_{11}={\cdot}2534{,}4250\\
&-\tfrac{1}{6}\theta\phi\,(4+\phi)\,(\psi\delta^2 z_{10}+\chi\delta^2 z_{11})+2781\\
&+\tfrac{1}{6}\theta\phi\,(1+\phi)\,(\psi\delta^2 z_{20}+\chi\delta^2 z_{21})-\ 865\\
&-\tfrac{1}{6}\psi\chi\,(1+\psi)\,(\phi\delta'^2 z_{00}+\theta\delta'^2 z_{10})+\ 858\\
&-\tfrac{1}{6}\chi\psi\,(1+\chi)\,(\phi\delta'^2 z_{01}+\theta\delta'^2 z_{11})-\ 220
\end{aligned}$$

$d_1/N={\cdot}2534{,}6804$

$r={\cdot}55$

$z_{00}={\cdot}279{,}133,\quad z_{01}={\cdot}256{,}963$
$z_{10}={\cdot}264{,}313,\quad z_{11}={\cdot}243{,}907$
$\delta^2 z_{10}=-845,\quad \delta'^2 z_{00}=-319$
$\delta^2 z_{11}=-821,\quad \delta'^2 z_{10}=-391$
$\delta^2 z_{20}=-678,\quad \delta'^2 z_{01}=-\ 46$
$\delta^2 z_{21}=-679,\quad \delta'^2 z_{11}=-137$

The five several contributions to $z_{\theta,\chi}$ are:

	·2619,3067
+	3247
−	1050
+	1280
+	279
	·2619,6843

$\dfrac{d_1}{N}={\cdot}2534{,}6804+\dfrac{189}{500}(850089)$
$={\cdot}2566{,}8118$

$d_1=256{\cdot}681$

If we used only the hyperbolic terms:
$d_1=256{\cdot}651$

But $n_x=a_1$
$=354-256{\cdot}681$
$=97{\cdot}319$ from γ *bis*
$=354-256{\cdot}651$
$=97{\cdot}349$ from α

(ii)

$\tfrac{1}{2}(1-\alpha_h)={\cdot}363,\quad \tfrac{1}{2}(1-\alpha_k)={\cdot}354$

$h={\cdot}35045,\quad k={\cdot}37454$

Neither h nor k finial

Use Formula β.

$\theta={\cdot}5045,\quad \chi={\cdot}7454$
$\phi={\cdot}4955,\quad \psi={\cdot}2546$
$\phi\psi={\cdot}12615,\quad \phi\chi={\cdot}36935$
$\theta\psi={\cdot}12845,\quad \theta\chi={\cdot}37606$
$\tfrac{1}{6}\theta\phi={\cdot}04166,\quad \tfrac{1}{6}\chi\psi={\cdot}03163$

$r=-{\cdot}50$

$z_{00}={\cdot}072{,}4876,\quad z_{01}={\cdot}061{,}5434$
$z_{10}={\cdot}061{,}5434,\quad z_{11}={\cdot}052{,}0367$
$\delta^2 z_{00}=11101,\quad \delta'^2 z_{00}=11101$
$\delta^2 z_{01}=10164,\quad \delta'^2 z_{10}=10164$
$\delta^2 z_{10}=11309,\quad \delta'^2 z_{01}=11309$
$\delta^2 z_{11}=10248,\quad \delta'^2 z_{11}=10248$

$$\begin{aligned}
&\phi\psi z_{00}+\phi\chi z_{01}+\theta\psi z_{10}+\theta\chi z_{11}={\cdot}0593{,}4901\\
&-\tfrac{1}{6}\theta\phi\,(1+\phi)\,(\psi\delta^2 z_{00}+\chi\delta^2 z_{01})-6481\\
&-\tfrac{1}{6}\theta\phi\,(1+\theta)\,(\psi\delta^2 z_{10}+\chi\delta^2 z_{11})-6592\\
&-\tfrac{1}{6}\chi\psi\,(1+\psi)\,(\phi\delta'^2 z_{00}+\theta\delta'^2 z_{10})-4218\\
&-\tfrac{1}{6}\chi\psi\,(1+\chi)\,(\phi\delta'^2 z_{01}+\theta\delta'^2 z_{11})-5948
\end{aligned}$$

$d_2/N={\cdot}0591{,}1662$

$r=-{\cdot}55$

$z_{00}={\cdot}064{,}7508,\quad z_{01}={\cdot}054{,}3306$
$z_{10}={\cdot}054{,}3306,\quad z_{11}={\cdot}064{,}3610$
$\delta^2 z_{00}=11859,\quad \delta'^2 z_{00}=11859$
$\delta^2 z_{01}=10801,\quad \delta'^2 z_{10}=10801$
$\delta^2 z_{10}=11876,\quad \delta'^2 z_{01}=11876$
$\delta^2 z_{11}=10696,\quad \delta'^2 z_{11}=10696$

The five several contributions to $z_{\theta,\chi}$ are:

	·0522,7209
−	6897
−	6892
−	4494
−	6228
	·0520,2698

$\dfrac{d_2}{N}={\cdot}0591{,}1662-\dfrac{189}{500}(706967)$
$={\cdot}0564{,}3872$

$d_2=56{\cdot}437$

If we used only the hyperbolic terms:
$d_2=56{\cdot}674$

But $n_x'=d_2$
$=56{\cdot}437$
from β
$=56{\cdot}674$
from α

(iii)

$\tfrac{1}{2}(1-\alpha_h)={\cdot}498,\quad \tfrac{1}{2}(1-\alpha_k)={\cdot}465$

$h={\cdot}00501,\quad k={\cdot}08784$

Doubly finial panel

Use Formula δ.

$\theta={\cdot}0501,\quad \chi={\cdot}8784$
$\phi={\cdot}9499,\quad \psi={\cdot}1216$
$\phi\psi={\cdot}11551,\quad \phi\chi={\cdot}83439$
$\theta\psi={\cdot}00609,\quad \theta\chi={\cdot}04401$
$\tfrac{1}{6}\theta\phi={\cdot}00793,\quad \tfrac{1}{6}\chi\psi={\cdot}01780$

$r=-{\cdot}50$

$z_{00}={\cdot}166{,}6667,\quad z_{01}={\cdot}147{,}2109$
$z_{10}={\cdot}147{,}2109,\quad z_{11}={\cdot}129{,}5818$
$\delta^2 z_{10}=10992=\delta'^2 z_{01}$
$\delta^2 z_{11}=10689=\delta'^2 z_{11}$
$\delta^2 z_{20}=12466=\delta'^2 z_{02}$
$\delta^2 z_{21}=11936=\delta'^2 z_{12}$

$$\begin{aligned}
&\phi\psi z_{00}+\phi\chi z_{01}+\theta\psi z_{10}+\theta\chi z_{11}={\cdot}1486{,}8238\\
&-\tfrac{1}{6}\theta\phi\,(4+\phi)\,(\psi\delta^2 z_{10}+\chi\delta^2 z_{11})-4210\\
&+\tfrac{1}{6}\theta\phi\,(1+\phi)\,(\psi\delta^2 z_{20}+\chi\delta^2 z_{21})+1656\\
&-\tfrac{1}{6}\psi\chi\,(4+\psi)\,(\phi\delta'^2 z_{01}+\theta\delta'^2 z_{11})-8053\\
&+\tfrac{1}{6}\psi\chi\,(1+\psi)\,(\phi\delta'^2 z_{02}+\theta\delta'^2 z_{12})+2484
\end{aligned}$$

$d_3/N={\cdot}1486{,}0315$

$r=-{\cdot}55$

$z_{00}={\cdot}157{,}3139,\quad z_{01}={\cdot}137{,}9225$
$z_{10}={\cdot}137{,}9225,\quad z_{11}={\cdot}120{,}4240$
$\delta^2 z_{10}=12254=\delta'^2 z_{01}$
$\delta^2 z_{11}=11911=\delta'^2 z_{11}$
$\delta^2 z_{20}=13648=\delta'^2 z_{02}$
$\delta^2 z_{21}=13054=\delta'^2 z_{12}$

The five several contributions to $z_{\theta,\chi}$ are:

	·1393,9229
−	4692
+	2030
−	8977
+	2719
	·1393,0309

$\dfrac{d_3}{N}={\cdot}1486{,}0315-\dfrac{189}{500}(930006)$
$={\cdot}1450{,}8773$

$d_3=145{\cdot}088$

If we used only the hyperbolic terms:
$d_3=145{\cdot}171$

But $n_y=a_3$
$=502-465+145{\cdot}088$
$=182{\cdot}088$ from δ
$=182{\cdot}171$
from α

(iv)

$\tfrac{1}{2}(1-\alpha_h)={\cdot}363,\quad \tfrac{1}{2}(1-\alpha_k)={\cdot}465$

$h={\cdot}35045,\quad k={\cdot}08784$

Finial panel in k

Use Formula γ.

$\theta={\cdot}5045,\quad \chi={\cdot}8784$
$\phi={\cdot}4955,\quad \psi={\cdot}1216$
$\phi\psi={\cdot}06025,\quad \phi\chi={\cdot}43525$
$\theta\psi={\cdot}06135,\quad \theta\chi={\cdot}44315$
$\tfrac{1}{6}\theta\phi={\cdot}04166,\quad \tfrac{1}{6}\chi\psi={\cdot}01780$

$r={\cdot}50$

$z_{00}={\cdot}270{,}344,\quad z_{01}={\cdot}255{,}392$
$z_{10}={\cdot}248{,}599,\quad z_{11}={\cdot}235{,}345$
$\delta^2 z_{00}=-213,\quad \delta'^2 z_{01}=-722$
$\delta^2 z_{01}=-278,\quad \delta'^2 z_{11}=-704$
$\delta^2 z_{10}=+\ 44,\quad \delta'^2 z_{02}=-556$
$\delta^2 z_{11}=-\ 38,\quad \delta'^2 z_{12}=-561$

$$\begin{aligned}
&\phi\psi z_{00}+\phi\chi z_{01}+\theta\psi z_{10}+\theta\chi z_{11}={\cdot}2469{,}9167\\
&-\tfrac{1}{6}\theta\phi\,(1+\phi)\,(\psi\delta^2 z_{00}+\chi\delta^2 z_{01})+1683\\
&-\tfrac{1}{6}\theta\phi\,(1+\theta)\,(\psi\delta^2 z_{10}+\chi\delta^2 z_{11})+\ 176\\
&-\tfrac{1}{6}\psi\chi\,(4+\psi)\,(\phi\delta'^2 z_{01}+\theta\delta'^2 z_{11})+5230\\
&+\tfrac{1}{6}\psi\chi\,(1+\psi)\,(\phi\delta'^2 z_{02}+\theta\delta'^2 z_{12})-1115
\end{aligned}$$

$d_4/N={\cdot}2470{,}5141$

$r={\cdot}55$

$z_{00}={\cdot}279{,}133,\quad z_{01}={\cdot}264{,}313$
$z_{10}={\cdot}256{,}963,\quad z_{11}={\cdot}243{,}907$
$\delta^2 z_{00}=-319,\quad \delta'^2 z_{01}=-845$
$\delta^2 z_{01}=-391,\quad \delta'^2 z_{11}=-821$
$\delta^2 z_{10}=-\ 46,\quad \delta'^2 z_{02}=-678$
$\delta^2 z_{11}=-137,\quad \delta'^2 z_{12}=-679$

The five several contributions to $z_{\theta,\chi}$ are:

	·2557,1206
+	2381
+	789
+	6110
−	1355
	·2557,9131

$\dfrac{d_4}{N}={\cdot}2470{,}5141+\dfrac{189}{500}(873990)$
$={\cdot}2503{,}5509$

$d_4=250{\cdot}355$

If we used only the hyperbolic terms:
$d_4=250{\cdot}288$

But $n_y'=b_4$
$=363-250{\cdot}355$
$=112{\cdot}645$ from γ
$=112{\cdot}712$
from α

Hence $n_{xy}=182{\cdot}088-97{\cdot}319-112{\cdot}645+56{\cdot}437=28{\cdot}561$; or from β, $n_{xy}=182{\cdot}171-97{\cdot}349-112{\cdot}712+56{\cdot}674=28{\cdot}784$.

The following table shows the order of differences from the observed values:

	d_1	d_2	d_3	d_4
(i) From β and γ formulae	256·681	56·437	145·968	250·355
(ii) From α formula ...	256·651	56·674	145·171	250·288
(iii) Observed values ...	277	43	132	255
(iv) Difference (i)—(ii) ...	+ ·030	− ·237	− ·203	+ ·067
(v) Difference (i)—(iii) ...	− 20·319	+ 13·437	+ 13·968	− 4·645
(vi) S.D. of (i)	13·813	7·297	11·137	13·700
(vii) Ratio of (v) to (vi) ...	− 1·47	+ 1·84	+ 1·18	− ·34

The ratio (vii) is in no case beyond the bounds of random sampling, and since n_u contains n_u', n_v' contains n_u' and n_v contains n_u, n_u' and n_v' we should expect a high correlation between all these deviations. If we consider the actual number 27 in the chosen cell it is clearly an easy random sample from a population containing either 28·4 or 28·8 in this cell.

We will now take illustrations of the reverse process of finding r from the observed d/N.

Illustration III. The following table indicates the relation between Athletic Capacity and Intelligence in 1708 Schoolboys:

	"Intelligent" and above	"Slow Intelligent" and below	Totals
Athletic	581·25	566·75	1148
Non-athletic	209·25	350·75	560
Totals	790·5	917·5	1708

Re-arranged in standard form:

$a=566·75$ $c=350·75$	$b=581·25$ $d=209·25$	1148 560
917·5	790·5	1708

and the correlation in this form is *negative*, i.e. in the original table it is positive or the more intelligent boys are the more athletic.

We have:

$$d/N = \cdot 122{,}5117\,; \quad \tfrac{1}{2}(1-\alpha_h) = \cdot 462{,}822, \quad \tfrac{1}{2}(1-\alpha_k) = \cdot 327{,}869.$$

Hence by linear interpolation from Table XXIX of *Tables for Statisticians*,

$$h = \cdot 09333, \qquad k = \cdot 44580.$$

Our present tables show that, for d/N lying between ·115 and ·125, and h between ·0 and ·1 and k between ·4 and ·5, we must deal with the values of r, $-\cdot 20$ and $-\cdot 25$. We have first then to find the value of d/N for the above values of h and k when $r = -\cdot 20$ and $-\cdot 25$.

We need here a "single finial" formula for x (h) because we are for this variate on the border of our table. The appropriate formula is γ *bis**, or:

$$\begin{aligned} z_{\theta,\chi} = {} & \phi\psi z_{0,0} + \phi\chi z_{0,1} + \theta\psi z_{1,0} + \theta\chi z_{1,1} \\ & - \tfrac{1}{6}\theta\phi \left\{(4+\phi)(\psi\delta^2 z_{1,0} + \chi\delta^2 z_{1,1}) - (1+\phi)(\psi\delta^2 z_{2,0} + \chi\delta^2 z_{2,1})\right\} \\ & - \tfrac{1}{6}\psi\chi \left\{(1+\psi)(\phi\delta'^2 z_{0,0} + \theta\delta'^2 z_{1,0}) + (1+\chi)(\phi\delta'^2 z_{0,1} + \theta\delta'^2 z_{1,1})\right\}. \end{aligned}$$

In our case:

$$\theta = \cdot 9333, \quad \phi = \cdot 0667\,; \quad \chi = \cdot 4580, \quad \psi = \cdot 5420\,;$$

$$\phi\psi = \cdot 03615, \quad \phi\chi = \cdot 03055, \quad \theta\psi = \cdot 50585, \quad \theta\chi = \cdot 42745\,;$$

$$\tfrac{1}{6}\theta\phi = \cdot 01038, \quad \tfrac{1}{6}\psi\chi = \cdot 04137.$$

For $r = -\cdot 20$:

$$z_{00} = \cdot 142{,}7384, \quad z_{10} = \cdot 129{,}2840, \quad z_{01} = \cdot 126{,}0358, \quad z_{11} = \cdot 114{,}0334\,;$$

$$\delta^2 z_{10} = 4265, \quad \delta^2 z_{20} = 5421, \quad \delta^2 z_{11} = 3979, \quad \delta^2 z_{21} = 4999,$$

$$\delta'^2 z_{00} = 9853, \quad \delta'^2 z_{10} = 9221, \quad \delta'^2 z_{01} = 10914, \quad \delta'^2 z_{11} = 10163.$$

For $r = -\cdot 25$:

$$z_{00} = \cdot 135{,}2305, \quad z_{10} = \cdot 121{,}8861, \quad z_{01} = \cdot 118{,}8755, \quad z_{11} = \cdot 106{,}9947\,;$$

$$\delta^2 z_{10} = 5012, \quad \delta^2 z_{20} = 6113, \quad \delta^2 z_{11} = 4684, \quad \delta^2 z_{21} = 5644,$$

$$\delta'^2 z_{00} = 10508, \quad \delta'^2 z_{10} = 9851, \quad \delta'^2 z_{01} = 11470, \quad \delta'^2 z_{11} = 10691.$$

From these two sets of values we can write down the values of $z_{\theta,\chi}$, i.e. those of d/N, for the above formula. There results the following numbers:

$z_{\theta,\chi} = d/N =$	When $r = -\cdot 20$	When $r = -\cdot 25$
$\phi\psi z_{0,0} + \phi\chi z_{0,1} + \theta\psi z_{1,0} + \theta\chi z_{1,1}$	$=\ \cdot 1231{,}5228$	$=\ \cdot 1159{,}1120$
$-\tfrac{1}{6}\theta\phi\ \{(4+\phi)(\psi\delta^2 z_{1,0} + \chi\delta^2 z_{1,1})\}$	$-$ 1745	$-$ 2052
$+\tfrac{1}{6}\theta\phi\ \{(1+\phi)(\psi\delta^2 z_{2,0} + \chi\delta^2 z_{2,1})\}$	$+$ 579	$+$ 653
$-\tfrac{1}{6}\psi\chi\ \{(1+\psi)(\phi\delta'^2 z_{0,0} + \theta\delta'^2 z_{1,0})\}$	$-$ 5909	$-$ 6312
$-\tfrac{1}{6}\psi\chi\ \{(1+\chi)(\phi\delta'^2 z_{0,1} + \theta\delta'^2 z_{1,1})\}$	$-$ 6160	$-$ 6480
	·1230,1993	·1157,6929

* See *Biometrika*, Vol. xix. p. 858, and Diagram, p. 857.

Hence by linear interpolation:

$$r = -\cdot 20 - \frac{5082}{72501} \times \cdot 05 = -\cdot 2035.$$

If we use only the first term, the hyperbolic formula, we have:

$$r = -\cdot 20 - \frac{6406}{72411} \times \cdot 05 = -\cdot 2044.$$

This is not so close to the full interpolation formula value, but would be a sufficiently close value of r for most practical purposes.

Illustration IV. The following table illustrates the influence of Wage of Father on the nature of the Mother's Employment:

Employment of Mother.

Wage of Father	Homework	Outwork	Totals
Under 22/-	144	106	250
22/- and over	168	39	207
Totals	312	145	457

Clearly d is the category 39, and the correlation in the table thus arranged is negative. We have:

$$\tfrac{1}{2}(1-\alpha_h) = \frac{145}{457} = \cdot 817{,}287; \quad \tfrac{1}{2}(1-\alpha_k) = \frac{207}{457} = \cdot 452{,}954.$$

Accordingly

$$d/N = \cdot 085{,}8392, \text{ and } h = \cdot 47530, \; k = \cdot 11821.$$

$$\theta = \cdot 7580, \quad \phi = \cdot 2470; \quad \chi = \cdot 1821, \quad \psi = \cdot 8170,$$

$$\phi\psi = \cdot 20202, \quad \phi\chi = \cdot 04498, \quad \theta\psi = \cdot 61588, \quad \theta\chi = \cdot 13712;$$

$$\tfrac{1}{2}\theta\phi = \cdot 03100, \quad \tfrac{1}{2}\psi\chi = \cdot 02482.$$

The above value of d/N, for a value of h between ·40 and ·50, and of k between ·10 and ·20, lies between the values for $r = -\cdot 40$ and $r = -\cdot 45$:

$$r = -\cdot 40 \begin{cases} z_{0,0} = \cdot 099{,}2408, & z_{1,0} = \cdot 085{,}5431, \quad z_{0,1} = \cdot 087{,}0997, \quad z_{1,1} = \cdot 074{,}8716; \\ \delta^2 z_{0,0} = 11853, & \delta^2 z_{1,0} = 12827, \quad \delta^2 z_{0,1} = 11026, \quad \delta^2 z_{1,1} = 11882, \\ \delta'^2 z_{0,0} = 7394, & \delta'^2 z_{1,0} = 6895, \quad \delta'^2 z_{0,1} = 8268, \quad \delta'^2 z_{1,1} = 7616. \end{cases}$$

$$r = -\cdot 45 \begin{cases} z_{0,0} = \cdot 091{,}4776, & z_{1,0} = \cdot 078{,}2330, \quad z_{0,1} = \cdot 079{,}6341, \quad z_{1,1} = \cdot 067{,}8782; \\ \delta^2 z_{0,0} = 12573, & \delta^2 z_{1,0} = 12899, \quad \delta^2 z_{0,1} = 11675, \quad \delta^2 z_{1,1} = 11878, \\ \delta'^2 z_{0,0} = 8255, & \delta'^2 z_{1,0} = 7680, \quad \delta'^2 z_{0,1} = 9021, \quad \delta'^2 z_{1,1} = 8289. \end{cases}$$

Accordingly we have the following values for:

$z_{\theta,\chi}=d/N=$	When $r=-\cdot 40$	When $r=-\cdot 45$
$\phi\psi z_{0,0}+\phi\chi z_{0,1}+\theta\psi z_{1,0}+\theta\chi z_{1,1}$	$=\cdot 0869{,}1705$	$=\cdot 0795{,}5185$
$-\frac{1}{8}\theta\phi\,\{(1+\phi)(\psi\delta^2 z_{0,0}+\chi\delta^2 z_{0,1})\}$	$-\quad 4524$	$-\quad 4797$
$-\frac{1}{8}\theta\phi\,\{(1+\theta)(\psi\delta^2 z_{1,0}+\chi\delta^2 z_{1,1})\}$	$-\quad 6605$	$-\quad 6909$
$-\frac{1}{8}\chi\psi\,\{(1+\psi)(\phi\delta'^2 z_{0,0}+\theta\delta'^2 z_{1,0})\}$	$-\quad 3167$	$-\quad 3529$
$-\frac{1}{8}\chi\psi\,\{(1+\chi)(\phi\delta'^2 z_{0,1}+\theta\delta'^2 z_{1,1})\}$	$-\quad 2281$	$-\quad 2485$
	$\cdot 0867{,}5128$	$\cdot 0793{,}7465$

Hence by linear interpolation

$$r=-\cdot 40-\frac{14121}{73766}\times\cdot 05=-\cdot 4096,$$

or the correlation of Mother's increasing Outwork with Father's decreasing Wage is $\cdot 4096$.

Had we used only the hyperbolic formula, or the first line of the above expression for $z_{\theta,\chi}$, we should have found

$$r=-\cdot 40-\frac{15778}{73652}\times\cdot 05=-\cdot 40-\cdot 0107$$

$$=-\cdot 4107, \text{ as above.}$$

Thus, in all the examples tried in this introduction, it would appear as if the hyperbolic formula were adequate for either finding a cell content, or determining the value of the coefficient of correlation.

k	d/N for $r = \cdot 00$									k
	$h = 0{\cdot}0$	$h = 0{\cdot}1$	$h = 0{\cdot}2$	$h = 0{\cdot}3$	$h = 0{\cdot}4$	$h = 0{\cdot}5$	$h = 0{\cdot}6$	$h = 0{\cdot}7$	$h = 0{\cdot}8$	
0·0	·2500000	·2300861	·2103702	·1910443	·1722891	·1542688	·1371266	·1209818	·1059277	*0·0*
0·1	·2300861	·2117584	·1936130	·1758265+	·1585653	·1419804	·1262037	·1113449	·0974900	*0·1*
0·2	·2103702	·1936130	·1770224	·1607601	·1449780	·1298142	·1153893	·1018039	·0891361	*0·2*
0·3	·1910443	·1758265+	·1607601	·1459917	·1316594	·1178887	·1047890	·0924515+	·0809475+	*0·3*
0·4	·1722891	·1585653	·1449780	·1316594	·1187342	·1063153	·0945017	·0833754	·0730008	*0·4*
0·5	·1542688	·1419804	·1298142	·1178887	·1063153	·0951954	·0846174	·0746549	·0653653	*0·5*
0·6	·1371266	·1262037	·1153893	·1047890	·0945017	·0846174	·0752148	·0663593	·0581020	*0·6*
0·7	·1209818	·1113449	·1018039	·0924515+	·0833754	·0746549	·0663593	·0585464	·0512613	*0·7*
0·8	·1059277	·0974900	·0891361	·0809475+	·0730008	·0653653	·0581020	·0512613	·0448827	*0·8*
0·9	·0920301	·0846993	·0774415+	·0703273	·0634231	·0567895−	·0504791	·0445359	·0389941	*0·9*
1·0	·0793276	·0730087	·0667527	·0606204	·0546692	·0489511	·0435117	·0383888	·0336120	*1·0*
1·1	·0678330	·0624297	·0570802	·0518365−	·0467476	·0418581	·0372068	·0328263	·0287416	*1·1*
1·2	·0575348	·0529519	·0484144	·0439668	·0396505+	·0355033	·0315582	·0278427	·0243781	*1·2*
1·3	·0484002	·0445449	·0407279	·0369864	·0333553	·0298666	·0265478	·0234222	·0205077	*1·3*
1·4	·0403783	·0371620	·0339776	·0308562	·0278270	·0249165	·0221478	·0195402	·0171087	*1·4*
1·5	·0334036	·0307428	·0281085−	·0255263	·0230203	·0206125+	·0183221	·0161649	·0141535	*1·5*
1·6	·0273996	·0252171	·0230563	·0209382	·0188826	·0169076	·0150289	·0132504	·0116095+	*1·6*
1·7	·0222827	·0205078	·0187505−	·0170280	·0153663	·0137501	·0122222	·0107832	·0094414	*1·7*
1·8	·0179652	·0165341	·0151173	·0137286	·0123808	·0110859	·0098540	·0086938	·0076120	*1·8*
1·9	·0143583	·0132146	·0120822	·0109723	·0098951	·0088601	·0078756	·0069484	·0060838	*1·9*
2·0	·0113751	·0104690	·0095719	·0086926	·0078392	·0070193	·0062393	·0055047	·0048197	*2·0*
2·1	·0089322	·0082207	·0075163	·0068258	·0061557	·0055118	·0048994	·0043225+	·0037847	*2·1*
2·2	·0069517	·0063980	·0058497	·0053123	·0047908	·0042897	·0038131	·0033641	·0029455+	*2·2*
2·3	·0053621	·0049349	·0045121	·0040976	·0036953	·0033088	·0029411	·0025948	·0022720	*2·3*
2·4	·0040988	·0037723	·0034490	·0031322	·0028247	·0025292	·0022482	·0019835+	·0017367	*2·4*
2·5	·0031048	·0028575+	·0026127	·0023726	·0021397	·0019159	·0017030	·0015025+	·0013156	*2·5*
2·6	·0023306	·0021449	·0019611	·0017810	·0016061	·0014382	·0012783	·0011278	·0009875−	*2·6*

k	d/N for $r = \cdot 00$									k
	$h = 0{\cdot}9$	$h = 1{\cdot}0$	$h = 1{\cdot}1$	$h = 1{\cdot}2$	$h = 1{\cdot}3$	$h = 1{\cdot}4$	$h = 1{\cdot}5$	$h = 1{\cdot}6$	$h = 1{\cdot}7$	
0·0	·0920301	·0793276	·0678330	·0575348	·0484002	·0403783	·0334036	·0273996	·0222827	*0·0*
0·1	·0846993	·0730087	·0624297	·0529519	·0445449	·0371620	·0307428	·0252171	·0205078	*0·1*
0·2	·0774415+	·0667527	·0570802	·0484144	·0407279	·0339776	·0281085−	·0230563	·0187505	*0·2*
0·3	·0703273	·0606204	·0518365−	·0439668	·0369864	·0308562	·0255263	·0209382	·0170280	*0·3*
0·4	·0634231	·0546692	·0467476	·0396505+	·0333553	·0278270	·0230203	·0188826	·0153663	*0·4*
0·5	·0567895−	·0489511	·0418581	·0355033	·0298666	·0249165−	·0206125+	·0169076	·0137501	*0·5*
0·6	·0504791	·0435117	·0372068	·0315582	·0265478	·0221478	·0183221	·0150289	·0122222	*0·6*
0·7	·0445359	·0383888	·0328263	·0278427	·0234222	·0195402	·0161649	·0132504	·0107832	*0·7*
0·8	·0389941	·0336120	·0287416	·0243781	·0205077	·0171087	·0141535−	·0116095+	·0094414	*0·8*
0·9	·0338781	·0292021	·0249707	·0211797	·0178171	·0148641	·0122965+	·0100864	·0082027	*0·9*
1·0	·0292021	·0251715−	·0215241	·0182564	·0153579	·0128125−	·0105993	·0086942	·0070705+	*1·0*
1·1	·0249707	·0215241	·0184053	·0156110	·0131325+	·0109559	·0090635−	·0074344	·0060460	*1·1*
1·2	·0211797	·0182564	·0156110	·0132410	·0111388	·0092926	·0076875−	·0063057	·0051281	*1·2*
1·3	·0178171	·0153579	·0131325+	·0111388	·0093708	·0078173	·0064670	·0053046	·0043140	*1·3*
1·4	·0148641	·0128125−	·0109559	·0092926	·0078173	·0065216	·0053951	·0044254	·0035990	*1·4*
1·5	·0122965+	·0105993	·0090635−	·0076875−	·0064670	·0053951	·0044632	·0036610	·0029773	*1·5*
1·6	·0100864	·0086942	·0074344	·0063057	·0053046	·0044254	·0036610	·0030030	·0024422	*1·6*
1·7	·0082027	·0070705+	·0060460	·0051281	·0043140	·0035990	·0029773	·0024422	·0019861	*1·7*
1·8	·0066133	·0057005+	·0048745+	·0041345−	·0034781	·0029016	·0024004	·0019690	·0016013	*1·8*
1·9	·0052856	·0045560	·0038959	·0033044	·0027798	·0023191	·0019185−	·0015736	·0012798	*1·9*
2·0	·0041874	·0036094	·0030864	·0026179	·0022022	·0018372	·0015199	·0012467	·0010139	*2·0*
2·1	·0032881	·0028343	·0024236	·0020557	·0017293	·0014427	·0011935−	·0009790	·0007961	*2·1*
2·2	·0025591	·0022059	·0018862	·0015999	·0013459	·0011228	·0009288	·0007619	·0006196	*2·2*
2·3	·0019739	·0017014	·0014549	·0012340	·0010381	·0008660	·0007164	·0005877	·0004779	*2·3*
2·4	·0015088	·0013006	·0011121	·0009433	·0007935+	·0006620	·0005477	·0004492	·0003653	*2·4*
2·5	·0011430	·0009852	·0008424	·0007145+	·0006011	·0005015−	·0004149	·0003403	·0002767	*2·5*
2·6	·0008579	·0007395+	·0006324	·0005364	·0004512	·0003764	·0003114	·0002554	·0002077	*2·6*

k	d/N for $r = \cdot 00$									k
	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	
0·0	·0179652	·0143583	·0113751	·0089322	·0069517	·0053621	·0040988	·0031048	·0023306	*0·0*
0·1	·0165341	·0132146	·0104690	·0082207	·0063980	·0049349	·0037723	·0028575+	·0021449	*0·1*
0·2	·0151173	·0120822	·0095719	·0075163	·0058497	·0045121	·0034490	·0026127	·0019611	*0·2*
0·3	·0137286	·0109723	·0086926	·0068258	·0053123	·0040976	·0031322	·0023726	·0017810	*0·3*
0·4	·0123808	·0098951	·0078392	·0061557	·0047908	·0036953	·0028247	·0021397	·0016061	*0·4*
0·5	·0110859	·0088601	·0070193	·0055118	·0042897	·0033088	·0025292	·0019159	·0014382	*0·5*
0·6	·0098540	·0078756	·0062393	·0048994	·0038131	·0029411	·0022482	·0017030	·0012783	*0·6*
0·7	·0086938	·0069484	·0055047	·0043225+	·0033641	·0025948	·0019835+	·0015025+	·0011278	*0·7*
0·8	·0076120	·0060838	·0048197	·0037847	·0029455+	·0022720	·0017367	·0013156	·0009875−	*0·8*
0·9	·0066133	·0052856	·0041874	·0032881	·0025591	·0019739	·0015088	·0011430	·0008579	*0·9*
1·0	·0057005+	·0045560	·0036094	·0028343	·0022059	·0017014	·0013006	·0009852	·0007395+	*1·0*
1·1	·0048745+	·0038959	·0030864	·0024236	·0018862	·0014549	·0011121	·0008424	·0006324	*1·1*
1·2	·0041345−	·0033044	·0026179	·0020557	·0015999	·0012340	·0009433	·0007145+	·0005364	*1·2*
1·3	·0034781	·0027798	·0022022	·0017293	·0013459	·0010381	·0007935+	·0006011	·0004512	*1·3*
1·4	·0029016	·0023191	·0018372	·0014427	·0011228	·0008660	·0006620	·0005015−	·0003764	*1·4*
1·5	·0024004	·0019185−	·0015199	·0011935−	·0009288	·0007164	·0005477	·0004149	·0003114	*1·5*
1·6	·0019690	·0015736	·0012467	·0009790	·0007619	·0005877	·0004492	·0003403	·0002554	*1·6*
1·7	·0016013	·0012798	·0010139	·0007961	·0006196	·0004779	·0003653	·0002767	·0002077	*1·7*
1·8	·0012910	·0010318	·0008174	·0006419	·0004996	·0003853	·0002945+	·0002231	·0001675−	*1·8*
1·9	·0010318	·0008246	·0006533	·0005130	·0003993	·0003080	·0002354	·0001783	·0001339	*1·9*
2·0	·0008174	·0006533	·0005176	·0004064	·0003163	·0002440	·0001865	·0001413	·0001060	*2·0*
2·1	·0006419	·0005130	·0004064	·0003191	·0002484	·0001916	·0001464	·0001109	·0000833	*2·1*
2·2	·0004996	·0003993	·0003163	·0002484	·0001933	·0001491	·0001140	·0000863	·0000648	*2·2*
2·3	·0003853	·0003080	·0002440	·0001916	·0001491	·0001150+	·0000879	·0000666	·0000500−	*2·3*
2·4	·0002945+	·0002354	·0001865−	·0001464	·0001140	·0000879	·0000672	·0000509	·0000382	*2·4*
2·5	·0002231	·0001783	·0001413	·0001109	·0000863	·0000666	·0000509	·0000386	·0000289	*2·5*
2·6	·0001675−	·0001339	·0001060	·0000833	·0000648	·0000500−	·0000382	·0000289	·0000217	*2·6*

k	d/N for $r = - \cdot 05$									k
	$h = 0{\cdot}0$	$h = 0{\cdot}1$	$h = 0{\cdot}2$	$h = 0{\cdot}3$	$h = 0{\cdot}4$	$h = 0{\cdot}5$	$h = 0{\cdot}6$	$h = 0{\cdot}7$	$h = 0{\cdot}8$	
0·0	·2420380	·2221648	·2025669	·1834338	·1649406	·1472439	·1304779	·1147519	·1001483	*0·0*
0·1	·2221648	·2038786	·1858525+	·1682597	·1512608	·1349993	·1195981	·1051670	·0917509	*0·1*
0·2	·2025669	·1858525+	·1693814	·1533116	·1377895+	·1229457	·1088920	·0957188	·0834939	*0·2*
0·3	·1834338	·1682597	·1533116	·1387327	·1246556	·1111983	·0984617	·0865272	·0754557	*0·3*
0·4	·1649406	·1512608	·1377895+	·1246556	·1119782	·0998633	·0884013	·0776650+	·0677086	*0·4*
0·5	·1472439	·1349993	·1229457	·1111983	·0998633	·0890353	·0787944	·0692055−	·0603163	*0·5*
0·6	·1304779	·1195981	·1088920	·0984617	·0884013	·0787944	·0697119	·0612108	·0533329	*0·6*
0·7	·1147519	·1051670	·0957188	·0865272	·0776650+	·0692055−	·0612108	·0537306	·0468015−	*0·7*
0·8	·1001483	·0917509	·0834939	·0754557	·0677086	·0603163	·0533329	·0468015−	·0407636	*0·8*
0·9	·0867220	·0794297	·0722620	·0652871	·0585674	·0521580	·0461055−	·0404469	·0352093	*0·9*
1·0	·0745010	·0682182	·0620453	·0560408	·0502582	·0447449	·0395407	·0346772	·0301773	*1·0*
1·1	·0634870	·0581182	·0528445+	·0477168	·0427806	·0380762	·0336373	·0294907	·0256557	*1·1*
1·2	·0536621	·0491100	·0446412	·0402978	·0361184	·0321368	·0283815+	·0248749	·0216332	*1·2*
1·3	·0449829	·0411567	·0374000	·0337512	·0302417	·0268906	·0237489	·0208080	·0180903	*1·3*
1·4	·0373929	·0342018	·0310717	·0280320	·0251095+	·0223277	·0197062	·0172603	·0150011	*1·4*
1·5	·0308214	·0281832	·0255904	·0230854	·0206723	·0183763	·0162135−	·0141964	·0123341	*1·5*
1·6	·0251885+	·0230258	·0209062	·0188496	·0168740	·0149951	·0132259	·0115767	·0100547	*1·6*
1·7	·0204082	·0186505+	·0169286	·0152586	·0136551	·0121307	·0106960	·0093591	·0081259	*1·7*
1·8	·0163918	·0149757	·0135890	·0122446	·0109544	·0097283	·0085749	·0075006	·0065100	*1·8*
1·9	·0130509	·0119198	·0108128	·0097401	·0087109	·0077334	·0068143	·0059585+	·0051699	*1·9*
2·0	·0102994	·0094041	·0085281	·0076796	·0068660	·0060936−	·0053674	·0046918	·0040693	*2·0*
2·1	·0080561	·0073535+	·0066665+	·0060013	·0053638	·0047587	·0041903	·0036616	·0031747	*2·1*
2·2	·0062452	·0056989	·0051648	·0046480	·0041528	·0036832	·0032421	·0028320	·0024546	*2·2*
2·3	·0047980	·0043769	·0039655+	·0035676	·0031865−	·0028251	·0024860	·0021708	·0018808	*2·3*
2·4	·0036529	·0033313	·0030173	·0027136	·0024229	·0021474	·0018890	·0016489	·0014281	*2·4*
2·5	·0027560	·0025125+	·0022749	·0020453	·0018256	·0016175−	·0014223	·0012411	·0010745+	*2·5*
2·6	·0020603	·0018775−	·0016996	·0015276	·0013630	·0012072	·0010612	·0009257	·0008011	*2·6*

k	d/N for $r = -\cdot 05$									k
	$h = 0{\cdot}9$	$h = 1{\cdot}0$	$h = 1{\cdot}1$	$h = 1{\cdot}2$	$h = 1{\cdot}3$	$h = 1{\cdot}4$	$h = 1{\cdot}5$	$h = 1{\cdot}6$	$h = 1{\cdot}7$	
0·0	·0867220	·0745010	·0634879	·0536021	·0449829	·0373929	·0308214	·0251885+	·0204082	*0·0*
0·1	·0794297	·0682182	·0581182	·0491100	·0411557	·0342018	·0281832	·0230258	·0186505+	*0·1*
0·2	·0722620	·0620453	·0528445+	·0446412	·0374000	·0310717	·0255964	·0209062	·0169246	*0·2*
0·3	·0652871	·0560408	·0477168	·0402978	·0337512	·0280320	·0230854	·0188496	·0152586	*0·3*
0·4	·0585674	·0502582	·0427806	·0361184	·0302417	·0251095+	·0206723	·0168740	·0136551	*0·4*
0·5	·0521580	·0447449	·0380762	·0321368	·0268996	·0223277	·0183763	·0149951	·0121307	*0·5*
0·6	·0461055−	·0395407	·0336373	·0283815+	·0237489	·0197062	·0162135	·0132259	·0106960	*0·6*
0·7	·0404469	·0346772	·0294907	·0248749	·0208080	·0172603	·0141964	·0115767	·0093591	*0·7*
0·8	·0352093	·0301773	·0256557	·0216332	·0180903	·0150011	·0123341	·0100547	·0081259	*0·8*
0·9	·0304097	·0260554	·0221442	·0186661	·0156040	·0129350−	·0106318	·0086640	·0069995+	*0·9*
1·0	·0260554	·0223172	·0189610	·0159776	·0133520	·0110644	·0090911	·0074059	·0059810	*1·0*
1·1	·0221442	·0189610	·0161041	·0135656	·0113325+	·0093877	·0077107	·0062792	·0050693	*1·1*
1·2	·0186661	·0159776	·0135656	·0114234	·0095396	·0078997	·0064863	·0052802	·0042612	*1·2*
1·3	·0156040	·0133520	·0113325+	·0095396	·0079637	·0065924	·0054109	·0044032	·0035522	*1·3*
1·4	·0129350−	·0110644	·0093877	·0078997	·0065924	·0054552	·0044759	·0036410	·0029362	*1·4*
1·5	·0106318	·0090911	·0077107	·0064863	·0054109	·0044759	·0036711	·0029852	·0024065−	*1·5*
1·6	·0086640	·0074059	·0062792	·0052802	·0044032	·0036410	·0029852	·0024265+	·0019554	*1·6*
1·7	·0069995+	·0059810	·0050693	·0042612	·0035522	·0029362	·0024065−	·0019554	·0015751	*1·7*
1·8	·0056057	·0047883	·0040569	·0034090	·0028407	·0023473	·0019230	·0015620	·0012578	*1·8*
1·9	·0044501	·0037999	·0032183	·0027033	·0022518	·0018600	·0015232	·0012368	·0009955+	*1·9*
2·0	·0035016	·0029889	·0025305−	·0021248	·0017693	·0014608	·0011959	·0009706	·0007810	*2·0*
2·1	·0027307	·0023301	·0019720	·0016552	·0013778	·0011371	·0009305+	·0007550	·0006072	*2·1*
2·2	·0021106	·0018003	·0015230	·0012779	·0010633	·0008773	·0007176	·0005820	·0004679	*2·2*
2·3	·0016166	·0013784	·0011657	·0009777	·0008132	·0006707	·0005484	·0004446	·0003573	*2·3*
2·4	·0012271	·0010459	·0008842	·0007413	·0006163	·0005081	·0004153	·0003366	·0002704	*2·4*
2·5	·0009229	·0007864	·0006645+	·0005569	·0004629	·0003814	·0003117	·0002525	·0002027	*2·5*
2·6	·0006879	·0005859	·0004949	·0004146	·0003445−	·0002837	·0002317	·0001878	·0001506	*2·6*

k	d/N for $r = -\cdot 05$									k
	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	
0·0	·0163918	·0130509	·0102994	·0080561	·0062452	·0047980	·0036529	·0027560	·0020603	*0·0*
0·1	·0149757	·0119198	·0094041	·0073535+	·0056989	·0043769	·0033313	·0025125+	·0018775−	*0·1*
0·2	·0135890	·0108128	·0085281	·0066665+	·0051648	·0039655+	·0030173	·0022749	·0016996	*0·2*
0·3	·0122446	·0097401	·0076796	·0060013	·0046480	·0035676	·0027136	·0020453	·0015276	*0·3*
0·4	·0109544	·0087109	·0068660	·0053638	·0041528	·0031885−	·0024229	·0018266	·0013630	*0·4*
0·5	·0097283	·0077334	·0060935−	·0047587	·0036832	·0028251	·0021474	·0016175−	·0012072	*0·5*
0·6	·0085749	·0068143	·0053674	·0041903	·0032421	·0024860	·0018890	·0014223	·0010612	*0·6*
0·7	·0075006	·0059585+	·0046918	·0036616	·0028320	·0021708	·0016489	·0012411	·0009257	*0·7*
0·8	·0065100	·0051699	·0040693	·0031747	·0024546	·0018808	·0014281	·0010745+	·0008011	*0·8*
0·9	·0056057	·0044501	·0035016	·0027307	·0021106	·0016166	·0012271	·0009229	·0006870	*0·9*
1·0	·0047883	·0037999	·0029889	·0023301	·0018003	·0013784	·0010459	·0007864	·0005859	*1·0*
1·1	·0040569	·0032183	·0025305−	·0019720	·0015230	·0011657	·0008842	·0006645+	·0004949	*1·1*
1·2	·0034090	·0027033	·0021248	·0016552	·0012779	·0009777	·0007413	·0005569	·0004146	*1·2*
1·3	·0028407	·0022518	·0017693	·0013778	·0010633	·0008132	·0006163	·0004629	·0003445	*1·3*
1·4	·0023473	·0018600	·0014608	·0011371	·0008773	·0006707	·0005081	·0003814	·0002837	*1·4*
1·5	·0019230	·0015232	·0011959	·0009305+	·0007176	·0005484	·0004153	·0003117	·0002317	*1·5*
1·6	·0015620	·0012368	·0009706	·0007550−	·0005820	·0004446	·0003366	·0002525−	·0001878	*1·6*
1·7	·0012578	·0009955+	·0007810	·0006072	·0004679	·0003573	·0002704	·0002027	·0001506	*1·7*
1·8	·0010039	·0007943	·0006229	·0004841	·0003729	·0002846	·0002153	·0001614	·0001198	*1·8*
1·9	·0007943	·0006282	·0004924	·0003826	·0002946	·0002247	·0001699	·0001273	·0000945+	*1·9*
2·0	·0006229	·0004924	·0003858	·0002996	·0002306	·0001759	·0001329	·0000996	·0000739	*2·0*
2·1	·0004841	·0003826	·0002996	·0002326	·0001790	·0001364	·0001031	·0000772	·0000572	*2·1*
2·2	·0003729	·0002946	·0002306	·0001790	·0001376	·0001049	·0000792	·0000593	·0000439	*2·2*
2·3	·0002846	·0002247	·0001759	·0001364	·0001049	·0000799	·0000603	·0000451	·0000334	*2·3*
2·4	·0002153	·0001699	·0001329	·0001031	·0000792	·0000603	·0000455+	·0000340	·0000252	*2·4*
2·5	·0001614	·0001273	·0000996	·0000772	·0000593	·0000451	·0000340	·0000254	·0000188	*2·5*
2·6	·0001198	·0000945+	·0000739	·0000572	·0000439	·0000334	·0000252	·0000188	·0000139	*2·6*

k	d/N for r = − ·10									k
	h = 0·0	*h = 0·1*	*h = 0·2*	*h = 0·3*	*h = 0·4*	*h = 0·5*	*h = 0·6*	*h = 0·7*	*h = 0·8*	
0·0	·2340579	·2142237	·1947447	·1758059	·1575766	·1402058	·1238186	·1085141	·0943638	*0·0*
0·1	·2142237	·1959834	·1780814	·1606873	·1439559	·1280229	·1130022	·0989832	·0860300	*0·1*
0·2	·1947447	·1780814	·1617382	·1458694	·1306156	·1160997	·1024242	·0898694	·0778926	*0·2*
0·3	·1758059	·1606873	·1458694	·1314918	·1176809	·1045474	·0921830	·0806592	·0700264	*0·3*
0·4	·1575766	·1439559	·1306156	·1176809	·1052651	·0934667	·0823672	·0720298	·0624986	*0·4*
0·5	·1402058	·1280229	·1160997	·1045474	·0934667	·0829447	·0730533	·0638480	·0553668	*0·5*
0·6	·1238186	·1130022	·1024242	·0921830	·0823672	·0730533	·0643044	·0561683	·0486780	*0·6*
0·7	·1085141	·0989832	·0898694	·0806592	·0720298	·0638480	·0561683	·0490321	·0424673	*0·7*
0·8	·0943638	·0860300	·0778926	·0700264	·0624986	·0553668	·0486780	·0424673	·0367586	*0·8*
0·9	·0814115+	·0741815+	·0671275−	·0603139	·0537987	·0476312	·0418512	·0364888	·0315637	*0·9*
1·0	·0696744	·0634520	·0573857	·0515310	·0459372	·0406461	·0356916	·0310987	·0268838	*1·0*
1·1	·0591451	·0538329	·0486583	·0436682	·0389044	·0344021	·0301896	·0262878	·0227100	*1·1*
1·2	·0497937	·0452957	·0409178	·0366996	·0326759	·0288763	·0253243	·0220369	·0190251	*1·2*
1·3	·0415716	·0377945+	·0341215+	·0305853	·0272151	·0240352	·0210650−	·0183184	·0158042	*1·3*
1·4	·0344147	·0312697	·0282138	·0252744	·0224752	·0198364	·0173736	·0150983	·0130171	*1·4*
1·5	·0282474	·0256509	·0231301	·0207074	·0184023	·0162311	·0142065+	·0123376	·0106297	*1·5*
1·6	·0229861	·0208607	·0187992	·0168195−	·0149376	·0131665+	·0115165−	·0099946	·0086051	*1·6*
1·7	·0185426	·0168179	·0151465−	·0135429	·0120197	·0105876	·0092545−	·0080260	·0069053	*1·7*
1·8	·0148273	·0134400	·0120987	·0108090	·0095871	·0084391	·0073715−	·0063885+	·0054926	*1·8*
1·9	·0117520	·0106459	·0095758	·0085509	·0075792	·0066672	·0058197	·0050401	·0043302	*1·9*
2·0	·0092319	·0083578	·0075129	·0067044	·0059386	·0052204	·0045537	·0039409	·0033834	*2·0*
2·1	·0071875+	·0065020	·0058417	·0052097	·0046115−	·0040510−	·0035311	·0030538	·0026198	*2·1*
2·2	·0055456	·0050142	·0045014	·0040117	·0035487	·0031152	·0027135+	·0023450−	·0020103	*2·2*
2·3	·0042401	·0038313	·0034373	·0030613	·0027061	·0023739	·0020663	·0017844	·0015286	*2·3*
2·4	·0032124	·0029009	·0026009	·0023148	·0020448	·0017925+	·0015591	·0013455−	·0011517	*2·4*
2·5	·0024117	·0021764	·0019500−	·0017343	·0015310	·0013411	·0011657	·0010052	·0008598	*2·5*
2·6	·0017939	·0016178	·0014486	·0012875−	·0011357	·0009942	·0008635+	·0007441	·0006360	*2·6*

k	d/N for r = − ·10									k
	h = 0·9	*h = 1·0*	*h = 1·1*	*h = 1·2*	*h = 1·3*	*h = 1·4*	*h = 1·5*	*h = 1·6*	*h = 1·7*	
0·0	·0814115+	·0696744	·0591451	·0497937	·0415716	·0344147	·0282474	·0229861	·0185426	*0·0*
0·1	·0741815+	·0634520	·0538329	·0452957	·0377945+	·0312697	·0256509	·0208607	·0168179	*0·1*
0·2	·0671275−	·0573857	·0486583	·0409178	·0341215+	·0282138	·0231301	·0187992	·0151465−	*0·2*
0·3	·0603139	·0515310	·0436682	·0366996	·0305853	·0252744	·0207074	·0168195−	·0135429	*0·3*
0·4	·0537987	·0459372	·0389044	·0326759	·0272151	·0224752	·0184023	·0149376	·0120197	*0·4*
0·5	·0476312	·0406461	·0344021	·0288763	·0240352	·0198364	·0162311	·0131665+	·0105876	*0·5*
0·6	·0418512	·0356916	·0301896	·0253243	·0210650−	·0173736	·0142065+	·0115165−	·0092545−	*0·6*
0·7	·0364888	·0310987	·0262878	·0220369	·0183184	·0150983	·0123376	·0099946	·0080260	*0·7*
0·8	·0315637	·0268838	·0227100	·0190251	·0158042	·0130171	·0106297	·0086051	·0069053	*0·8*
0·9	·0270855−	·0230544	·0194622	·0162932	·0135256	·0111327	·0090846	·0073490	·0058932	*0·9*
1·0	·0230544	·0196102	·0165435+	·0138403	·0114814	·0094435+	·0077007	·0062251	·0049883	*1·0*
1·1	·0194622	·0165435+	·0139468	·0116598	·0096658	·0079445+	·0064737	·0052294	·0041874	*1·1*
1·2	·0162932	·0138403	·0116598	·0097410	·0080693	·0066276	·0053967	·0043562	·0034856	*1·2*
1·3	·0135256	·0114814	·0096658	·0080693	·0066797	·0054823	·0044608	·0035981	·0028768	*1·3*
1·4	·0111327	·0094435+	·0079445+	·0066276	·0054823	·0044962	·0036557	·0029465+	·0023541	*1·4*
1·5	·0090846	·0077007	·0064737	·0053967	·0044608	·0036557	·0029701	·0023921	·0019097	*1·5*
1·6	·0073490	·0062251	·0052294	·0043562	·0035981	·0029465+	·0023921	·0019251	·0015357	*1·6*
1·7	·0058932	·0049883	·0041874	·0034856	·0028768	·0023541	·0019097	·0015357	·0012241	*1·7*
1·8	·0046842	·0039620	·0033234	·0027644	·0022799	·0018642	·0015111	·0012142	·0009671	*1·8*
1·9	·0036902	·0031190	·0026143	·0021729	·0017907	·0014631	·0011850+	·0009515−	·0007572	*1·9*
2·0	·0028812	·0024334	·0020382	·0016927	·0013939	·0011380	·0009210	·0007389	·0005876	*2·0*
2·1	·0022294	·0018815−	·0015747	·0013068	·0010753	·0008772	·0007094	·0005687	·0004519	*2·1*
2·2	·0017094	·0014416	·0012056	·0009998	·0008220	·0006700	·0005414	·0004337	·0003443	*2·2*
2·3	·0012988	·0010945+	·0009147	·0007579	·0006227	·0005071	·0004095−	·0003277	·0002600	*2·3*
2·4	·0009779	·0008234	·0006876	·0005693	·0004673	·0003808	·0003069	·0002454	·0001945+	*2·4*
2·5	·0007295+	·0006138	·0005121	·0004237	·0003475+	·0002826	·0002278	·0001820	·0001442	*2·5*
2·6	·0005392	·0004533	·0003779	·0003124	·0002561	·0002081	·0001676	·0001338	·0001059	*2·6*

k	d/N for $r = -\cdot10$									k
	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	
0·0	·0148273	·0117520	·0092319	·0071875+	·0055466	·0042401	·0032124	·0024117	·0017939	0·0
0·1	·0134400	·0106459	·0083578	·0065029	·0050142	·0038313	·0029009	·0021764	·0016178	0·1
0·2	·0120967	·0095758	·0075129	·0058417	·0045014	·0034373	·0026009	·0019500	·0014486	0·2
0·3	·0108090	·0085509	·0067044	·0052097	·0040117	·0030613	·0023148	·0017343	·0012875	0·3
0·4	·0095871	·0075792	·0059386	·0046115−	·0035487	·0027061	·0020448	·0015310	·0011357	0·4
0·5	·0084391	·0066672	·0052204	·0040510	·0031152	·0023739	·0017925+	·0013411	·0009942	0·5
0·6	·0073715−	·0058197	·0045537	·0035311	·0027135+	·0020663	·0015591	·0011657	·0008635+	0·6
0·7	·0063885+	·0050401	·0039400	·0030538	·0023450−	·0017844	·0013455−	·0010052	·0007441	0·7
0·8	·0054926	·0043302	·0033834	·0026198	·0020103	·0015286	·0011517	·0008598	·0006360	0·8
0·9	·0046842	·0036902	·0028812	·0022294	·0017094	·0012988	·0009779	·0007295+	·0005392	0·9
1·0	·0039620	·0031190	·0024334	·0018815−	·0014416	·0010945+	·0008234	·0006138	·0004533	1·0
1·1	·0033234	·0026143	·0020382	·0015747	·0012056	·0009147	·0006876	·0005121	·0003779	1·1
1·2	·0027644	·0021729	·0016927	·0013068	·0009998	·0007579	·0005693	·0004237	·0003124	1·2
1·3	·0022799	·0017907	·0013930	·0010753	·0008220	·0006227	·0004673	·0003475+	·0002561	1·3
1·4	·0018642	·0014631	·0011380	·0008772	·0006700	·0005071	·0003803	·0002826	·0002081	1·4
1·5	·0015111	·0011850+	·0009210	·0007094	·0005414	·0004095−	·0003069	·0002278	·0001676	1·5
1·6	·0012142	·0009515−	·0007389	·0005687	·0004337	·0003277	·0002454	·0001820	·0001338	1·6
1·7	·0009671	·0007572	·0005876	·0004519	·0003443	·0002600	·0001945+	·0001442	·0001059	1·7
1·8	·0007634	·0005973	·0004631	·0003558	·0002709	·0002044	·0001528	·0001132	·0000831	1·8
1·9	·0005973	·0004669	·0003618	·0002777	·0002113	·0001593	·0001190	·0000880	·0000646	1·9
2·0	·0004631	·0003618	·0002800	·0002148	·0001633	·0001230	·0000918	·0000679	·0000497	2·0
2·1	·0003558	·0002777	·0002148	·0001647	·0001251	·0000941	·0000702	·0000519	·0000379	2·1
2·2	·0002709	·0002113	·0001633	·0001251	·0000949	·0000714	·0000532	·0000393	·0000287	2·2
2·3	·0002044	·0001593	·0001230	·0000941	·0000714	·0000536	·0000399	·0000294	·0000215+	2·3
2·4	·0001528	·0001190	·0000918	·0000702	·0000532	·0000399	·0000297	·0000219	·0000160	2·4
2·5	·0001132	·0000880	·0000679	·0000519	·0000393	·0000294	·0000219	·0000161	·0000118	2·5
2·6	·0000831	·0000646	·0000497	·0000379	·0000287	·0000215+	·0000160	·0000118	·0000086	2·6

k	d/N for $r = -\cdot15$									k
	$h = 0{\cdot}0$	$h = 0{\cdot}1$	$h = 0{\cdot}2$	$h = 0{\cdot}3$	$h = 0{\cdot}4$	$h = 0{\cdot}5$	$h = 0{\cdot}6$	$h = 0{\cdot}7$	$h = 0{\cdot}8$	
0·0	·2260363	·2062428	·1868846	·1681429	·1501813	·1331410	·1171378	·1022603	·0885689	0·0
0·1	·2062428	·1880530	·1702809	·1530921	·1366354	·1210385−	·1064057	·0928162	·0803228	0·1
0·2	·1868846	·1702809	·1540752	·1384176	·1234424	·1092644	·0959769	·0836495+	·0723285+	0·2
0·3	·1681429	·1530921	·1384176	·1242545+	·1107231	·0979258	·0859451	·0748422	·0646567	0·3
0·4	·1501813	·1366354	·1234424	·1107231	·0985844	·0871169	·0763930	·0664657	·0573689	0·4
0·5	·1331410	·1210385−	·1092644	·0979258	·0871169	·0769171	·0673895−	·0585797	·0505160	0·5
0·6	·1171378	·1064057	·0959769	·0859451	·0763930	·0673895−	·0589890	·0512304	·0441371	0·6
0·7	·1022603	·0928162	·0836495+	·0748422	·0664657	·0585797	·0512304	·0444507	·0382597	0·7
0·8	·0885689	·0803228	·0723285+	·0646567	·0573689	·0505160	·0441371	·0382597	·0328990	0·8
0·9	·0760961	·0689532	·0620368	·0554073	·0491173	·0432098	·0377177	·0326634	·0280592	0·9
1·0	·0648481	·0587107	·0527752	·0470928	·0417081	·0366570	·0319669	·0276560	·0237330	1·0
1·1	·0548072	·0495769	·0445248	·0396943	·0351225+	·0308393	·0268672	·0232200	·0199075+	1·1
1·2	·0459342	·0415138	·0372495+	·0331774	·0293282	·0257265+	·0223907	·0193323	·0165567	1·2
1·3	·0381724	·0344680	·0308989	·0274950+	·0242816	·0212786	·0185009	·0159575−	·0136522	1·3
1·4	·0314513	·0283732	·0254115−	·0225904	·0199306	·0174483	·0151551	·0130581	·0111599	1·4
1·5	·0256900	·0231544	·0207178	·0183999	·0162174	·0141832	·0123085−	·0105925−	·0090431	1·5
1·6	·0208013	·0187307	·0167436	·0148557	·0130805+	·0114281	·0099056	·0085169	·0072633	1·6
1·7	·0166950+	·0150190	·0134126	·0118885+	·0104573	·0091268	·0079025+	·0067874	·0057821	1·7
1·8	·0132807	·0119360	·0106489	·0094294	·0082857	·0072240	·0062483	·0053608	·0045618	1·8
1·9	·0104705−	·0094012	·0083791	·0074121	·0065063	·0056666	·0048960	·0041961	·0035667	1·9
2·0	·0081807	·0073381	·0065338	·0057738	·0050630	·0044049	·0038018	·0032546	·0027634	2·0
2·1	·0063340	·0056760	·0050488	·0044569	·0039041	·0033930	·0029252	·0025015−	·0021215−	2·1
2·2	·0048595+	·0043504	·0038657	·0034090	·0029830	·0025897	·0022302	·0019050−	·0016138	2·2
2·3	·0036942	·0033039	·0029328	·0025836	·0022583	·0019584	·0016847	·0014374	·0012163	2·3
2·4	·0027826	·0024860	·0022045−	·0019399	·0016939	·0014673	·0012608	·0010746	·0009082	2·4
2·5	·0020765+	·0018533	·0016417	·0014432	·0012588	·0010892	·0009349	·0007959	·0006719	2·5
2·6	·0015352	·0013690	·0012113	·0010637	·0009267	·0008010	·0006867	·0005839	·0004924	2·6

k	*d/N* for *r* = − ·15									*k*
	h = *0·9*	*h* = *1·0*	*h* = *1·1*	*h* = *1·2*	*h* = *1·3*	*h* = *1·4*	*h* = *1·5*	*h* = *1·6*	*h* = *1·7*	
0·0	·0760961	·0648481	·0548072	·0460342	·0381724	·0314513	·0256900	·0208013	·0166950+	*0·0*
0·1	·0689532	·0587107	·0495769	·0415138	·0344680	·0283732	·0231544	·0187307	·0150190	*0·1*
0·2	·0620368	·0527752	·0445248	·0372495+	·0308989	·0254115−	·0207178	·0167436	·0134126	*0·2*
0·3	·0554073	·0470028	·0396043	·0331774	·0274950+	·0225904	·0183990	·0148557	·0118885+	*0·3*
0·4	·0491173	·0417081	·0351225+	·0293282	·0242816	·0199306	·0162174	·0130805+	·0104573	*0·4*
0·5	·0432098	·0366570	·0308393	·0257265+	·0212786	·0174483	·0141832	·0114281	·0091268	*0·5*
0·6	·0377177	·0319669	·0268672	·0223907	·0185009	·0151551	·0123065−	·0099056	·0079025+	*0·6*
0·7	·0326634	·0276560	·0232209	·0193323	·0159575−	·0130581	·0105925−	·0085169	·0067874	*0·7*
0·8	·0280592	·0237339	·0199075+	·0165567	·0136522	·0111509	·0090431	·0072633	·0057821	*0·8*
0·9	·0239075−	·0202016	·0169272	·0140633	·0115839	·0094591	·0076566	·0061430	·0048848	*0·0*
1·0	·0202016	·0170525+	·0142736	·0118461	·0097472	·0079506	·0064285+	·0051520	·0040922	*1·0*
1·1	·0169272	·0142736	·0119349	·0098945+	·0081325−	·0066263	·0053517	·0042842	·0033991	*1·1*
1·2	·0140633	·0118461	·0098945+	·0081940	·0067274	·0054754	·0044172	·0035321	·0027992	*1·2*
1·3	·0115839	·0097472	·0081325−	·0067274	·0055172	·0044853	·0036144	·0028868	·0022851	*1·3*
1·4	·0094591	·0079506	·0066263	·0054754	·0044853	·0036422	·0029317	·0023388	·0018492	*1·4*
1·5	·0076566	·0064285+	·0053517	·0044172	·0036144	·0029317	·0023570	·0018781	·0014832	*1·5*
1·6	·0061430	·0051520	·0042842	·0035321	·0028868	·0023388	·0018781	·0014948	·0011791	*1·6*
1·7	·0048848	·0040922	·0033991	·0027992	·0022851	·0018492	·0014832	·0011791	·0009289	*1·7*
1·8	·0038496	·0032213	·0026727	·0021984	·0017926	·0014489	·0011608	·0009217	·0007252	*1·8*
1·9	·0030065−	·0025129	·0020825+	·0017110	·0013935+	·0011250+	·0009002	·0007139	·0005611	*1·9*
2·0	·0023267	·0019425+	·0016080	·0013195+	·0010734	·0008656	·0006918	·0005479	·0004301	*2·0*
2·1	·0017842	·0014879	·0012302	·0010083	·0008193	·0006599	·0005267	·0004167	·0003267	*2·1*
2·2	·0013557	·0011292	·0009325+	·0007635−	·0006196	·0004984	·0003973	·0003140	·0002458	*2·2*
2·3	·0010206	·0008491	·0007004	·0005727	·0004642	·0003730	·0002970	·0002344	·0001833	*2·3*
2·4	·0007612	·0006326	·0005211	·0004256	·0003446	·0002765+	·0002199	·0001733	·0001354	*2·4*
2·5	·0005625−	·0004668	·0003842	·0003134	·0002534	·0002031	·0001613	·0001270	·0000991	*2·5*
2·6	·0004117	·0003413	·0002805+	·0002286	·0001846	·0001478	·0001172	·0000922	·0000718	*2·6*

k	*d/N* for *r* = − ·15									*k*
	h = *1·8*	*h* = *1·9*	*h* = *2·0*	*h* = *2·1*	*h* = *2·2*	*h* = *2·3*	*h* = *2·4*	*h* = *2·5*	*h* = *2·6*	
0·0	·0132807	·0104705−	·0081807	·0063340	·0048595+	·0036942	·0027826	·0020765+	·0015352	*0·0*
0·1	·0119360	·0094012	·0073381	·0056760	·0043504	·0033039	·0024860	·0018533	·0013690	*0·1*
0·2	·0106489	·0083791	·0065338	·0050488	·0038657	·0029328	·0022045−	·0016417	·0012113	*0·2*
0·3	·0094294	·0074121	·0057738	·0044569	·0034090	·0025836	·0019399	·0014432	·0010637	*0·3*
0·4	·0082857	·0065063	·0050630	·0039041	·0029830	·0022583	·0016939	·0012588	·0009267	*0·4*
0·5	·0072240	·0056666	·0044049	·0033930	·0025897	·0019584	·0014673	·0010892	·0008010	*0·5*
0·6	·0062483	·0048960	·0038018	·0029252	·0022302	·0016847	·0012608	·0009349	·0006867	*0·6*
0·7	·0053608	·0041961	·0032546	·0025015−	·0019050−	·0014374	·0010746	·0007959	·0005839	*0·7*
0·8	·0045618	·0035667	·0027634	·0021215−	·0016138	·0012163	·0009082	·0006719	·0004924	*0·8*
0·9	·0038496	·0030065−	·0023267	·0017842	·0013557	·0010206	·0007612	·0005625−	·0004117	*0·9*
1·0	·0032213	·0025129	·0019425+	·0014879	·0011292	·0008491	·0006326	·0004668	·0003413	*1·0*
1·1	·0026727	·0020825+	·0016080	·0012302	·0009325+	·0007004	·0005211	·0003842	·0002805+	*1·1*
1·2	·0021984	·0017110	·0013195+	·0010083	·0007635−	·0005727	·0004256	·0003134	·0002286	*1·2*
1·3	·0017926	·0013935+	·0010734	·0008193	·0006196	·0004642	·0003446	·0002534	·0001846	*1·3*
1·4	·0014489	·0011250+	·0008656	·0006599	·0004984	·0003730	·0002765+	·0002031	·0001478	*1·4*
1·5	·0011608	·0009002	·0006918	·0005267	·0003973	·0002970	·0002199	·0001613	·0001172	*1·5*
1·6	·0009217	·0007139	·0005479	·0004167	·0003140	·0002344	·0001733	·0001270	·0000922	*1·6*
1·7	·0007252	·0005611	·0004301	·0003267	·0002458	·0001833	·0001354	·0000991	·0000718	*1·7*
1·8	·0005655+	·0004370	·0003346	·0002538	·0001908	·0001420	·0001048	·0000766	·0000554	*1·8*
1·9	·0004370	·0003372	·0002579	·0001954	·0001467	·0001091	·0000804	·0000586	·0000424	*1·9*
2·0	·0003346	·0002579	·0001969	·0001490	·0001117	·0000830	·0000611	·0000445+	·0000321	*2·0*
2·1	·0002538	·0001954	·0001490	·0001126	·0000843	·0000626	·0000460	·0000335−	·0000241	*2·1*
2·2	·0001908	·0001467	·0001117	·0000843	·0000631	·0000467	·0000343	·0000249	·0000179	*2·2*
2·3	·0001420	·0001091	·0000830	·0000626	·0000467	·0000346	·0000253	·0000184	·0000132	*2·3*
2·4	·0001048	·0000804	·0000611	·0000460	·0000343	·0000253	·0000185+	·0000134	·0000097	*2·1*
2·5	·0000766	·0000586	·0000445+	·0000335−	·0000249	·0000184	·0000134	·0000097	·0000070	*2·5*
2·6	·0000554	·0000424	·0000321	·0000241	·0000179	·0000132	·0000097	·0000070	·0000050+	*2·6*

d/N for $r = -\cdot 20$

k	$h = 0\cdot 0$	$h = 0\cdot 1$	$h = 0\cdot 2$	$h = 0\cdot 3$	$h = 0\cdot 4$	$h = 0\cdot 5$	$h = 0\cdot 6$	$h = 0\cdot 7$	$h = 0\cdot 8$	k
0·0	·2179529	·1982010	·1789662	·1604263	·1427384	·1260358	·1104246	·0959826	·0827586	0·0
0·1	·1982010	·1800670	·1624318	·1454567	·1292840	·1140334	·0997991	·0866492	·0746254	0·1
0·2	·1789662	·1624318	·1463745−	·1309401	·1162561	·1024289	·0895418	·0776536	·0667991	0·2
0·3	·1604263	·1454567	·1309401	·1170068	·1037703	·0913243	·0797415	·0690724	·0593454	0·3
0·4	·1427384	·1292840	·1162561	·1037703	·0919265−	·0808068	·0704739	·0609705	·0523195	0·4
0·5	·1260358	·1140334	·1024289	·0913243	·0808068	·0709475	·0617999	·0533996	·0457647	0·5
0·6	·1104246	·0997991	·0895418	·0797415−	·0704739	·0617999	·0537647	·0463976	·0397124	0·6
0·7	·0959826	·0866492	·0776536	·0690724	·0609705−	·0533996	·0463976	·0399481	·0341813	0·7
0·8	·0827586	·0746254	·0667991	·0593454	·0523195−	·0457647	·0397124	·0341813	·0291784	0·8
0·9	·0707737	·0637436	·0569899	·0505683	·0445252	·0388967	·0337082	·0289744	·0246999	0·9
1·0	·0600228	·0539961	·0482160	·0427293	·0375747	·0327817	·0283709	·0243534	·0207319	1·0
1·1	·0504773	·0453540	·0404485+	·0358000	·0314402	·0273932	·0236752	·0202946	·0172523	1·1
1·2	·0420888	·0377702	·0336424	·0297375−	·0260814	·0226934	·0195863	·0167660	·0142324	1·2
1·3	·0347923	·0311833	·0277397	·0244877	·0214482	·0186365−	·0160624	·0137300	·0116383	1·3
1·4	·0285107	·0255208	·0226730	·0199882	·0174834	·0151704	·0130565+	·0111446	·0094330	1·4
1·5	·0231580	·0207028	·0183683	·0161714	·0141254	·0122394	·0105189	·0089655+	·0075774	1·5
1·6	·0186436	·0166452	·0147485+	·0129669	·0113106	·0097864	·0083986	·0071478	·0060321	1·6
1·7	·0148751	·0132631	·0117359	·0103039	·0089750+	·0077545+	·0066451	·0056471	·0047585−	1·7
1·8	·0117614	·0104729	·0092543	·0081138	·0070573	·0060887	·0052099	·0044208	·0037195−	1·8
1·9	·0092152	·0081945−	·0072310	·0063309	·0054986	·0047370	·0040473	·0034291	·0028807	1·9
2·0	·0071542	·0063531	·0055983	·0048944	·0042448	·0036516+	·0031151	·0026353	·0022104	2·0
2·1	·0055032	·0048802	·0042943	·0037490	·0032457	·0027887	·0023755	·0019965+	·0016803	2·1
2·2	·0041940	·0037141	·0032635+	·0028450−	·0024601	·0021099	·0017945+	·0015134	·0012655−	2·2
2·3	·0031666	·0028003	·0024571	·0021388	·0018467	·0015814	·0013430	·0011308	·0009441	2·3
2·4	·0023685+	·0020916	·0018326	·0015928	·0013732	·0011742	·0009956	·0008370	·0006977	2·4
2·5	·0017550−	·0015475+	·0013539	·0011751	·0010115+	·0008636	·0007311	·0006137	·0005107	2·5
2·6	·0012881	·0011342	·0009908	·0008587	·0007380	·0006291	·0005318	·0004456	·0003703	2·6

d/N for $r = -\cdot 20$

k	$h = 0\cdot 9$	$h = 1\cdot 0$	$h = 1\cdot 1$	$h = 1\cdot 2$	$h = 1\cdot 3$	$h = 1\cdot 4$	$h = 1\cdot 5$	$h = 1\cdot 6$	$h = 1\cdot 7$	k
0·0	·0707737	·0600228	·0504773	·0420888	·0347923	·0285107	·0231580	·0186436	·0148751	0·0
0·1	·0637436	·0539961	·0453540	·0377702	·0311833	·0255208	·0207028	·0166452	·0132631	0·1
0·2	·0569899	·0482160	·0404485+	·0336424	·0277397	·0226730	·0183683	·0147485+	·0117359	0·2
0·3	·0505683	·0427293	·0358000	·0297375−	·0244877	·0199882	·0161714	·0129669	·0103039	0·3
0·4	·0445252	·0375747	·0314402	·0260814	·0214482	·0174834	·0141254	·0113106	·0089750+	0·4
0·5	·0388967	·0327817	·0273932	·0226934	·0186365−	·0151704	·0122394	·0097864	·0077545+	0·5
0·6	·0337082	·0283709	·0236752	·0195863	·0160624	·0130565+	·0105189	·0083986	·0066451	0·6
0·7	·0289744	·0243534	·0202946	·0167660	·0137300	·0111446	·0089655+	·0071478	·0056471	0·7
0·8	·0246999	·0207319	·0172523	·0142324	·0116383	·0094330	·0075774	·0060321	·0047585−	0·8
0·9	·0208798	·0175008	·0145427	·0119797	·0097819	·0079166	·0063498	·0050472	·0039754	0·9
1·0	·0175008	·0146477	·0121542	·0099975+	·0081512	·0065870	·0052753	·0041867	·0032926	1·0
1·1	·0145427	·0121542	·0100704	·0082712	·0067336	·0054331	·0043445+	·0034427	·0027032	1·1
1·2	·0119797	·0099975+	·0082712	·0067832	·0055138	·0044421	·0035465+	·0028059	·0021997	1·2
1·3	·0097819	·0081512	·0067336	·0055138	·0044751	·0035996	·0028694	·0022666	·0017741	1·3
1·4	·0079166	·0065870	·0054331	·0044421	·0035996	·0028909	·0023008	·0018146	·0014180	1·4
1·5	·0063498	·0052753	·0043445+	·0035465+	·0028694	·0023008	·0018283	·0014395+	·0011231	1·5
1·6	·0050472	·0041867	·0034427	·0028059	·0022666	·0018146	·0014395+	·0011316	·0008814	1·6
1·7	·0039754	·0032926	·0027032	·0021997	·0017741	·0014180	·0011231	·0008814	·0006854	1·7
1·8	·0031026	·0025657	·0021031	·0017087	·0013759	·0010979	·0008681	·0006802	·0005281	1·8
1·9	·0023992	·0019809	·0016212	·0013150+	·0010571	·0008422	·0006648	·0005201	·0004030	1·9
2·0	·0018381	·0015152	·0012381	·0010026	·0008047	·0006400	·0005044	·0003939	·0003048	2·0
2·1	·0013951	·0011482	·0009367	·0007573	·0006068	·0004818	·0003791	·0002955+	·0002283	2·1
2·2	·0010490	·0008619	·0007020	·0005667	·0004533	·0003593	·0002822	·0002197	·0001694	2·2
2·3	·0007813	·0006410	·0005212	·0004200	·0003354	·0002654	·0002081	·0001617	·0001245−	2·3
2·4	·0005765−	·0004721	·0003833	·0003083	·0002458	·0001942	·0001520	·0001179	·0000906	2·4
2·5	·0004213	·0003445−	·0002792	·0002242	·0001785−	·0001408	·0001100	·0000852	·0000653	2·5
2·6	·0003049	·0002489	·0002014	·0001615−	·0001283	·0001010	·0000788	·0000609	·0000467	2·6

k	*d/N* for *r* = − ·20									*k*
	h = *1·8*	*h* = *1·9*	*h* = *2·0*	*h* = *2·1*	*h* = *2·2*	*h* = *2·3*	*h* = *2·4*	*h* = *2·5*	*h* = *2·6*	
0·0	·0117614	·0092152	·0071542	·0055032	·0041940	·0031666	·0023685+	·0017550−	·0012881	*0·0*
0·1	·0104729	·0081945−	·0063531	·0048802	·0037141	·0028003	·0020916	·0015475+	·0011342	*0·1*
0·2	·0092543	·0072310	·0055983	·0042943	·0032635+	·0024571	·0018326	·0013539	·0009908	*0·2*
0·3	·0081138	·0063300	·0048944	·0037490	·0028450−	·0021388	·0015928	·0011751	·0008587	*0·3*
0·4	·0070573	·0054986	·0042448	·0032467	·0024601	·0018467	·0013732	·0010115+	·0007380	*0·4*
0·5	·0060887	·0047370	·0036515+	·0027887	·0021099	·0015814	·0011742	·0008636	·0006291	*0·5*
0·6	·0052099	·0040473	·0031151	·0023755−	·0017945+	·0013430	·0009956	·0007311	·0005318	*0·6*
0·7	·0044208	·0034291	·0026353	·0020065+	·0015134	·0011308	·0008370	·0006137	·0004456	*0·7*
0·8	·0037195−	·0028807	·0022104	·0016803	·0012655−	·0009441	·0006977	·0005107	·0003703	*0·8*
0·9	·0031026	·0023992	·0018381	·0013951	·0010490	·0007813	·0005765−	·0004213	·0003049	*0·9*
1·0	·0025657	·0019800	·0015152	·0011482	·0008619	·0006410	·0004721	·0003445−	·0002489	*1·0*
1·1	·0021031	·0016212	·0012381	·0009367	·0007020	·0005212	·0003833	·0002792	·0002014	*1·1*
1·2	·0017087	·0013150+	·0010026	·0007573	·0005667	·0004200	·0003083	·0002242	·0001615−	*1·2*
1·3	·0013759	·0010571	·0008047	·0006068	·0004533	·0003354	·0002458	·0001785−	·0001283	*1·3*
1·4	·0010979	·0008422	·0006400	·0004818	·0003593	·0002654	·0001942	·0001408	·0001010	*1·4*
1·5	·0008681	·0006648	·0005044	·0003791	·0002822	·0002081	·0001520	·0001100	·0000788	*1·5*
1·6	·0006802	·0005201	·0003939	·0002955+	·0002197	·0001617	·0001179	·0000852	·0000609	*1·6*
1·7	·0005281	·0004030	·0003048	·0002283	·0001694	·0001245−	·0000906	·0000653	·0000467	*1·7*
1·8	·0004061	·0003095−	·0002336	·0001747	·0001294	·0000949	·0000690	·0000496	·0000354	*1·8*
1·9	·0003095−	·0002354	·0001774	·0001324	·0000979	·0000717	·0000520	·0000374	·0000266	*1·9*
2·0	·0002336	·0001774	·0001335−	·0000994	·0000734	·0000537	·0000389	·0000279	·0000198	*2·0*
2·1	·0001747	·0001324	·0000994	·0000740	·0000545+	·0000398	·0000288	·0000206	·0000146	*2·1*
2·2	·0001294	·0000979	·0000734	·0000545+	·0000401	·0000292	·0000211	·0000151	·0000107	*2·2*
2·3	·0000949	·0000717	·0000537	·0000398	·0000292	·0000212	·0000153	·0000109	·0000077	*2·3*
2·4	·0000690	·0000520	·0000389	·0000288	·0000211	·0000153	·0000110	·0000078	·0000055+	*2·4*
2·5	·0000496	·0000374	·0000279	·0000206	·0000151	·0000109	·0000078	·0000056	·0000039	*2·5*
2·6	·0000354	·0000266	·0000198	·0000146	·0000107	·0000077	·0000055+	·0000039	·0000028	*2·6*

k	*d/N* for *r* = − ·25									*k*
	h = *0·0*	*h* = *0·1*	*h* = *0·2*	*h* = *0·3*	*h* = *0·4*	*h* = *0·5*	*h* = *0·6*	*h* = *0·7*	*h* = *0·8*	
0·0	·2097847	·1900757	·1709684	·1526363	·1352305−	·1188755+	·1036675−	·0896727	·0769282	*0·0*
0·1	·1900757	·1720036	·1545135+	·1377626	·1218861	·1069947	·0931724	·0804759	·0689346	*0·1*
0·2	·1709684	·1545135+	·1386173	·1234203	·1090429	·0955823	·0831112	·0716771	·0613028	*0·2*
0·3	·1526363	·1377626	·1234203	·1097345−	·0968110	·0847343	·0735666	·0633471	·0540926	*0·3*
0·4	·1352305−	·1218861	·1090429	·0968110	·0852824	·0745302	·0646066	·0555433	·0473520	*0·4*
0·5	·1188755+	·1069947	·0955823	·0847343	·0745302	·0650321	·0562833	·0483089	·0411162	*0·5*
0·6	·1036675−	·0931724	·0831112	·0735660	·0646066	·0562833	·0486322	·0416726	·0354080	*0·6*
0·7	·0896727	·0804759	·0716771	·0633471	·0555433	·0483089	·0416726	·0356484	·0302373	*0·7*
0·8	·0769282	·0689346	·0613028	·0540926	·0473520	·0411162	·0354080	·0302373	·0256025+	*0·8*
0·9	·0654427	·0585527	·0519882	·0457995−	·0400260	·0346964	·0298281	·0254276	·0214917	*0·9*
1·0	·0551995+	·0493107	·0437121	·0384453	·0335425−	·0290263	·0249099	·0211972	·0178837	*1·0*
1·1	·0461592	·0411692	·0364354	·0319918	·0278643	·0240706	·0206204	·0175153	·0147502	*1·1*
1·2	·0382634	·0340717	·0301040	·0263877	·0229434	·0197847	·0169182	·0143444	·0120574	*1·2*
1·3	·0314390	·0279489	·0246526	·0215721	·0187233	·0161167	·0137565+	·0116420	·0097675−	*1·3*
1·4	·0256019	·0227218	·0200078	·0174771	·0151421	·0130103	·0110846	·0093632	·0078406	*1·4*
1·5	·0206614	·0183060	·0160915−	·0140313	·0121347	·0104072	·0088502	·0074616	·0062362	*1·5*
1·6	·0165231	·0146144	·0128239	·0111620	·0096357	·0082486	·0070012	·0058914	·0049143	*1·6*
1·7	·0130929	·0115604	·0101260	·0087978	·0075807	·0064772	·0054873	·0046085+	·0038365+	*1·7*
1·8	·0102794	·0090602	·0079218	·0068701	·0059086	·0050390	·0042606	·0035713	·0029672	*1·8*
1·9	·0079956	·0070347	·0061396	·0053147	·0045623	·0038833	·0032771	·0027414	·0022732	*1·9*
2·0	·0061611	·0054110	·0047138	·0040729	·0034896	·0029645+	·0024968	·0020845+	·0017250+	*2·0*
2·1	·0047029	·0041228	·0035850+	·0030916	·0026438	·0022416	·0018842	·0015700	·0012965+	*2·1*
2·2	·0035560	·0031116	·0027006	·0023245+	·0019840	·0016789	·0014084	·0011711	·0009652	*2·2*
2·3	·0026632	·0023261	·0020151	·0017311	·0014746	·0012458	·0010426	·0008652	·0007116	*2·3*
2·4	·0019755+	·0017222	·0014891	·0012768	·0010855−	·0009149	·0007644	·0006330	·0005195+	*2·4*
2·5	·0014514	·0012629	·0010898	·0009326	·0007913	·0006656	·0005550−	·0004586	·0003756	*2·5*
2·6	·0010560	·0009171	·0007899	·0006746	·0005713	·0004795+	·0003990	·0003291	·0002690	*2·6*

d/N for *r* = −·25

k	*h* = 0·9	*h* = 1·0	*h* = 1·1	*h* = 1·2	*h* = 1·3	*h* = 1·4	*h* = 1·5	*h* = 1·6	*h* = 1·7	*k*
0·0	·0654427	·0551995+	·0461592	·0382634	·0314300	·0256019	·0206814	·0165271	·0130920	0·0
0·1	·0585527	·0493107	·0411692	·0340717	·0279489	·0227218	·0183000	·0146144	·0115094	0·1
0·2	·0519882	·0437121	·0364354	·0301040	·0246526	·0200078	·0160015	·0128235	·0101260	0·2
0·3	·0457095−	·0384453	·0319918	·0263877	·0215721	·0174771	·0140313	·0111026	·0087978	0·3
0·4	·0400260	·0335425−	·0278643	·0229434	·0187233	·0151421	·0121347	·0096357	·0075607	0·4
0·5	·0346964	·0290263	·0240706	·0197847	·0161167	·0130103	·0104072	·0082496	·0064772	0·5
0·6	·0298281	·0249099	·0206204	·0169182	·0137565+	·0110846	·0088502	·0070012	·0054873	0·6
0·7	·0254276	·0211972	·0175153	·0143444	·0116420	·0093632	·0074616	·0058914	·0046065+	0·7
0·8	·0214917	·0178837	·0147502	·0120574	·0097675	·0078406	·0062362	·0049143	·0038365+	0·8
0·9	·0180082	·0149575+	·0123138	·0100468	·0081232	·0065081	·0051662	·0040631	·0031658	0·9
1·0	·0149575+	·0124005−	·0101894	·0082976	·0066959	·0053541	·0042418	·0033294	·0025889	1·0
1·1	·0123138	·0101894	·0083566	·0067919	·0054701	·0043653	·0034515	·0027036	·0020980	1·1
1·2	·0100468	·0082976	·0067919	·0055093	·0044284	·0035260	·0027830	·0021755+	·0016847	1·2
1·3	·0081232	·0066959	·0054701	·0044284	·0035524	·0028235	·0022234	·0017345	·0013404	1·3
1·4	·0065081	·0053541	·0043653	·0035260	·0028235	·0022306	·0017500	·0013701	·0010566	1·4
1·5	·0051662	·0042418	·0034515−	·0027830	·0022234	·0017500	·0013801	·0010722	·0008251	1·5
1·6	·0040631	·0033294	·0027036	·0021755+	·0017345	·0013701	·0010722	·0008312	·0006383	1·6
1·7	·0031658	·0025889	·0020980	·0016847	·0013404	·0010566	·0008251	·0006383	·0004891	1·7
1·8	·0024435−	·0019942	·0016127	·0012923	·0010261	·0008071	·0006289	·0004855	·0003712	1·8
1·9	·0018682	·0015215+	·0012279	·0009819	·0007780	·0006106	·0004748	·0003657	·0002790	1·9
2·0	·0014148	·0011499	·0009261	·0007390	·0005842	·0004576	·0003550+	·0002729	·0002077	2·0
2·1	·0010612	·0008607	·0006917	·0005508	·0004345+	·0003396	·0002620	·0002016	·0001532	2·1
2·2	·0007883	·0006380	·0005117	·0004066	·0003201	·0002496	·0001928	·0001475+	·0001118	2·2
2·3	·0005800	·0004684	·0003749	·0002972	·0002335	·0001817	·0001400	·0001069	·0000809	2·3
2·4	·0004226	·0003406	·0002720	·0002152	·0001687	·0001309	·0001007	·0000767	·0000579	2·4
2·5	·0003049	·0002452	·0001954	·0001542	·0001206	·0000935	·0000717	·0000545+	·0000410	2·5
2·6	·0002178	·0001748	·0001390	·0001095−	·0000854	·0000660	·0000506	·0000383	·0000288	2·6

d/N for *r* = −·25

k	*h* = 1·8	*h* = 1·9	*h* = 2·0	*h* = 2·1	*h* = 2·2	*h* = 2·3	*h* = 2·4	*h* = 2·5	*h* = 2·6	*k*
0·0	·0102794	·0079956	·0061611	·0047029	·0035560	·0026632	·0019735+	·0014514	·0010580	0·0
0·1	·0090602	·0070347	·0054110	·0041228	·0031116	·0023261	·0017222	·0012629	·0009171	0·1
0·2	·0079218	·0061396	·0047138	·0035850+	·0027006	·0020151	·0014891	·0010898	·0007809	0·2
0·3	·0068701	·0053147	·0040729	·0030916	·0023245+	·0017311	·0012768	·0009326	·0006746	0·3
0·4	·0059086	·0045623	·0034896	·0026438	·0019840	·0014746	·0010855−	·0007913	·0005713	0·4
0·5	·0050390	·0038833	·0029645+	·0022416	·0016789	·0012463	·0009149	·0006656	·0004796+	0·5
0·6	·0042606	·0032771	·0024968	·0018842	·0014084	·0010426	·0007644	·0005550	·0003990	0·6
0·7	·0035713	·0027414	·0020845+	·0015700	·0011711	·0008652	·0006330	·0004586	·0003291	0·7
0·8	·0029672	·0022732	·0017250+	·0012965+	·0009652	·0007115	·0005195+	·0003756	·0002690	0·8
0·9	·0024435−	·0018682	·0014148	·0010612	·0007883	·0005800	·0004226	·0003049	·0002178	0·9
1·0	·0019942	·0015215+	·0011499	·0008607	·0006380	·0004684	·0003406	·0002452	·0001748	1·0
1·1	·0016127	·0012279	·0009261	·0006917	·0005117	·0003749	·0002720	·0001954	·0001390	1·1
1·2	·0012923	·0009819	·0007390	·0005508	·0004066	·0002972	·0002152	·0001542	·0001095−	1·2
1·3	·0010261	·0007780	·0005842	·0004345+	·0003201	·0002335−	·0001687	·0001206	·0000854	1·3
1·4	·0008071	·0006106	·0004576	·0003396	·0002496	·0001817	·0001309	·0000935	·0000660	1·4
1·5	·0006289	·0004748	·0003550+	·0002620	·0001928	·0001400	·0001007	·0000717	·0000506	1·5
1·6	·0004855−	·0003657	·0002728	·0002016	·0001475+	·0001069	·0000767	·0000545+	·0000383	1·6
1·7	·0003712	·0002790	·0002077	·0001532	·0001118	·0000809	·0000579	·0000410	·0000288	1·7
1·8	·0002811	·0002108	·0001566	·0001152	·0000839	·0000606	·0000433	·0000306	·0000214	1·8
1·9	·0002108	·0001578	·0001170	·0000859	·0000624	·0000449	·0000320	·0000226	·0000158	1·9
2·0	·0001566	·0001170	·0000865−	·0000633	·0000459	·0000330	·0000235−	·0000165+	·0000115+	2·0
2·1	·0001152	·0000859	·0000633	·0000463	·0000335−	·0000240	·0000170	·0000120	·0000083	2·1
2·2	·0000839	·0000624	·0000459	·0000335−	·0000242	·0000173	·0000122	·0000086	·0000060	2·2
2·3	·0000606	·0000449	·0000330	·0000240	·0000173	·0000123	·0000087	·0000061	·0000042	2·3
2·4	·0000433	·0000320	·0000235−	·0000170	·0000122	·0000087	·0000061	·0000043	·0000030	2·4
2·5	·0000306	·0000226	·0000165+	·0000120	·0000086	·0000061	·0000043	·0000030	·0000021	2·5
2·6	·0000214	·0000158	·0000115+	·0000083	·0000060	·0000042	·0000030	·0000021	·0000014	2·6

k	d/N for $r = -\cdot 30$									k
	$h = 0{\cdot}0$	$h = 0{\cdot}1$	$h = 0{\cdot}2$	$h = 0{\cdot}3$	$h = 0{\cdot}4$	$h = 0{\cdot}5$	$h = 0{\cdot}6$	$h = 0{\cdot}7$	$h = 0{\cdot}8$	
0·0	·2015067	·1818424	·1628676	·1447517	·1276390	·1116448	·0968546	·0833226	·0710731	*0·0*
0·1	·1818424	·1638633	·1465053	·1299903	·1144249	·0999093	·0865163	·0742903	·0632482	*0·1*
0·2	·1628676	·1465053	·1307846	·1158412	·1017888	·0887140	·0766782	·0657166	·0558396	*0·2*
0·3	·1447517	·1299903	·1158412	·1024238	·0898340	·0781481	·0674161	·0576653	·0489003	*0·3*
0·4	·1276390	·1144249	·1017888	·0898340	·0786444	·0682825+	·0587896	·0501855+	·0424703	*0·4*
0·5	·1116448	·0999093	·0887140	·0781481	·0682825+	·0591692	·0508408	·0433110	·0365760	*0·5*
0·6	·0968546	·0865163	·0766782	·0674161	·0587896	·0508408	·0435950−	·0370605+	·0312307	*0·6*
0·7	·0833226	·0742903	·0657166	·0576653	·0501855+	·0433110	·0370605+	·0314383	·0264353	*0·7*
0·8	·0710731	·0632482	·0558396	·0489003	·0424703	·0365760	·0312307	·0264353	·0221793	*0·8*
0·9	·0601022	·0533813	·0470347	·0411056	·0356261	·0306163	·0260852	·0220311	·0184427	*0·9*
1·0	·0503807	·0446585−	·0392691	·0342476	·0296193	·0253991	·0215924	·0181957	·0151972	*1·0*
1·1	·0418580	·0370290	·0324931	·0282782	·0244039	·0208808	·0177116	·0148913	·0124087	*1·1*
1·2	·0344655−	·0304268	·0266436	·0231378	·0199240	·0170096	·0143952	·0120751	·0100384	*1·2*
1·3	·0281215+	·0247746	·0216479	·0187585+	·0161171	·0137285−	·0115916	·0097007	·0080453	*1·3*
1·4	·0227354	·0199871	·0174269	·0150675+	·0129167	·0109772	·0092470	·0077202	·0063874	*1·4*
1·5	·0182110	·0159753	·0138984	·0119899	·0102550−	·0086949	·0073071	·0060861	·0050232	*1·5*
1·6	·0144511	·0126494	·0109805+	·0094512	·0080649	·0068220	·0057195−	·0047522	·0039127	*1·6*
1·7	·0113598	·0099216	·0085931	·0073794	·0062823	·0053015−	·0044340	·0036751	·0030183	*1·7*
1·8	·0088452	·0077081	·0066608	·0057067	·0048469	·0040804	·0034044	·0028148	·0023060	*1·8*
1·9	·0068216	·0059312	·0051136	·0043708	·0037034	·0031102	·0025886	·0021349	·0017446	*1·9*
2·0	·0052104	·0045200	·0038878	·0033153	·0028023	·0023476	·0019401	·0016035−	·0013069	*2·0*
2·1	·0039414	·0034112	·0029272	·0024902	·0020997	·0017547	·0014532	·0011925−	·0009695+	*2·1*
2·2	·0029525−	·0025494	·0021825−	·0018521	·0015579	·0012087	·0010728	·0008781	·0007121	*2·2*
2·3	·0021901	·0018866	·0016113	·0013641	·0011445+	·0009517	·0007842	·0006402	·0005178	*2·3*
2·4	·0016086	·0013825−	·0011778	·0009946	·0008325−	·0006905−	·0005675−	·0004621	·0003728	*2·4*
2·5	·0011699	·0010030	·0008524	·0007181	·0005905+	·0004960	·0004066	·0003302	·0002657	*2·5*
2·6	·0008424	·0007205−	·0006108	·0005133	·0004274	·0003527	·0002884	·0002336	·0001875−	*2·6*

k	d/N for $r = -\cdot 30$									k
	$h = 0{\cdot}9$	$h = 1{\cdot}0$	$h = 1{\cdot}1$	$h = 1{\cdot}2$	$h = 1{\cdot}3$	$h = 1{\cdot}4$	$h = 1{\cdot}5$	$h = 1{\cdot}6$	$h = 1{\cdot}7$	
0·0	·0601022	·0503807	·0418580	·0344655−	·0281215+	·0227354	·0182110	·0144511	·0113598	*0·0*
0·1	·0533813	·0446585−	·0370290	·0304268	·0247746	·0199871	·0159753	·0126494	·0099216	*0·1*
0·2	·0470347	·0392691	·0324031	·0266436	·0216479	·0174269	·0138984	·0109805+	·0085931	*0·2*
0·3	·0411056	·0342476	·0282782	·0231378	·0187585+	·0150675+	·0119899	·0094512	·0073794	*0·3*
0·4	·0356261	·0296193	·0244039	·0199240	·0161171	·0129167	·0102550−	·0080649	·0062823	*0·4*
0·5	·0306163	·0253991	·0208808	·0170096	·0137285−	·0109772	·0086949	·0068220	·0053015−	*0·5*
0·6	·0260852	·0215924	·0177116	·0143952	·0115916	·0092470	·0073071	·0057195−	·0044340	*0·6*
0·7	·0220311	·0181957	·0148913	·0120751	·0097007	·0077202	·0060861	·0047522	·0036751	*0·7*
0·8	·0184427	·0151972	·0124087	·0100384	·0080453	·0063874	·0050232	·0039127	·0030183	*0·8*
0·9	·0153006	·0125788	·0102467	·0082697	·0066118	·0052366	·0041080	·0031919	·0024562	*0·9*
1·0	·0125788	·0103171	·0083842	·0067503	·0053839	·0042535+	·0033285+	·0025797	·0019801	*1·0*
1·1	·0102467	·0083842	·0067970	·0054591	·0043433	·0034229	·0026718	·0020655−	·0015813	*1·1*
1·2	·0082697	·0067503	·0054591	·0043737	·0034711	·0027286	·0021241	·0016381	·0012509	*1·2*
1·3	·0066118	·0053839	·0043433	·0034711	·0027478	·0021546	·0016732	·0012869	·0009801	*1·3*
1·4	·0052366	·0042535+	·0034229	·0027286	·0021546	·0016851	·0013052	·0010012	·0007605+	*1·4*
1·5	·0041080	·0033285+	·0026718	·0021241	·0016732	·0013052	·0010084	·0007714	·0005845−	*1·5*
1·6	·0031919	·0025797	·0020655−	·0016381	·0012869	·0010012	·0007714	·0005886	·0004448	*1·6*
1·7	·0024562	·0019801	·0015813	·0012509	·0009801	·0007605+	·0005845−	·0004448	·0003352	*1·7*
1·8	·0018717	·0015051	·0011989	·0009459	·0007392	·0005721	·0004384	·0003328	·0002501	*1·8*
1·9	·0014125−	·0011329	·0009000	·0007083	·0005520	·0004261	·0003257	·0002465−	·0001847	*1·9*
2·0	·0010554	·0008443	·0006690	·0005251	·0004081	·0003142	·0002395−	·0001808	·0001351	*2·0*
2·1	·0007808	·0006231	·0004924	·0003854	·0002988	·0002294	·0001744	·0001313	·0000978	*2·1*
2·2	·0005720	·0004552	·0003588	·0002801	·0002166	·0001658	·0001257	·0000944	·0000701	*2·2*
2·3	·0004149	·0003293	·0002588	·0002015+	·0001554	·0001186	·0000897	·0000671	·0000498	*2·3*
2·4	·0002979	·0002368	·0001849	·0001435+	·0001104	·0000840	·0000634	·0000473	·0000350−	*2·4*
2·5	·0002118	·0001672	·0001307	·0001012	·0000776	·0000589	·0000448	·0000330	·0000243	*2·5*
2·6	·0001490	·0001173	·0000915−	·0000706	·0000540	·0000409	·0000307	·0000228	·0000167	*2·6*

d/N for *r* = − ·30

k	*h* = 1·8	*h* = 1·9	*h* = 2·0	*h* = 2·1	*h* = 2·2	*h* = 2·3	*h* = 2·4	*h* = 2·5	*h* = 2·6	*k*
0·0	·0088452	·0068216	·0052104	·0039414	·0029525−	·0021901	·0016088	·0011860	·0008424	*0·0*
0·1	·0077081	·0059312	·0045200	·0034112	·0025494	·0018866	·0013825	·0010030	·0007205	*0·1*
0·2	·0066608	·0051136	·0038878	·0029272	·0021825	·0016113	·0011778	·0008524	·0006108	*0·2*
0·3	·0057067	·0043708	·0033153	·0024902	·0018521	·0013641	·0009946	·0007181	·0005133	*0·3*
0·4	·0048469	·0037034	·0028023	·0020997	·0015579	·0011445+	·0008325	·0005985+	·0004274	*0·4*
0·5	·0040804	·0031102	·0023470	·0017547	·0012987	·0009517	·0006865	·0004980	·0003527	*0·5*
0·6	·0034044	·0025886	·0019491	·0014532	·0010728	·0007842	·0005675	·0004086	·0002884	*0·6*
0·7	·0028148	·0021349	·0016035−	·0011925−	·0008781	·0006402	·0004621	·0003302	·0002336	*0·7*
0·8	·0023060	·0017446	·0013060	·0009695+	·0007121	·0005178	·0003728	·0002657	·0001875	*0·8*
0·9	·0018717	·0014125−	·0010554	·0007808	·0005720	·0004149	·0002979	·0002118	·0001490	*0·9*
1·0	·0015051	·0011329	·0008443	·0006231	·0004552	·0003203	·0002358	·0001672	·0001173	*1·0*
1·1	·0011989	·0009000	·0006690	·0004924	·0003588	·0002588	·0001840	·0001307	·0000915	*1·1*
1·2	·0009459	·0007083	·0005251	·0003854	·0002801	·0002015+	·0001435+	·0001012	·0000708	*1·2*
1·3	·0007392	·0005520	·0004081	·0002988	·0002166	·0001554	·0001104	·0000776	·0000540	*1·3*
1·4	·0005721	·0004261	·0003142	·0002294	·0001658	·0001180	·0000840	·0000589	·0000408	*1·4*
1·5	·0004384	·0003257	·0002395−	·0001744	·0001257	·0000897	·0000634	·0000443	·0000307	*1·5*
1·6	·0003328	·0002465−	·0001808	·0001313	·0000944	·0000671	·0000473	·0000330	·0000228	*1·6*
1·7	·0002501	·0001847	·0001351	·0000978	·0000701	·0000498	·0000350−	·0000243	·0000167	*1·7*
1·8	·0001861	·0001371	·0001000	·0000722	·0000516	·0000365+	·0000258	·0000177	·0000122	*1·8*
1·9	·0001371	·0001007	·0000733	·0000527	·0000376	·0000266	·0000185+	·0000128	·0000088	*1·9*
2·0	·0001000	·0000733	·0000531	·0000382	·0000271	·0000191	·0000133	·0000092	·0000063	*2·0*
2·1	·0000722	·0000527	·0000382	·0000273	·0000194	·0000136	·0000094	·0000065	·0000044	*2·1*
2·2	·0000516	·0000376	·0000271	·0000194	·0000137	·0000096	·0000066	·0000045+	·0000031	*2·2*
2·3	·0000365+	·0000266	·0000191	·0000136	·0000096	·0000067	·0000046	·0000032	·0000021	*2·3*
2·4	·0000256	·0000185+	·0000133	·0000094	·0000066	·0000046	·0000032	·0000022	·0000015	*2·4*
2·5	·0000177	·0000128	·0000092	·0000065−	·0000045+	·0000032	·0000022	·0000015	·0000010	*2·5*
2·6	·0000122	·0000088	·0000063	·0000044	·0000031	·0000021	·0000015−	·0000010	·0000006	*2·6*

d/N for *r* = − ·35

k	*h* = 0·0	*h* = 0·1	*h* = 0·2	*h* = 0·3	*h* = 0·4	*h* = 0·5	*h* = 0·6	*h* = 0·7	*h* = 0·8	*k*
0·0	·1930908	·1734734	·1546379	·1367490	·1199435−	·1043271	·0899735−	·0769240	·0651894	*0·0*
0·1	·1734734	·1555469	·1383798	·1221186	·1068829	·0927635+	·0798213	·0680873	·0575650+	*0·1*
0·2	·1546379	·1383798	·1228525−	·1081845−	·0944794	·0818138	·0702367	·0597702	·0504113	*0·2*
0·3	·1367490	·1221186	·1081845−	·0950584	·0828288	·0715590	·0612874	·0520282	·0437730	*0·3*
0·4	·1199435−	·1068829	·0944794	·0828288	·0720054	·0620606	·0530233	·0449010	·0376812	*0·4*
0·5	·1043271	·0927635+	·0818138	·0715590	·0620606	·0533593	·0454760	·0384122	·0321524	*0·5*
0·6	·0899735−	·0798213	·0702367	·0612874	·0530233	·0454760	·0386590	·0325807	·0271903	*0·6*
0·7	·0769240	·0680873	·0597702	·0520282	·0449010	·0384122	·0325807	·0273072	·0227858	*0·7*
0·8	·0651894	·0575650+	·0504113	·0437730	·0376812	·0321524	·0271903	·0227858	·0189196	*0·8*
0·9	·0547521	·0482324	·0421344	·0364938	·0313341	·0266664	·0224905−	·0187959	·0155635−	*0·9*
1·0	·0455699	·0400453	·0348946	·0301457	·0258156	·0219113	·0184297	·0153596	·0126824	*1·0*
1·1	·0375802	·0329419	·0286315+	·0246704	·0210705−	·0178352	·0149599	·0124328	·0102364	*1·1*
1·2	·0307040	·0268460	·0232727	·0199998	·0170352	·0143798	·0120277	·0099675+	·0081830	*1·2*
1·3	·0248509	·0216723	·0187380	·0160594	·0136414	·0114829	·0095774	·0079140	·0064781	*1·3*
1·4	·0199230	·0173292	·0149429	·0127718	·0108186	·0090810	·0075523	·0062224	·0050784	*1·4*
1·5	·0158197	·0137235+	·0118016	·0100765−	·0084967	·0071115+	·0058972	·0048444	·0039419	*1·5*
1·6	·0124404	·0107629	·0092301	·0078451	·0066078	·0055145+	·0045595−	·0037343	·0030295−	*1·6*
1·7	·0096879	·0083586	·0071482	·0060584	·0050881	·0042339	·0034902	·0028500−	·0023049	*1·7*
1·8	·0074704	·0064276	·0054813	·0046323	·0038791	·0032183	·0026450+	·0021533	·0017364	*1·8*
1·9	·0057037	·0048938	·0041614	·0035067	·0029278	·0024218	·0019844	·0016105−	·0012945−	*1·9*
2·0	·0043116	·0036888	·0031278	·0026279	·0021876	·0018040	·0014737	·0011923	·0009554	*2·0*
2·1	·0032267	·0027527	·0023272	·0019495+	·0016180	·0013303	·0010833	·0008735−	·0006979	*2·1*
2·2	·0023905+	·0020334	·0017141	·0014316	·0011846	·0009709	·0007882	·0006337	·0005046	*2·2*
2·3	·0017532	·0014870	·0012497	·0010406	·0008584	·0007014	·0005676	·0004549	·0003611	*2·3*
2·4	·0012727	·0010762	·0009018	·0007486	·0006156	·0005015−	·0004046	·0003232	·0002557	*2·4*
2·5	·0009144	·0007710	·0006441	·0005331	·0004370	·0003548	·0002854	·0002272	·0001792	*2·5*
2·6	·0006503	·0005467	·0004553	·0003756	·0003070	·0002485−	·0001992	·0001581	·0001243	*2·6*

k	d/N for r = − ·35									k
	h = 0·9	*h = 1·0*	*h = 1·1*	*h = 1·2*	*h = 1·3*	*h = 1·4*	*h = 1·5*	*h = 1·6*	*h = 1·7*	
0·0	·0547521	·0455690	·0375802	·0307040	·0248509	·0199230	·0158197	·0124404	·0096879	*0·0*
0·1	·0482324	·0400453	·0329419	·0268460	·0216723	·0173292	·0137235+	·0107629	·0083586	*0·1*
0·2	·0421344	·0348946	·0286315+	·0232727	·0187380	·0149429	·0118016	·0092301	·0071482	*0·2*
0·3	·0364938	·0301457	·0246704	·0199998	·0160594	·0127718	·0100765−	·0078451	·0060584	*0·3*
0·4	·0313341	·0258156	·0210705−	·0170352	·0136414	·0108186	·0084967	·0066078	·0050881	*0·4*
0·5	·0266664	·0219113	·0178352	·0143798	·0114829	·0090810	·0071115+	·0055145+	·0042330	*0·5*
0·6	·0224905−	·0184297	·0149599	·0120277	·0095774	·0075523	·0058972	·0045595−	·0034902	*0·6*
0·7	·0187959	·0153596	·0124328	·0099675+	·0079140	·0062224	·0048444	·0037343	·0028500−	*0·7*
0·8	·0155635−	·0126824	·0102364	·0081830	·0064781	·0050784	·0039419	·0030295−	·0023049	*0·8*
0·9	·0127667	·0103736	·0083488	·0066544	·0052524	·0041052	·0031768	·0024340	·0018462	*0·9*
1·0	·0103736	·0084048	·0067444	·0053597	·0042178	·0032866	·0025356	·0019367	·0014644	*1·0*
1·1	·0083488	·0067444	·0053960	·0042753	·0033542	·0026056	·0020040	·0015259	·0011502	*1·1*
1·2	·0066544	·0053597	·0042753	·0033771	·0026414	·0020455+	·0015683	·0011904	·0008944	*1·2*
1·3	·0052524	·0042178	·0033542	·0026414	·0020596	·0015900	·0012152	·0009194	·0006886	*1·3*
1·4	·0041052	·0032866	·0026056	·0020455+	·0015900	·0012236	·0009322	·0007031	·0005249	*1·4*
1·5	·0031768	·0025356	·0020040	·0015683	·0012152	·0009322	·0007079	·0005322	·0003960	*1·5*
1·6	·0024340	·0019367	·0015259	·0011904	·0009194	·0007031	·0005322	·0003988	·0002958	*1·6*
1·7	·0018462	·0014644	·0011502	·0008944	·0006886	·0005249	·0003960	·0002958	·0002186	*1·7*
1·8	·0013863	·0010961	·0008582	·0006652	·0005105+	·0003878	·0002916	·0002171	·0001600	*1·8*
1·9	·0010304	·0008121	·0006338	·0004897	·0003746	·0002836	·0002126	·0001577	·0001158	*1·9*
2·0	·0007581	·0005956	·0004633	·0003568	·0002720	·0002053	·0001534	·0001134	·0000830	*2·0*
2·1	·0005520	·0004323	·0003352	·0002573	·0001955+	·0001471	·0001095+	·0000807	·0000589	*2·1*
2·2	·0003978	·0003105+	·0002400	·0001836	·0001391	·0001043	·0000774	·0000568	·0000413	*2·2*
2·3	·0002838	·0002208	·0001701	·0001297	·0000979	·0000731	·0000541	·0000396	·0000287	*2·3*
2·4	·0002003	·0001553	·0001193	·0000906	·0000682	·0000508	·0000374	·0000273	·0000197	*2·4*
2·5	·0001399	·0001081	·0000828	·0000627	·0000470	·0000349	·0000257	·0000186	·0000134	*2·5*
2·6	·0000967	·0000745+	·0000568	·0000429	·0000321	·0000237	·0000174	·0000126	·0000090	*2·6*

k	d/N for r = − ·35									k
	h = 1·8	*h = 1·9*	*h = 2·0*	*h = 2·1*	*h = 2·2*	*h = 2·3*	*h = 2·4*	*h = 2·5*	*h = 2·6*	
0·0	·0074704	·0057037	·0043116	·0032267	·0023905+	·0017532	·0012727	·0009144	·0006503	*0·0*
0·1	·0064276	·0048938	·0036888	·0027527	·0020334	·0014870	·0010762	·0007710	·0005467	*0·1*
0·2	·0054813	·0041614	·0031278	·0023272	·0017141	·0012497	·0009018	·0006441	·0004553	*0·2*
0·3	·0046323	·0035067	·0026279	·0019495+	·0014316	·0010406	·0007486	·0005331	·0003756	*0·3*
0·4	·0038791	·0029278	·0021876	·0016180	·0011846	·0008584	·0006156	·0004370	·0003070	*0·4*
0·5	·0032183	·0024218	·0018040	·0013303	·0009709	·0007014	·0005015−	·0003548	·0002485−	*0·5*
0·6	·0026450+	·0019844	·0014737	·0010833	·0007882	·0005676	·0004046	·0002854	·0001992	*0·6*
0·7	·0021533	·0016105−	·0011923	·0008735−	·0006337	·0004549	·0003232	·0002272	·0001581	*0·7*
0·8	·0017364	·0012945−	·0009554	·0006979	·0005046	·0003611	·0002557	·0001792	·0001243	*0·8*
0·9	·0013863	·0010304	·0007581	·0005520	·0003978	·0002838	·0002003	·0001399	·0000967	*0·9*
1·0	·0010961	·0008121	·0005956	·0004323	·0003105+	·0002208	·0001553	·0001081	·0000745+	*1·0*
1·1	·0008582	·0006338	·0004633	·0003352	·0002400	·0001701	·0001193	·0000828	·0000568	*1·1*
1·2	·0006652	·0004897	·0003568	·0002573	·0001836	·0001297	·0000906	·0000627	·0000429	*1·2*
1·3	·0005105+	·0003746	·0002720	·0001955+	·0001391	·0000979	·0000682	·0000470	·0000321	*1·3*
1·4	·0003878	·0002836	·0002053	·0001471	·0001043	·0000731	·0000508	·0000349	·0000237	*1·4*
1·5	·0002916	·0002126	·0001534	·0001095+	·0000774	·0000541	·0000374	·0000257	·0000174	*1·5*
1·6	·0002171	·0001577	·0001134	·0000807	·0000568	·0000396	·0000273	·0000186	·0000126	*1·6*
1·7	·0001600	·0001158	·0000830	·0000589	·0000413	·0000287	·0000197	·0000134	·0000090	*1·7*
1·8	·0001166	·0000842	·0000601	·0000425−	·0000297	·0000206	·0000141	·0000095+	·0000064	*1·8*
1·9	·0000842	·0000605+	·0000431	·0000303	·0000212	·0000146	·0000100	·0000067	·0000045−	*1·9*
2·0	·0000601	·0000431	·0000306	·0000215−	·0000149	·0000102	·0000070	·0000047	·0000031	*2·0*
2·1	·0000425−	·0000303	·0000215−	·0000150−	·0000104	·0000071	·0000048	·0000032	·0000021	*2·1*
2·2	·0000297	·0000212	·0000149	·0000104	·0000072	·0000049	·0000033	·0000022	·0000015−	*2·2*
2·3	·0000206	·0000146	·0000102	·0000071	·0000049	·0000033	·0000022	·0000015−	·0000010	*2·3*
2·4	·0000141	·0000100	·0000070	·0000048	·0000033	·0000022	·0000015+	·0000010	·00000065+	*2·4*
2·5	·0000095+	·0000067	·0000047	·0000032	·0000022	·0000015−	·0000010	·00000066	·00000043	*2·5*
2·6	·0000064	·0000045−	·0000031	·0000021	·0000015−	·0000010	·00000065+	·00000043	·00000028	*2·6*

d/N for $r = -\cdot 40$

k	$h = 0\cdot 0$	$h = 0\cdot 1$	$h = 0\cdot 2$	$h = 0\cdot 3$	$h = 0\cdot 4$	$h = 0\cdot 5$	$h = 0\cdot 6$	$h = 0\cdot 7$	$h = 0\cdot 8$	k
0·0	·1845051	·1649375+	·1462498	·1286015−	·1121213	·0969041	·0830109	·0704688	·0592739	0·0
0·1	·1649375+	·1470976	·1301130	·1141238	·0992408	·0855431	·0730781	·0618629	·0518859	0·1
0·2	·1462498	·1301130	·1147093	·1004304	·0870997	·0748716	·0637817	·0538376	·0450220	0·2
0·3	·1286015−	·1141238	·1004304	·0876248	·0757849	·0649617	·0551797	·0464393	·0387180	0·3
0·4	·1121213	·0992408	·0870997	·0757849	·0653598	·0558634	·0473112	·0396968	·0329946	0·4
0·5	·0969041	·0855431	·0748716	·0649617	·0558634	·0476054	·0401955+	·0336222	·0278577	0·5
0·6	·0830109	·0730781	·0637817	·0551797	·0473112	·0401955+	·0338341	·0282120	·0233001	0·6
0·7	·0704688	·0618629	·0538376	·0464393	·0396968	·0336222	·0282120	·0234486	·0193028	0·7
0·8	·0592739	·0518859	·0450220	·0387180	·0329946	·0278577	·0233001	·0193028	·0158373	0·8
0·9	·0493939	·0431106	·0372953	·0319746	·0271625−	·0228601	·0190578	·0157359	·0128673	0·9
1·0	·0407727	·0354798	·0305998	·0261523	·0221454	·0185770	·0154358	·0127023	·0103511	1·0
1·1	·0333350−	·0289192	·0248638	·0211823	·0178787	·0149482	·0123788	·0101519	·0082441	1·1
1·2	·0269907	·0233427	·0200057	·0169884	·0142916	·0119090	·0098283	·0080323	·0065000	1·2
1·3	·0216403	·0186565−	·0159379	·0134897	·0113103	·0093927	·0077249	·0062911	·0050728	1·3
1·4	·0171794	·0147632	·0125707	·0106043	·0088609	·0073332	·0060100	·0048771	·0039185+	1·4
1·5	·0135022	·0115654	·0098152	·0082518	·0068715+	·0056670	·0046280	·0037422	·0029957	1·5
1·6	·0105054	·0089689	·0075860	·0063559	·0052743	·0043344	·0035270	·0028416	·0022664	1·6
1·7	·0080910	·0068845−	·0058032	·0048453	·0040067	·0032809	·0026601	·0021353	·0016967	1·7
1·8	·0061678	·0052303	·0043936	·0036556	·0030121	·0024676	·0019853	·0015878	·0012570	1·8
1·9	·0046534	·0039326	·0032920	·0027293	·0022408	·0018217	·0014662	·0011082	·0009213	1·9
2·0	·0034746	·0029262	·0024409	·0020164	·0016495+	·0013360	·0010713	·0008514	·0006681	2·0
2·1	·0025673	·0021545+	·0017908	·0014741	·0012014	·0009695+	·0007745	·0006124	·0004793	2·1
2·2	·0018771	·0015697	·0013000	·0010662	·0008658	·0006960	·0005539	·0004364	·0003402	2·2
2·3	·0013580	·0011316	·0009338	·0007630	·0006173	·0004944	·0003919	·0003076	·0002389	2·3
2·4	·0009720	·0008070	·0006635+	·0005401	·0004354	·0003474	·0002743	·0002144	·0001659	2·4
2·5	·0006884	·0005695−	·0004665−	·0003783	·0003038	·0002414	·0001899	·0001479	·0001140	2·5
2·6	·0004823	·0003975+	·0003244	·0002621	·0002097	·0001660	·0001301	·0001009	·0000774	2·6

d/N for $r = -\cdot 40$

k	$h = 0\cdot 9$	$h = 1\cdot 0$	$h = 1\cdot 1$	$h = 1\cdot 2$	$h = 1\cdot 3$	$h = 1\cdot 4$	$h = 1\cdot 5$	$h = 1\cdot 6$	$h = 1\cdot 7$	k
0·0	·0493939	·0407727	·0333350−	·0269907	·0216403	·0171794	·0135022	·0105054	·0080910	0·0
0·1	·0431106	·0354798	·0289192	·0233427	·0186565−	·0147632	·0115654	·0089689	·0068845−	0·1
0·2	·0372953	·0305998	·0248638	·0200057	·0159379	·0125707	·0098152	·0075860	·0058032	0·2
0·3	·0319746	·0261523	·0211823	·0169884	·0134897	·0106043	·0082518	·0063559	·0048453	0·3
0·4	·0271625−	·0221454	·0178787	·0142916	·0113103	·0088609	·0068715+	·0052743	·0040067	0·4
0·5	·0228601	·0185770	·0149482	·0119090	·0093927	·0073332	·0056670	·0043344	·0032809	0·5
0·6	·0190578	·0154358	·0123788	·0098283	·0077249	·0060100	·0046280	·0035270	·0026601	0·6
0·7	·0157359	·0127023	·0101519	·0080323	·0062911	·0048771	·0037422	·0028416	·0021353	0·7
0·8	·0128673	·0103511	·0082441	·0065000−	·0050728	·0039185+	·0029957	·0022664	·0016967	0·8
0·9	·0104184	·0083521	·0062286	·0052076	·0040496	·0031168	·0023740	·0017894	·0013347	0·9
1·0	·0083521	·0066720	·0052764	·0041303	·0032002	·0024539	·0018622	·0013984	·0010390	1·0
1·1	·0062286	·0052764	·0041576	·0032427	·0025032	·0019123	·0014457	·0010815+	·0008005+	1·1
1·2	·0052076	·0041303	·0032427	·0025198	·0019379	·0014749	·0011108	·0008278	·0006104	1·2
1·3	·0040496	·0032002	·0025032	·0019379	·0014848	·0011258	·0008446	·0006270	·0004605+	1·3
1·4	·0031168	·0024539	·0019123	·0014749	·0011258	·0008503	·0006355−	·0004699	·0003438	1·4
1·5	·0023740	·0018622	·0014457	·0011108	·0008446	·0006355−	·0004731	·0003485−	·0002539	1·5
1·6	·0017894	·0013984	·0010815+	·0008278	·0006270	·0004699	·0003485−	·0002557	·0001856	1·6
1·7	·0013347	·0010390	·0008005+	·0006104	·0004605+	·0003438	·0002539	·0001856	·0001342	1·7
1·8	·0009850−	·0007639	·0005863	·0004453	·0003347	·0002488	·0001831	·0001332	·0000959	1·8
1·9	·0007192	·0005556	·0004248	·0003214	·0002406	·0001782	·0001305+	·0000946	·0000679	1·9
2·0	·0005196	·0003998	·0003045−	·0002295−	·0001711	·0001262	·0000921	·0000665−	·0000475−	2·0
2·1	·0003713	·0002846	·0002159	·0001621	·0001203	·0000884	·0000643	·0000462	·0000329	2·1
2·2	·0002625+	·0002004	·0001514	·0001132	·0000837	·0000613	·0000443	·0000318	·0000225+	2·2
2·3	·0001836	·0001396	·0001051	·0000782	·0000576	·0000420	·0000303	·0000216	·0000152	2·3
2·4	·0001270	·0000962	·0000721	·0000535−	·0000392	·0000280	·0000204	·0000145+	·0000102	2·4
2·5	·0000869	·0000656	·0000489	·0000361	·0000264	·0000191	·0000137	·0000097	·0000068	2·5
2·6	·0000588	·0000442	·0000329	·0000242	·0000176	·0000127	·0000090	·0000063	·0000044	2·6

k	d/N for $r = -·40$									*k*
	h = 1·8	*h = 1·9*	*h = 2·0*	*h = 2·1*	*h = 2·2*	*h = 2·3*	*h = 2·4*	*h = 2·5*	*h = 2·6*	
0·0	·0061678	·0046534	·0034746	·0025673	·0018771	·0013580	·0009720	·0006884	·0004823	*0·0*
0·1	·0052303	·0039326	·0029262	·0021545+	·0015697	·0011316	·0008070	·0005695−	·0003975+	*0·1*
0·2	·0043936	·0032920	·0024409	·0017908	·0013000	·0009338	·0006635+	·0004665−	·0003244	*0·2*
0·3	·0036556	·0027293	·0020164	·0014741	·0010662	·0007630	·0005401	·0003783	·0002621	*0·3*
0·4	·0030121	·0022408	·0016495+	·0012014	·0008658	·0006173	·0004354	·0003038	·0002097	*0·4*
0·5	·0024576	·0018217	·0013360	·0009695+	·0006960	·0004944	·0003474	·0002414	·0001660	*0·5*
0·6	·0019853	·0014662	·0010713	·0007745−	·0005530	·0003919	·0002743	·0001899	·0001301	*0·6*
0·7	·0015878	·0011682	·0008504	·0006124	·0004364	·0003076	·0002144	·0001479	·0001009	*0·7*
0·8	·0012570	·0009213	·0006681	·0004793	·0003402	·0002389	·0001659	·0001140	·0000774	*0·8*
0·9	·0009850−	·0007192	·0005196	·0003713	·0002625+	·0001836	·0001270	·0000869	·0000588	*0·9*
1·0	·0007639	·0005556	·0003998	·0002846	·0002004	·0001396	·0000962	·0000656	·0000442	*1·0*
1·1	·0005863	·0004248	·0003045−	·0002150	·0001514	·0001051	·0000721	·0000489	·0000329	*1·1*
1·2	·0004453	·0003214	·0002295−	·0001621	·0001132	·0000782	·0000535−	·0000361	·0000242	*1·2*
1·3	·0003347	·0002406	·0001711	·0001203	·0000837	·0000576	·0000392	·0000264	·0000176	*1·3*
1·4	·0002488	·0001782	·0001262	·0000884	·0000613	·0000420	·0000280	·0000191	·0000127	*1·4*
1·5	·0001831	·0001305+	·0000921	·0000643	·0000443	·0000303	·0000204	·0000137	·0000090	*1·5*
1·6	·0001332	·0000946	·0000665−	·0000462	·0000318	·0000216	·0000145+	·0000097	·0000063	*1·6*
1·7	·0000959	·0000679	·0000475−	·0000329	·0000225+	·0000152	·0000102	·0000068	·0000044	*1·7*
1·8	·0000683	·0000481	·0000336	·0000231	·0000158	·0000106	·0000071	·0000047	·0000030	*1·8*
1·9	·0000481	·0000338	·0000234	·0000161	·0000109	·0000073	·0000049	·0000032	·0000021	*1·9*
2·0	·0000336	·0000234	·0000162	·0000111	·0000075−	·0000050+	·0000033	·0000022	·0000014	*2·0*
2·1	·0000231	·0000161	·0000111	·0000075+	·0000051	·0000034	·0000022	·0000014	·00000093	*2·1*
2·2	·0000158	·0000109	·0000075−	·0000051	·0000034	·0000023	·0000015	·00000096	·00000061	*2·2*
2·3	·0000106	·0000073	·0000050+	·0000034	·0000023	·0000015−	·00000097	·00000063	·00000040	*2·3*
2·4	·0000071	·0000049	·0000033	·0000022	·0000015−	·00000097	·00000063	·00000041	·00000026	*2·4*
2·5	·0000047	·0000032	·0000022	·0000014	·00000096	·00000063	·00000041	·00000026	·00000016	*2·5*
2·6	·0000030	·0000021	·0000014	·00000093	·00000061	·00000040	·00000026	·00000016	·00000010	*2·6*

k	d/N for $r = -·45$									*k*
	h = 0·0	*h = 0·1*	*h = 0·2*	*h = 0·3*	*h = 0·4*	*h = 0·5*	*h = 0·6*	*h = 0·7*	*h = 0·8*	
0·0	·1757120	·1561980	·1376688	·1202786	·1041466	·0893558	·0759529	·0639494	·0533250−	*0·0*
0·1	·1561980	·1384562	·1216724	·1059795−	·0914776	·0782330	·0662783	·0556144	·0462137	*0·1*
0·2	·1376688	·1216724	·1065976	·0925575−	·0796341	·0678782	·0573101	·0479215−	·0396794	*0·2*
0·3	·1202786	·1059795−	·0925575−	·0801067	·0686927	·0583523	·0490952	·0409059	·0337468	*0·3*
0·4	·1041466	·0914776	·0796341	·0686927	·0587040	·0496929	·0416600	·0345840	·0284250+	*0·4*
0·5	·0893558	·0782330	·0678782	·0583523	·0496929	·0419144	·0350104	·0289554	·0237083	*0·5*
0·6	·0759529	·0662783	·0573101	·0490952	·0416600	·0350104	·0291344	·0240040	·0195779	*0·6*
0·7	·0639494	·0556144	·0479215−	·0409059	·0345840	·0289554	·0240040	·0197002	·0160044	*0·7*
0·8	·0533250−	·0462137	·0396794	·0337468	·0284250+	·0237083	·0195779	·0160044	·0129498	*0·8*
0·9	·0440315−	·0380241	·0325291	·0275627	·0231280	·0192155+	·0158053	·0128686	·0103701	*0·9*
1·0	·0359976	·0309738	·0263994	·0222842	·0186265−	·0154146	·0126282	·0102400	·0082177	*1·0*
1·1	·0291344	·0249759	·0212060	·0178322	·0148467	·0122375−	·0099846	·0080629	·0064436	*1·1*
1·2	·0233403	·0199337	·0168606	·0141220	·0117108	·0096136	·0078114	·0062816	·0049987	*1·2*
1·3	·0185067	·0157451	·0132658	·0110669	·0091402	·0074725−	·0060464	·0048417	·0038363	*1·3*
1·4	·0145219	·0123069	·0103280	·0085813	·0070583	·0057464	·0046300	·0036916	·0029124	*1·4*
1·5	·0112758	·0095183	·0079557	·0065832	·0053923	·0043715+	·0035072	·0027843	·0021870	*1·5*
1·6	·0086629	·0072834	·0060629	·0049962	·0040752	·0032896	·0026278	·0020770	·0016242	*1·6*
1·7	·0065845+	·0055136	·0045707	·0037508	·0030463	·0024485+	·0019474	·0015324	·0011929	*1·7*
1·8	·0049511	·0041288	·0034084	·0027852	·0022524	·0018025−	·0014273	·0011181	·0008665+	*1·8*
1·9	·0036827	·0030583	·0025139	·0020455+	·0016470	·0013123	·0010345−	·0008068	·0006224	*1·9*
2·0	·0027094	·0022406	·0018340	·0014857	·0011911	·0009448	·0007414	·0005756	·0004420	*2·0*
2·1	·0019715+	·0016235−	·0013231	·0010672	·0008518	·0006726	·0005254	·0004061	·0003104	*2·1*
2·2	·0014188	·0011633	·0009440	·0007580	·0006023	·0004735−	·0003682	·0002832	·0002155−	*2·2*
2·3	·0010097	·0008243	·0006660	·0005324	·0004211	·0003295+	·0002551	·0001953	·0001479	*2·3*
2·4	·0007106	·0005776	·0004646	·0003697	·0002911	·0002268	·0001747	·0001332	·0001004	*2·4*
2·5	·0004945+	·0004002	·0003204	·0002539	·0001990	·0001543	·0001183	·0000897	·0000674	*2·5*
2·6	·0003403	·0002741	·0002185+	·0001723	·0001344	·0001038	·0000792	·0000598	·0000446	*2·6*

k	d/N for $r = -\cdot 45$									k
	$h = 0{\cdot}9$	$h = 1{\cdot}0$	$h = 1{\cdot}1$	$h = 1{\cdot}2$	$h = 1{\cdot}3$	$h = 1{\cdot}4$	$h = 1{\cdot}5$	$h = 1{\cdot}6$	$h = 1{\cdot}7$	
0·0	·0440315−	·0359976	·0291344	·0233403	·0185067	·0145219	·0112758	·0086629	·0065845+	0·0
0·1	·0380241	·0309738	·0249759	·0199337	·0157451	·0123069	·0095183	·0072834	·0055136	0·1
0·2	·0325291	·0263994	·0212069	·0168606	·0132658	·0103280	·0079557	·0060629	·0045707	0·2
0·3	·0275627	·0222842	·0178322	·0141220	·0110669	·0085813	·0065832	·0049962	·0037508	0·3
0·4	·0231280	·0186265−	·0148467	·0117108	·0091402	·0070583	·0053923	·0040752	·0030463	0·4
0·5	·0192155+	·0154146	·0122375−	·0096130	·0074725−	·0057464	·0043715+	·0032896	·0024485+	0·5
0·6	·0158053	·0126282	·0099846	·0078114	·0060464	·0046300	·0035072	·0026278	·0019474	0·6
0·7	·0128686	·0102400	·0080629	·0062816	·0048417	·0036016	·0027843	·0020770	·0015324	0·7
0·8	·0103701	·0082177	·0064436	·0049987	·0038363	·0029124	·0021870	·0016242	·0011929	0·8
0·9	·0082700	·0065261	·0050954	·0039359	·0030075+	·0022732	·0016994	·0012565−	·0009187	0·9
1·0	·0065261	·0051281	·0039867	·0030661	·0023326	·0017552	·0013063	·0009615−	·0006998	1·0
1·1	·0050954	·0039867	·0030859	·0023629	·0017896	·0013406	·0009932	·0007277	·0005272	1·1
1·2	·0039359	·0030661	·0023629	·0018012	·0013681	·0010128	·0007469	·0005447	·0003928	1·2
1·3	·0030075+	·0023326	·0017896	·0013681	·0010194	·0007567	·0005555+	·0004032	·0002804	1·3
1·4	·0022732	·0017552	·0013406	·0010128	·0007567	·0005591	·0004086	·0002952	·0002100	1·4
1·5	·0016994	·0013063	·0009932	·0007469	·0005555+	·0004086	·0002971	·0002137	·0001519	1·5
1·6	·0012565−	·0009615−	·0007277	·0005447	·0004032	·0002952	·0002137	·0001529	·0001082	1·6
1·7	·0009187	·0006998	·0005272	·0003928	·0002804	·0002100	·0001519	·0001082	·0000761	1·7
1·8	·0006643	·0005037	·0003777	·0002801	·0002054	·0001490	·0001068	·0000757	·0000531	1·8
1·9	·0004749	·0003584	·0002675+	·0001975−	·0001441	·0001040	·0000742	·0000524	·0000365+	1·9
2·0	·0003357	·0002522	·0001874	·0001376	·0001000	·0000718	·0000510	·0000358	·0000248	2·0
2·1	·0002347	·0001754	·0001297	·0000948	·0000686	·0000490	·0000346	·0000242	·0000167	2·1
2·2	·0001622	·0001207	·0000888	·0000646	·0000465−	·0000331	·0000232	·0000162	·0000111	2·2
2·3	·0001108	·0000820	·0000601	·0000435+	·0000311	·0000220	·0000154	·0000107	·0000073	2·3
2·4	·0000748	·0000551	·0000402	·0000290	·0000206	·0000145+	·0000101	·0000070	·0000047	2·4
2·5	·0000499	·0000366	·0000266	·0000191	·0000135+	·0000095−	·0000066	·0000046−	·0000030	2·5
2·6	·0000330	·0000234	·0000174	·0000124	·0000087	·0000061	·0000042	·0000029	·0000019	2·6

k	d/N for $r = -\cdot 45$									k
	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	
0·0	·0049511	·0036827	·0027094	·0019715+	·0014188	·0010097	·0007106	·0004945+	·0003403	0·0
0·1	·0041288	·0030583	·0022406	·0016235−	·0011633	·0008243	·0005776	·0004002	·0002741	0·1
0·2	·0034084	·0025189	·0018340	·0013231	·0009440	·0006660	·0004646	·0003204	·0002185+	0·2
0·3	·0027852	·0020455+	·0014857	·0010672	·0007580	·0005324	·0003697	·0002539	·0001723	0·3
0·4	·0022524	·0016470	·0011911	·0008518	·0006023	·0004211	·0002911	·0001990	·0001344	0·4
0·5	·0018025−	·0013123	·0009448	·0006726	·0004735−	·0003295+	·0002268	·0001543	·0001038	0·5
0·6	·0014273	·0010345−	·0007414	·0005254	·0003682	·0002551	·0001747	·0001183	·0000792	0·6
0·7	·0011181	·0008068	·0005756	·0004061	·0002832	·0001953	·0001332	·0000897	·0000598	0·7
0·8	·0008665+	·0006224	·0004420	·0003104	·0002155−	·0001479	·0001004	·0000674	·0000446	0·8
0·9	·0006643	·0004749	·0003357	·0002347	·0001622	·0001108	·0000748	·0000499	·0000330	0·9
1·0	·0005037	·0003584	·0002522	·0001754	·0001207	·0000820	·0000551	·0000366	·0000234	1·0
1·1	·0003777	·0002675+	·0001874	·0001297	·0000888	·0000601	·0000402	·0000266	·0000174	1·1
1·2	·0002801	·0001975−	·0001376	·0000948	·0000646	·0000435+	·0000290	·0000191	·0000124	1·2
1·3	·0002054	·0001441	·0001000	·0000686	·0000465−	·0000311	·0000206	·0000135+	·0000087	1·3
1·4	·0001490	·0001040	·0000718	·0000490	·0000331	·0000220	·0000145+	·0000095−	·0000061	1·4
1·5	·0001068	·0000742	·0000510	·0000346	·0000232	·0000154	·0000101	·0000066	·0000042	1·5
1·6	·0000757	·0000524	·0000358	·0000242	·0000162	·0000107	·0000070	·0000045−	·0000029	1·6
1·7	·0000531	·0000365+	·0000248	·0000167	·0000111	·0000073	·0000047	·0000030	·0000019	1·7
1·8	·0000368	·0000252	·0000170	·0000114	·0000075+	·0000049	·0000032	·0000020	·0000013	1·8
1·9	·0000252	·0000172	·0000116	·0000077	·0000051	·0000033	·0000021	·0000013	·00000085−	1·9
2·0	·0000170	·0000116	·0000078	·0000051	·0000034	·0000022	·0000014	·00000088	·00000055+	2·0
2·1	·0000114	·0000077	·0000051	·0000034	·0000022	·0000014	·00000090	·00000057	·00000037	2·1
2·2	·0000075+	·0000051	·0000034	·0000022	·0000014	·00000092	·00000058	·00000036	·00000022	2·2
2·3	·0000049	·0000033	·0000022	·0000014	·00000092	·00000059	·00000037	·00000023	·00000014	2·3
2·4	·0000032	·0000021	·0000014	·00000090	·00000058	·00000037	·00000023	·00000014	·00000009	2·4
2·5	·0000020	·0000013	·00000088	·00000057	·00000036	·00000023	·00000014	·00000009	·00000005+	2·5
2·6	·0000013	·00000085−	·00000055+	·00000037	·00000022	·00000014	·00000009	·00000005+	·00000002	2·6

k	d/N for $r = -\cdot 50$									k
	$h = 0\cdot 0$	$h = 0\cdot 1$	$h = 0\cdot 2$	$h = 0\cdot 3$	$h = 0\cdot 4$	$h = 0\cdot 5$	$h = 0\cdot 6$	$h = 0\cdot 7$	$h = 0\cdot 8$	
0·0	·1666667	·1472109	·1288543	·1117443	·0959807	·0816598	·0687848	·0573588	·0473431	*0·0*
0·1	·1472109	·1295818	·1130216	·0976550+	·0835700	·0708178	·0594141	·0493418	·0405553	*0·1*
0·2	·1288543	·1130216	·0982164	·0845419	·0720665−	·0608253	·0508211	·0420282	·0343956	*0·2*
0·3	·1117443	·0976550+	·0845419	·0724876	·0615434	·0517301	·0430398	·0354399	·0288762	*0·3*
0·4	·0959807	·0835700	·0720665−	·0615434	·0520367	·0435548	·0360817	·0295794	·0239929	*0·4*
0·5	·0816598	·0708178	·0608253	·0517301	·0435548	·0362982	·0299375+	·0244321	·0197268	*0·5*
0·6	·0687848	·0594141	·0508211	·0430398	·0360817	·0299375+	·0245803	·0199680	·0160471	*0·6*
0·7	·0573588	·0493418	·0420282	·0354399	·0295794	·0244321	·0199680	·0161454	·0129134	*0·7*
0·8	·0473431	·0405553	·0343956	·0288762	·0239929	·0197268	·0160471	·0129134	·0102785+	*0·8*
0·9	·0386718	·0329853	·0278526	·0232782	·0192530	·0157559	·0127561	·0102154	·0080912	*0·9*
1·0	·0312570	·0265442	·0223135+	·0185637	·0152821	·0124469	·0100285−	·0079918	·0062984	*1·0*
1·1	·0249952	·0211319	·0176829	·0146428	·0119973	·0097244	·0077966	·0061823	·0048478	*1·1*
1·2	·0197727	·0166407	·0138601	·0114231	·0093142	·0075127	·0059935−	·0047286	·0036890	*1·2*
1·3	·0154710	·0129603	·0107439	·0088123	·0071504	·0057388	·0045553	·0035756	·0027751	*1·3*
1·4	·0119721	·0099821	·0082354	·0067219	·0054273	·0043340	·0034227	·0026728	·0020636	*1·4*
1·5	·0091615+	·0076023	·0062416	·0050694	·0040726	·0032357	·0025422	·0019748	·0015167	*1·5*
1·6	·0069322	·0057246	·0046769	·0037796	·0030210	·0023879	·0018663	·0014421	·0011017	*1·6*
1·7	·0051861	·0042617	·0034644	·0027856	·0022150−	·0017417	·0013541	·0010408	·0007909	*1·7*
1·8	·0038356	·0031363	·0025367	·0020292	·0016052	·0012556	·0009710	·0007424	·0005610	*1·8*
1·9	·0028042	·0022814	·0018359	·0014610	·0011497	·0008945+	·0006881	·0005232	·0003932	*1·9*
2·0	·0020265−	·0016403	·0013132	·0010396	·0008138	·0006298	·0004818	·0003644	·0002723	*2·0*
2·1	·0014474	·0011656	·0009283	·0007310	·0005692	·0004381	·0003334	·0002507	·0001864	*2·1*
2·2	·0010217	·0008185+	·0006484	·0005079	·0003934	·0003011	·0002279	·0001704	·0001260	*2·2*
2·3	·0007128	·0005680	·0004476	·0003487	·0002686	·0002045+	·0001539	·0001144	·0000841	*2·3*
2·4	·0004913	·0003895−	·0003053	·0002365+	·0001812	·0001372	·0001027	·0000759	·0000555+	*2·4*
2·5	·0003347	·0002639	·0002058	·0001586	·0001208	·0000909	·0000677	·0000498	·0000362	*2·5*
2·6	·0002253	·0001767	·0001370	·0001050−	·0000795−	·0000596	·0000441	·0000323	·0000233	*2·6*

k	d/N for $r = -\cdot 50$									k
	$h = 0\cdot 9$	$h = 1\cdot 0$	$h = 1\cdot 1$	$h = 1\cdot 2$	$h = 1\cdot 3$	$h = 1\cdot 4$	$h = 1\cdot 5$	$h = 1\cdot 6$	$h = 1\cdot 7$	
0·0	·0386718	·0312570	·0249952	·0197727	·0154710	·0119721	·0091615+	·0069322	·0051861	*0·0*
0·1	·0329853	·0265442	·0211319	·0166407	·0129603	·0099821	·0076023	·0057246	·0042617	*0·1*
0·2	·0278526	·0223135+	·0176829	·0138601	·0107439	·0082354	·0062416	·0046769	·0034644	*0·2*
0·3	·0232782	·0185637	·0146428	·0114231	·0088123	·0067219	·0050694	·0037796	·0027856	*0·3*
0·4	·0192530	·0152821	·0119973	·0093142	·0071504	·0054273	·0040726	·0030210	·0022150−	*0·4*
0·5	·0157559	·0124469	·0097244	·0075127	·0057388	·0043340	·0032357	·0023879	·0017417	*0·5*
0·6	·0127561	·0100285−	·0077966	·0059935−	·0045553	·0034227	·0025422	·0018663	·0013541	*0·6*
0·7	·0102154	·0079918	·0061823	·0047286	·0035756	·0026728	·0019748	·0014421	·0010408	*0·7*
0·8	·0080912	·0062984	·0048478	·0036890	·0027751	·0020636	·0015167	·0011017	·0007909	*0·8*
0·9	·0063376	·0049085+	·0037587	·0028455−	·0021294	·0015751	·0011515−	·0008319	·0005940	*0·9*
1·0	·0049085+	·0037823	·0028814	·0021699	·0016162	·0011884	·0008641	·0006209	·0004409	*1·0*
1·1	·0037587	·0028814	·0021836	·0016357	·0012111	·0008863	·0006409	·0004580	·0003234	*1·1*
1·2	·0028455−	·0021699	·0016357	·0012188	·0008975−	·0006532	·0004698	·0003339	·0002345−	*1·2*
1·3	·0021294	·0016152	·0012111	·0008975−	·0006574	·0004758	·0003403	·0002405+	·0001680	*1·3*
1·4	·0015751	·0011884	·0008863	·0006532	·0004758	·0003425+	·0002436	·0001712	·0001189	*1·4*
1·5	·0011515−	·0008641	·0006409	·0004698	·0003403	·0002436	·0001723	·0001204	·0000831	*1·5*
1·6	·0008319	·0006209	·0004580	·0003339	·0002405+	·0001712	·0001204	·0000837	·0000574	*1·6*
1·7	·0005940	·0004409	·0003234	·0002345−	·0001680	·0001189	·0000831	·0000574	·0000392	*1·7*
1·8	·0004190	·0003093	·0002257	·0001627	·0001159	·0000815+	·0000567	·0000389	·0000264	*1·8*
1·9	·0002921	·0002144	·0001556	·0001115+	·0000790	·0000553	·0000383	·0000261	·0000176	*1·9*
2·0	·0002012	·0001469	·0001059	·0000755−	·0000532	·0000370	·0000254	·0000173	·0000116	*2·0*
2·1	·0001369	·0000994	·0000713	·0000505+	·0000354	·0000245−	·0000167	·0000113	·0000075+	*2·1*
2·2	·0000920	·0000664	·0000474	·0000334	·0000232	·0000160	·0000108	·0000073	·0000048	*2·2*
2·3	·0000611	·0000439	·0000311	·0000218	·0000151	·0000103	·0000070	·0000046	·0000031	*2·3*
2·4	·0000401	·0000286	·0000201	·0000140	·0000097	·0000066	·0000044	·0000029	·0000019	*2·4*
2·5	·0000260	·0000184	·0000129	·0000089	·0000061	·0000042	·0000028	·0000018	·0000012	*2·5*
2·6	·0000166	·0000117	·0000082	·0000056	·0000039	·0000026	·0000017	·0000011	·00000072	*2·6*

k	d/N for $r = -\cdot 50$									k
	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	
0·0	·0038356	·0028042	·0020265−	·0014474	·0010217	·0007128	·0004913	·0003347	·0002253	0·0
0·1	·0031363	·0022814	·0016403	·0011656	·0008185+	·0005680	·0003895−	·0002639	·0001767	0·1
0·2	·0025367	·0018359	·0013132	·0009283	·0006484	·0004476	·0003053	·0002058	·0001370	0·2
0·3	·0020292	·0014610	·0010396	·0007310	·0005070	·0003487	·0002365+	·0001586	·0001050−	0·3
0·4	·0016052	·0011497	·0008138	·0005692	·0003934	·0002686	·0001812	·0001208	·0000795−	0·4
0·5	·0012556	·0008945+	·0006298	·0004381	·0003011	·0002045+	·0001372	·0000909	·0000596	0·5
0·6	·0009710	·0006881	·0004818	·0003334	·0002279	·0001539	·0001027	·0000677	·0000441	0·6
0·7	·0007424	·0005232	·0003644	·0002507	·0001704	·0001144	·0000759	·0000498	·0000323	0·7
0·8	·0005610	·0003932	·0002723	·0001864	·0001260	·0000841	·0000555+	·0000362	·0000233	0·8
0·9	·0004190	·0002921	·0002012	·0001369	·0000920	·0000611	·0000401	·0000260	·0000166	0·9
1·0	·0003093	·0002144	·0001469	·0000994	·0000664	·0000439	·0000286	·0000184	·0000117	1·0
1·1	·0002257	·0001556	·0001059	·0000713	·0000474	·0000311	·0000201	·0000120	·0000082	1·1
1·2	·0001627	·0001115+	·0000755−	·0000505+	·0000334	·0000218	·0000140	·0000089	·0000056	1·2
1·3	·0001159	·0000790	·0000532	·0000354	·0000232	·0000151	·0000097	·0000061	·0000039	1·3
1·4	·0000815+	·0000553	·0000370	·0000245−	·0000160	·0000103	·0000066	·0000042	·0000026	1·4
1·5	·0000567	·0000383	·0000254	·0000167	·0000108	·0000070	·0000044	·0000028	·0000017	1·5
1·6	·0000389	·0000261	·0000173	·0000113	·0000073	·0000046	·0000029	·0000018	·0000011	1·6
1·7	·0000264	·0000176	·0000116	·0000075+	·0000048	·0000031	·0000019	·0000012	·00000072	1·7
1·8	·0000177	·0000117	·0000077	·0000050−	·0000032	·0000020	·0000012	·00000076	·00000046	1·8
1·9	·0000117	·0000077	·0000050+	·0000032	·0000020	·0000013	·00000079	·00000048	·00000025−	1·9
2·0	·0000077	·0000050+	·0000032	·0000021	·0000013	·00000081	·00000050+	·00000029	·00000018	2·0
2·1	·0000050−	·0000032	·0000021	·0000013	·00000082	·00000051	·00000031	·00000019	·00000011	2·1
2·2	·0000032	·0000020	·0000013	·00000082	·00000051	·00000031	·00000019	·00000011	·00000007	2·2
2·3	·0000020	·0000013	·00000081	·00000051	·00000031	·00000020	·00000012	·00000007	·00000004	2·3
2·4	·0000012	·00000079	·00000050+	·00000031	·00000019	·00000012	·00000007	·00000004	·00000002	2·4
2·5	·00000076	·00000048	·00000029	·00000019	·00000011	·00000007	·00000004	·00000002	·00000001	2·5
2·6	·00000046	·00000025−	·00000018	·00000011	·00000007	·00000004	·00000002	·00000001	·00000000	2·6

k	d/N for $r = -\cdot 55$									k
	$h = 0{\cdot}0$	$h = 0{\cdot}1$	$h = 0{\cdot}2$	$h = 0{\cdot}3$	$h = 0{\cdot}4$	$h = 0{\cdot}5$	$h = 0{\cdot}6$	$h = 0{\cdot}7$	$h = 0{\cdot}8$	
0·0	·1573139	·1379225+	·1197565−	·1029553	·0876156	·0737906	·0614915−	·0506918	·0413324	0·0
0·1	·1379225+	·1204240	·1041166	·0891146	·0754916	·0632813	·0524796	·0430487	·0349227	0·1
0·2	·1197565−	·1041166	·0896199	·0763569	·0643803	·0537065−	·0443183	·0361695+	·0291896	0·2
0·3	·1029553	·0891146	·0763569	·0647508	·0543306	·0450980	·0370254	·0300603	·0241304	0·3
0·4	·0876156	·0754916	·0643803	·0543306	·0453610	·0374610	·0305953	·0247076	·0197260	0·4
0·5	·0737906	·0632813	·0537065−	·0450980	·0374610	·0307756	·0250014	·0200805+	·0159430	0·5
0·6	·0614915−	·0524796	·0443183	·0370254	·0305953	·0250014	·0202001	·0161344	·0127378	0·6
0·7	·0506918	·0430487	·0361695+	·0300603	·0247076	·0200805+	·0161344	·0128143	·0100586	0·7
0·8	·0413324	·0349227	·0291896	·0241304	·0197260	·0159430	·0127378	·0100586	·0078495+	0·8
0·9	·0333271	·0280128	·0232900	·0191491	·0155672	·0125110	·0099384	·0078024	·0060528	0·9
1·0	·0265696	·0222148	·0183697	·0150202	·0121420	·0097023	·0076624	·0059790	·0046112	1·0
1·1	·0209407	·0174141	·0143206	·0116436	·0093586	·0074347	·0058368	·0045280	·0034704	1·1
1·2	·0163136	·0134918	·0110328	·0089192	·0071273	·0056287	·0043925+	·0033869	·0025798	1·2
1·3	·0125604	·0103298	·0083901	·0067507	·0053626	·0042098	·0032653	·0025022	·0018942	1·3
1·4	·0095566	·0078148	·0063174	·0050477	·0039859	·0031101	·0023975+	·0018258	·0013734	1·4
1·5	·0071843	·0058410	·0046942	·0037284	·0029263	·0022693	·0017386	·0013157	·0009834	1·5
1·6	·0053359	·0043129	·0034455−	·0027201	·0021218	·0016353	·0012450−	·0009362	·0006953	1·6
1·7	·0039150−	·0031456	·0024970	·0019600	·0015195−	·0011637	·0008803	·0006577	·0004853	1·7
1·8	·0028373	·0022660	·0017884	·0013947	·0010744	·0008176	·0006146	·0004563	·0003345−	1·8
1·9	·0020309	·0016122	·0012645+	·0009800	·0007502	·0005673	·0004237	·0003125−	·0002276	1·9
2·0	·0014357	·0011327	·0008829	·0006799	·0005172	·0003886	·0002883	·0002112	·0001528	2·0
2·1	·0010022	·0007858	·0006087	·0004657	·0003520	·0002627	·0001937	·0001409	·0001013	2·1
2·2	·0006908	·0005382	·0004142	·0003150−	·0002365−	·0001754	·0001284	·0000928	·0000663	2·2
2·3	·0004702	·0003639	·0002784	·0002103	·0001569	·0001156	·0000841	·0000603	·0000428	2·3
2·4	·0003159	·0002429	·0001845+	·0001386	·0001027	·0000752	·0000543	·0000387	·0000273	2·4
2·5	·0002092	·0001602	·0001209	·0000902	·0000663	·0000482	·0000346	·0000245+	·0000172	2·5
2·6	·0001372	·0001042	·0000781	·0000579	·0000423	·0000306	·0000218	·0000149	·0000106	2·6

k	d/N for $r = -\cdot 55$									k
	$h = 0{\cdot}9$	$h = 1{\cdot}0$	$h = 1{\cdot}1$	$h = 1{\cdot}2$	$h = 1{\cdot}3$	$h = 1{\cdot}4$	$h = 1{\cdot}5$	$h = 1{\cdot}6$	$h = 1{\cdot}7$	
0·0	·0333271	·0265696	·0209407	·0163136	·0125604	·0095566	·0071843	·0053359	·0039150−	0·0
0·1	·0280128	·0222148	·0174141	·0134918	·0103208	·0078148	·0058410	·0043129	·0031456	0·1
0·2	·0232000	·0183697	·0143206	·0110328	·0083991	·0063174	·0046942	·0034455−	·0024979	0·2
0·3	·0191491	·0150202	·0116436	·0089192	·0067507	·0050477	·0037284	·0027201	·0019600	0·3
0·4	·0155672	·0121420	·0093586	·0071273	·0053626	·0039859	·0029263	·0021218	·0015195−	0·4
0·5	·0125110	·0097023	·0074347	·0056287	·0042098	·0031101	·0022693	·0016353	·0011637	0·5
0·6	·0099384	·0076624	·0058368	·0043925+	·0032653	·0023975+	·0017386	·0012450−	·0008803	0·6
0·7	·0078024	·0059799	·0045280	·0033869	·0025022	·0018258	·0013157	·0009362	·0006577	0·7
0·8	·0060528	·0046112	·0034704	·0025798	·0018942	·0013734	·0009834	·0006953	·0004853	0·8
0·9	·0046393	·0035129	·0026275+	·0019411	·0014163	·0010205−	·0007260	·0005099	·0003536	0·9
1·0	·0035129	·0026436	·0019651	·0014426	·0010458	·0007487	·0005292	·0003693	·0002544	1·0
1·1	·0026275+	·0019651	·0014515−	·0010588	·0007627	·0005424	·0003809	·0002641	·0001807	1·1
1·2	·0019411	·0014426	·0010588	·0007674	·0005492	·0003881	·0002707	·0001864	·0001267	1·2
1·3	·0014163	·0010458	·0007627	·0005492	·0003905−	·0002741	·0001899	·0001299	·0000877	1·3
1·4	·0010205−	·0007487	·0005424	·0003881	·0002741	·0001911	·0001316	·0000894	·0000600	1·4
1·5	·0007260	·0005292	·0003809	·0002707	·0001899	·0001316	·0000900	·0000607	·0000404	1·5
1·6	·0005099	·0003693	·0002641	·0001864	·0001299	·0000894	·0000607	·0000407	·0000269	1·6
1·7	·0003536	·0002544	·0001807	·0001267	·0000877	·0000600	·0000404	·0000269	·0000177	1·7
1·8	·0002421	·0001730	·0001221	·0000850+	·0000585−	·0000397	·0000266	·0000176	·0000115−	1·8
1·9	·0001636	·0001162	·0000814	·0000563	·0000385−	·0000259	·0000172	·0000113	·0000073	1·9
2·0	·0001091	·0000770	·0000536	·0000368	·0000250−	·0000167	·0000110	·0000072	·0000046	2·0
2·1	·0000719	·0000503	·0000348	·0000238	·0000160	·0000106	·0000070	·0000045+	·0000029	2·1
2·2	·0000467	·0000325−	·0000223	·0000151	·0000101	·0000066	·0000043	·0000028	·0000018	2·2
2·3	·0000300	·0000207	·0000141	·0000095−	·0000063	·0000041	·0000027	·0000017	·0000011	2·3
2·4	·0000191	·0000130	·0000088	·0000059	·0000039	·0000025+	·0000016	·0000010	·00000065−	2·4
2·5	·0000118	·0000081	·0000054	·0000036	·0000024	·0000015+	·00000097	·00000061	·00000038	2·5
2·6	·0000073	·0000049	·0000033	·0000022	·0000014	·00000091	·00000058	·00000036	·00000022	2·6

k	d/N for $r = -\cdot 55$									k
	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	
0·0	·0028373	·0020309	·0014357	·0010022	·0006908	·0004702	·0003159	·0002092	·0001372	0·0
0·1	·0022660	·0016122	·0011327	·0007858	·0005382	·0003639	·0002429	·0001602	·0001042	0·1
0·2	·0017884	·0012645+	·0008829	·0006087	·0004142	·0002784	·0001845+	·0001209	·0000781	0·2
0·3	·0013947	·0009800	·0006799	·0004657	·0003150−	·0002103	·0001386	·0000902	·0000579	0·3
0·4	·0010744	·0007502	·0005172	·0003520	·0002365−	·0001569	·0001027	·0000663	·0000423	0·4
0·5	·0008176	·0005673	·0003886	·0002627	·0001754	·0001156	·0000752	·0000482	·0000306	0·5
0·6	·0006146	·0004237	·0002883	·0001937	·0001284	·0000841	·0000543	·0000346	·0000218	0·6
0·7	·0004563	·0003125−	·0002112	·0001409	·0000928	·0000603	·0000387	·0000245+	·0000149	0·7
0·8	·0003345−	·0002276	·0001528	·0001013	·0000663	·0000428	·0000273	·0000172	·0000106	0·8
0·9	·0002421	·0001636	·0001091	·0000719	·0000467	·0000300	·0000191	·0000118	·0000073	0·9
1·0	·0001730	·0001162	·0000770	·0000503	·0000325−	·0000207	·0000130	·0000081	·0000049	1·0
1·1	·0001221	·0000814	·0000536	·0000348	·0000223	·0000141	·0000088	·0000054	·0000033	1·1
1·2	·0000850+	·0000563	·0000368	·0000238	·0000151	·0000095−	·0000059	·0000036	·0000022	1·2
1·3	·0000585−	·0000385−	·0000250−	·0000160	·0000101	·0000063	·0000039	·0000024	·0000014	1·3
1·4	·0000397	·0000259	·0000167	·0000106	·0000066	·0000041	·0000025+	·0000015+	·00000091	1·4
1·5	·0000266	·0000172	·0000110	·0000070	·0000043	·0000027	·0000016	·00000097	·00000058	1·5
1·6	·0000176	·0000113	·0000072	·0000045+	·0000028	·0000017	·0000010	·00000061	·00000036	1·6
1·7	·0000115−	·0000073	·0000046	·0000029	·0000018	·0000011	·00000065−	·00000038	·00000022	1·7
1·8	·0000074	·0000047	·0000029	·0000018	·0000011	·00000067	·00000040	·00000023	·00000013	1·8
1·9	·0000047	·0000030	·0000018	·0000011	·00000069	·00000041	·00000024	·00000015−	·00000008	1·9
2·0	·0000029	·0000018	·0000011	·00000070	·00000042	·00000025+	·00000015−	·00000008	·00000005−	2·0
2·1	·0000018	·0000011	·00000070	·00000042	·00000025+	·00000015−	·00000009	·00000005−	·00000003	2·1
2·2	·0000011	·00000069	·00000042	·00000025+	·00000015−	·00000009	·00000005−	·00000003	·00000002	2·2
2·3	·00000067	·00000041	·00000025+	·00000015−	·00000009	·00000005+	·00000003	·00000002	·00000001	2·3
2·4	·00000040	·00000024	·00000015−	·00000009	·00000005−	·00000003	·00000002	·00000001	·00000001	2·4
2·5	·00000023	·00000015−	·00000008	·00000005−	·00000003	·00000002	·00000001	·00000001	·00000000	2·5
2·6	·00000013	·00000008	·00000005−	·00000003	·00000002	·00000001	·00000001	·00000000		2·6

d/N for *r* = − ·60

k	*h* = 0·0	*h* = 0·1	*h* = 0·2	*h* = 0·3	*h* = 0·4	*h* = 0·5	*h* = 0·6	*h* = 0·7	*h* = 0·8	*k*
0·0	·1475836	·1282648	·1103130	·0938584	·0789827	·0657195+	·0540578	·0439466	·0353029	0·0
0·1	·1282648	·1109204	·0949035−	·0803157	·0672130	·0556079	·0454726	·0367452	·0293361	0·1
0·2	·1103130	·0949035−	·0807649	·0679724	·0565592	·0465192	·0378115−	·0303662	·0240902	0·2
0·3	·0938584	·0803157	·0679724	·0568800	·0470515+	·0384660	·0310726	·0247964	·0195447	0·3
0·4	·0789827	·0672130	·0565592	·0470515+	·0380867	·0314321	·0252301	·0200039	·0156633	0·4
0·5	·0657195+	·0556079	·0465192	·0384660	·0314321	·0253763	·0202375+	·0159396	·0123970	0·5
0·6	·0540578	·0454726	·0378115−	·0310726	·0252301	·0202375+	·0160328	·0125430	·0096885+	0·6
0·7	·0439466	·0367452	·0303662	·0247964	·0200039	·0159396	·0125430	·0097456	·0074754	0·7
0·8	·0353029	·0293361	·0240902	·0195447	·0156633	·0123970	·0096885+	·0074754	·0056935−	0·8
0·9	·0280172	·0231352	·0188757	·0152133	·0121102	·0095193	·0073878	·0056599	·0042798	0·9
1·0	·0219629	·0180192	·0146050+	·0116923	·0092438	·0072156	·0055604	·0042294	·0031749	1·0
1·1	·0170032	·0138586	·0111576	·0088714	·0069649	·0053984	·0041302	·0031188	·0023239	1·1
1·2	·0129980	·0105234	·0084147	·0066442	·0051795−	·0039859	·0030274	·0022692	·0016783	1·2
1·3	·0098099	·0078883	·0062640	·0049112	·0038012	·0029040	·0021894	·0016288	·0011956	1·3
1·4	·0073086	·0058363	·0046020	·0035824	·0027526	·0020875−	·0015621	·0011534	·0008402	1·4
1·5	·0053743	·0042616	·0033364	·0025784	·0019667	·0014803	·0010995−	·0008067	·0005824	1·5
1·6	·0039001	·0030706	·0023866	·0018309	·0013862	·0010356	·0007633	·0005550+	·0003981	1·6
1·7	·0027928	·0021830	·0016843	·0012826	·0009638	·0007145+	·0005226	·0003771	·0002684	1·7
1·8	·0019732	·0015312	·0011726	·0008863	·0006609	·0004862	·0003529	·0002527	·0001784	1·8
1·9	·0013754	·0010594	·0008053	·0006040	·0004470	·0003263	·0002350−	·0001669	·0001169	1·9
2·0	·0009458	·0007230	·0005455−	·0004060	·0002981	·0002160	·0001543	·0001087	·0000755+	2·0
2·1	·0006415−	·0004867	·0003644	·0002691	·0001961	·0001410	·0000999	·0000698	·0000481	2·1
2·2	·0004291	·0003231	·0002400	·0001759	·0001272	·0000907	·0000638	·0000442	·0000302	2·2
2·3	·0002831	·0002115+	·0001559	·0001134	·0000813	·0000575+	·0000401	·0000276	·0000187	2·3
2·4	·0001842	·0001365+	·0000999	·0000720	·0000513	·0000360	·0000240	·0000170	·0000114	2·4
2·5	·0001181	·0000869	·0000631	·0000451	·0000319	·0000222	·0000152	·0000103	·0000069	2·5
2·6	·0000747	·0000545+	·0000393	·0000279	·0000195+	·0000135−	·0000092	·0000062	·0000041	2·6

d/N for *r* = − ·60

k	*h* = 0·9	*h* = 1·0	*h* = 1·1	*h* = 1·2	*h* = 1·3	*h* = 1·4	*h* = 1·5	*h* = 1·6	*h* = 1·7	*k*
0·0	·0280172	·0219629	·0170032	·0129980	·0098099	·0073086	·0053743	·0039001	·0027928	0·0
0·1	·0231352	·0180192	·0138586	·0105234	·0078883	·0058363	·0042616	·0030706	·0021830	0·1
0·2	·0188757	·0146050+	·0111576	·0084147	·0062640	·0046020	·0033364	·0023866	·0016843	0·2
0·3	·0152133	·0116923	·0088714	·0066442	·0049112	·0035824	·0025784	·0018309	·0012826	0·3
0·4	·0121102	·0092438	·0069649	·0051795−	·0038012	·0027526	·0019667	·0013862	·0009638	0·4
0·5	·0095193	·0072156	·0053984	·0039859	·0029040	·0020875−	·0014803	·0010356	·0007145+	0·5
0·6	·0073878	·0055604	·0041302	·0030274	·0021894	·0015621	·0010995−	·0007633	·0005226	0·6
0·7	·0056599	·0042294	·0031188	·0022692	·0016288	·0011534	·0008067	·0005550+	·0003771	0·7
0·8	·0042798	·0031749	·0023239	·0016783	·0011956	·0008402	·0005824	·0003981	·0002684	0·8
0·9	·0031938	·0023518	·0017086	·0012246	·0008658	·0006038	·0004153	·0002816	·0001884	0·9
1·0	·0023518	·0017189	·0012394	·0008816	·0006185−	·0004280	·0002920	·0001965+	·0001304	1·0
1·1	·0017086	·0012394	·0008869	·0006260	·0004358	·0002992	·0002025+	·0001352	·0000890	1·1
1·2	·0012246	·0008816	·0006260	·0004384	·0003028	·0002062	·0001385+	·0000917	·0000599	1·2
1·3	·0008658	·0006185−	·0004358	·0003028	·0002075−	·0001402	·0000934	·0000614	·0000397	1·3
1·4	·0006038	·0004280	·0002992	·0002062	·0001402	·0000940	·0000621	·0000405−	·0000260	1·4
1·5	·0004153	·0002920	·0002025+	·0001385+	·0000934	·0000621	·0000407	·0000263	·0000168	1·5
1·6	·0002816	·0001965+	·0001352	·0000917	·0000614	·0000405−	·0000263	·0000169	·0000107	1·6
1·7	·0001884	·0001304	·0000890	·0000599	·0000397	·0000260	·0000168	·0000107	·0000067	1·7
1·8	·0001242	·0000853	·0000577	·0000385+	·0000254	·0000164	·0000105+	·0000066	·0000041	1·8
1·9	·0000807	·0000550+	·0000369	·0000245+	·0000160	·0000103	·0000065−	·0000041	·0000025+	1·9
2·0	·0000516	·0000350−	·0000233	·0000153	·0000099	·0000063	·0000040	·0000025−	·0000015+	2·0
2·1	·0000327	·0000219	·0000145−	·0000094	·0000060	·0000038	·0000024	·0000015−	·00000089	2·1
2·2	·0000204	·0000135+	·0000090	·0000057	·0000036	·0000023	·0000014	·00000086	·00000052	2·2
2·3	·0000125+	·0000082	·0000054	·0000034	·0000022	·0000013	·00000082	·00000050+	·00000030	2·3
2·4	·0000076	·0000050−	·0000032	·0000020	·0000013	·00000078	·00000048	·00000029	·00000017	2·4
2·5	·0000045+	·0000029	·0000019	·0000012	·00000073	·00000045−	·00000027	·00000016	·00000010	2·5
2·6	·0000027	·0000017	·0000011	·00000067	·00000042	·00000026+	·00000015+	·00000009	·00000005+	2·6

k	d/N for $r = -\cdot60$									k
	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	
0·0	·0019732	·0013754	·0009458	·0006415-	·0004201	·0002831	·0001842	·0001181	·0000747	0·0
0·1	·0015312	·0010594	·0007230	·0004867	·0003231	·0002115+	·0001365+	·0000869	·0000545+	0·1
0·2	·0011726	·0008053	·0005455-	·0003644	·0002400	·0001559	·0000999	·0000631	·0000393	0·2
0·3	·0008863	·0006040	·0004060	·0002691	·0001759	·0001134	·0000720	·0000451	·0000279	0·3
0·4	·0006609	·0004470	·0002981	·0001961	·0001272	·0000813	·0000513	·0000319	·0000195+	0·4
0·5	·0004862	·0003263	·0002160	·0001410	·0000907	·0000575+	·0000360	·0000222	·0000135-	0·5
0·6	·0003529	·0002350-	·0001543	·0000999	·0000638	·0000401	·0000249	·0000152	·0000092	0·6
0·7	·0002527	·0001669	·0001087	·0000698	·0000442	·0000276	·0000170	·0000103	·0000062	0·7
0·8	·0001784	·0001169	·0000755+	·0000481	·0000302	·0000187	·0000114	·0000069	·0000041	0·8
0·9	·0001242	·0000807	·0000516	·0000327	·0000204	·0000125+	·0000076	·0000045+	·0000027	0·9
1·0	·0000853	·0000550+	·0000350-	·0000219	·0000135+	·0000082	·0000049	·0000029	·0000017	1·0
1·1	·0000577	·0000369	·0000233	·0000145-	·0000090	·0000054	·0000032	·0000019	·0000011	1·1
1·2	·0000385+	·0000245+	·0000153	·0000094	·0000057	·0000034	·0000020	·0000012	·00000067	1·2
1·3	·0000254	·0000160	·0000099	·0000060	·0000036	·0000022	·0000013	·00000073	·00000042	1·3
1·4	·0000164	·0000103	·0000063	·0000038	·0000023	·0000013	·00000078	·00000045-	·00000025+	1·4
1·5	·0000105+	·0000065-	·0000040	·0000024	·0000014	·00000082	·00000048	·00000027	·00000015+	1·5
1·6	·0000066	·0000041	·0000025-	·0000015-	·00000086	·00000050+	·00000029	·00000016	·00000009	1·6
1·7	·0000041	·0000025+	·0000015+	·00000089	·00000052	·00000030	·00000017	·00000010	·00000005+	1·7
1·8	·0000025+	·0000015+	·00000091	·00000053	·00000031	·00000018	·00000010	·00000005+	·00000002	1·8
1·9	·0000015+	·00000091	·00000054	·00000031	·00000018	·00000010	·00000006	·00000003	·00000002	1·9
2·0	·00000091	·00000054	·00000032	·00000018	·00000010	·00000006	·00000003	·00000002	·00000001	2·0
2·1	·00000053	·00000031	·00000018	·00000011	·00000006	·00000003	·00000002	·00000001	·00000000	2·1
2·2	·00000031	·00000018	·00000010	·00000006	·00000003	·00000002	·00000001	·00000001		2·2
2·3	·00000018	·00000010	·00000006	·00000003	·00000002	·00000001	·00000001	·00000000		2·3
2·4	·00000010	·00000006	·00000003	·00000002	·00000001	·00000001	·00000000			2·4
2·5	·00000005+	·00000003	·00000002	·00000001	·00000000	·00000000				2·5
2·6	·00000002	·00000002	·00000001	·00000000						2·6

k	d/N for $r = -\cdot65$									k
	$h = 0{\cdot}0$	$h = 0{\cdot}1$	$h = 0{\cdot}2$	$h = 0{\cdot}3$	$h = 0{\cdot}4$	$h = 0{\cdot}5$	$h = 0{\cdot}6$	$h = 0{\cdot}7$	$h = 0{\cdot}8$	
0·0	·13738444	·11814907	·10044363	·08438644	·07004026	·05741457	·04647053	·03712815-	·02927486	0·0
0·1	·11814907	·10099015+	·08531418	·07120653	·05870105+	·04778352	·03839747	·03045178	·02382912	0·1
0·2	·10044363	·08531418	·07159960	·05935509	·04858909	·03926776	·03132135-	·02465184	·01914109	0·2
0·3	·08438644	·07120653	·05935509	·04886063	·03971028	·03185499	·02521607	·01969265+	·01516933	0·3
0·4	·07004026	·05870105+	·04858909	·03971028	·03203480	·02550301	·02003120	·01551943	·01185795+	0·4
0·5	·05741457	·04778352	·03926776	·03185499	·02550301	·02014534	·01569751	·01206339	·00914128	0·5
0·6	·04647053	·03839747	·03132135-	·02521607	·02003120	·01569751	·01213266	·00924691	·00694822	0·6
0·7	·03712815-	·03045178	·02465184	·01969265+	·01551943	·01206339	·00924691	·00698838	·00520632	0·7
0·8	·02927486	·02382912	·01914109	·01516933	·01185795+	·00914128	·00694822	·00520632	·00384510	0·8
0·9	·02277480	·01839480	·01465894	·01152332	·00893361	·00682912	·00514649	·00382288	·00279856	0·9
1·0	·01747795-	·01400509	·01107066	·00863094	·00663515+	·00502887	·00375699	·00276623	·00200701	1·0
1·1	·01322869	·01051473	·00824328	·00637283	·00485747	·00364069	·00270268	·00197225+	·00141806	1·1
1·2	·00987308	·00778314	·00605076	·00463800	·00350460	·00261009	·00191565+	·00138534	·00098699	1·2
1·3	·00726479	·00567916	·00437757	·00332650+	·00249156	·00183913	·00133767	·00095855-	·00067663	1·3
1·4	·00526934	·00408430	·00312109	·00235094	·00174523	·00127665+	·00092010	·00065326	·00045685-	1·4
1·5	·00376692	·00289463	·00219265-	·00163695+	·00120428	·00087293	·00062335-	·00043846	·00030375+	1·5
1·6	·00265370	·00202140	·00151762	·00112284	·00081855+	·00058788	·00041590	·00028980	·00019886	1·6
1·7	·00184202	·00139071	·00103476	·00075864	·00054797	·00038990	·00027325+	·00018860	·00012819	1·7
1·8	·00125967	·00094254	·00069494	·00050483	·00036126	·00025464	·00017677	·00012085-	·00008135-	1·8
1·9	·00084858	·00062919	·00045966	·00033082	·00023453	·00016375-	·00011259	·00007623	·00005082	1·9
2·0	·00056305+	·00041367	·00029941	·00021348	·00014991	·00010367	·00007060	·00004734	·00003125-	2·0
2·1	·00036794	·00026783	·00019205-	·00013563	·00009434	·00006462	·00004358	·00002894	·00001891	2·1
2·2	·00023678	·00017075-	·00012128	·00008484	·00005845-	·00003965-	·00002648	·00001741	·00001127	2·2
2·3	·00015004	·00010718	·00007541	·00005225-	·00003565-	·00002394	·00001583	·00001031	·00000661	2·3
2·4	·00009361	·00006624	·00004616	·00003167	·00002140	·00001423	·00000932	·00000601	·00000381	2·4
2·5	·00005750+	·00004030	·00002781	·00001890	·00001264	·00000833	·00000540	·00000345-	·00000216	2·5
2·6	·00003477	·00002413	·00001649	·00001110	·00000735+	·00000479	·00000308	·00000194	·00000121	2·6

k	*d/N* for *r* = − ·65									k
	h = *0·9*	*h* = *1·0*	*h* = *1·1*	*h* = *1·2*	*h* = *1·3*	*h* = *1·4*	*h* = *1·5*	*h* = *1·6*	*h* = *1·7*	
0·0	·02277480	·01747795−	·01322869	·00987308	·00726479	·00526934	·00376692	·00263370	·00184202	*0·0*
0·1	·01839480	·01400509	·01051473	·00778314	·00567916	·00408430	·00289463	·00202140	·00139071	*0·1*
0·2	·01465894	·01107066	·00824328	·00605076	·00437757	·00312109	·00219265	·00151762	·00103476	*0·2*
0·3	·01152332	·00863004	·00637263	·00463800	·00332650+	·00235094	·00163605+	·00112284	·00075864	*0·3*
0·4	·00893361	·00663515+	·00485747	·00350460	·00249156	·00174523	·00120428	·00081865+	·00054797	*0·4*
0·5	·00682912	·00502887	·00364969	·00261009	·00183913	·00127665+	·00087203	·00058788	·00038900	*0·5*
0·6	·00514649	·00375699	·00270268	·00191565+	·00133767	·00092010	·00062335−	·00041590	·00027325+	*0·6*
0·7	·00382288	·00276623	·00197225+	·00138534	·00095855−	·00065326	·00043846	·00028080	·00018860	*0·7*
0·8	·00279856	·00200701	·00141806	·00098690	·00067663	·00045685	·00030375+	·00019886	·00012810	*0·8*
0·9	·00201873	·00143471	·00100446	·00069268	·00047045+	·00031466	·00020723	·00013438	·00008579	*0·9*
1·0	·00143471	·00101035−	·00070084	·00047881	·00032214	·00021342	·00013922	·00008941	·00005652	*1·0*
1·1	·00100446	·00070084	·00048163	·00032595+	·00021722	·00014254	·00009208	·00005856	·00003666	*1·1*
1·2	·00069268	·00047881	·00032595+	·00021851	·00014423	·00009373	·00005997	·00003767	·00002341	*1·2*
1·3	·00047045+	·00032214	·00021722	·00014423	·00009428	·00006068	·00003844	·00002397	·00001471	*1·3*
1·4	·00031466	·00021342	·00014254	·00009373	·00006068	·00003867	·00002426	·00001498	·00000910	*1·4*
1·5	·000207230	·000139216	·$0^4$92084	·$0^4$59965+	·$0^4$38442	·$0^4$24258	·$0^4$15067	·$0^4$09211	·$0^4$055414	*1·5*
1·6	·000134377	·$0^4$89405+	·$0^4$58564	·$0^4$37665+	·$0^4$23972	·$0^4$14978	·$0^4$09211	·$0^4$05574	·$0^4$033201	*1·6*
1·7	·$0^4$85786	·$0^4$56523	·$0^4$36664	·$0^4$23411	·$0^4$14714	·$0^4$09102	·$0^4$055414	·$0^4$033201	·$0^4$019575	*1·7*
1·8	·$0^4$53913	·$0^4$35176	·$0^4$22593	·$0^4$14284	·$0^4$08888	·$0^4$05444	·$0^4$032808	·$0^4$019459	·$0^4$011357	*1·8*
1·9	·$0^4$33352	·$0^4$21547	·$0^4$13703	·$0^4$08577	·$0^4$05284	·$0^4$03204	·$0^4$019114	·$0^4$011222	·$0^4$006483	*1·9*
2·0	·$0^4$20308	·$0^4$12991	·$0^4$08180	·$0^4$05069	·$0^4$03091	·$0^4$018554	·$0^4$010958	·$0^4$006368	·$0^4$003641	*2·0*
2·1	·$0^4$12170	·$0^4$07708	·$0^4$04805−	·$0^4$02948	·$0^4$01780	·$0^4$010573	·$0^4$006181	·$0^4$003555+	·$0^4$002012	*2·1*
2·2	·$0^4$07178	·$0^4$04501	·$0^4$02778	·$0^4$01687	·$0^4$010081	·$0^4$005928	·$0^4$003430	·$0^4$001953	·$0^4$001094	*2·2*
2·3	·$0^4$04166	·$0^4$02586	·$0^4$015800	·$0^4$009498	·$0^4$005619	·$0^4$003271	·$0^4$001873	·$0^4$001055+	·$0^4$000585+	*2·3*
2·4	·$0^4$02380	·$0^4$01462	·$0^4$008843	·$0^4$005262	·$0^4$003081	·$0^4$001775+	·$0^4$001006	·$0^4$000561	·$0^4$000308	*2·4*
2·5	·$0^4$01337	·$0^4$008136	·$0^4$004870	·$0^4$002868	·$0^4$001662	·$0^4$000948	·$0^4$000532	·$0^4$000293	·$0^4$000159	*2·5*
2·6	·$0^4$00740	·$0^4$004453	·$0^4$002639	·$0^4$001538	·$0^4$000882	·$0^4$000498	·$0^4$000276	·$0^4$000151	·$0^4$0000811	*2·6*

k	*d/N* for *r* = − ·65									k
	h = *1·8*	*h* = *1·9*	*h* = *2·0*	*h* = *2·1*	*h* = *2·2*	*h* = *2·3*	*h* = *2·4*	*h* = *2·5*	*h* = *2·6*	
0·0	·00125967	·00084858	·00056305+	·00036794	·00023678	·00015004	·00009361	·00005750+	·00003477	*0·0*
0·1	·00094254	·00062919	·00041367	·00026783	·00017075−	·00010718	·00006624	·00004030	·00002413	*0·1*
0·2	·00069494	·00045966	·00029941	·00019205−	·00012128	·00007541	·00004616	·00002781	·00001649	*0·2*
0·3	·00050483	·00033082	·00021348	·00013563	·00008484	·00005225−	·00003167	·00001890	·00001110	*0·3*
0·4	·00036126	·00023453	·00014991	·00009434	·00005845	·00003565−	·00002140	·00001264	·00000735+	*0·4*
0·5	·000254641	·000163749	·000103673	·$0^4$64618	·$0^4$39647	·$0^4$23944	·$0^4$14233	·$0^4$08327	·$0^4$04794	*0·5*
0·6	·000176773	·000112592	·$0^4$70600	·$0^4$43579	·$0^4$26478	·$0^4$15835−	·$0^4$09320	·$0^4$05398	·$0^4$03077	*0·6*
0·7	·000120847	·$0^4$76232	·$0^4$47339	·$0^4$28936	·$0^4$17409	·$0^4$10308	·$0^4$06007	·$0^4$03445+	·$0^4$01944	*0·7*
0·8	·$0^4$81347	·$0^4$50818	·$0^4$31250−	·$0^4$18914	·$0^4$11267	·$0^4$06606	·$0^4$03811	·$0^4$02164	·$0^4$01209	*0·8*
0·9	·$0^4$53913	·$0^4$33352	·$0^4$20308	·$0^4$12170	·$0^4$07178	·$0^4$04166	·$0^4$02380	·$0^4$01337	·$0^4$00740	*0·9*
1·0	·$0^4$35176	·$0^4$21547	·$0^4$12991	·$0^4$07708	·$0^4$04501	·$0^4$02586	·$0^4$01462	·$0^4$008136	·$0^4$004453	*1·0*
1·1	·$0^4$22593	·$0^4$13703	·$0^4$08180	·$0^4$04805−	·$0^4$02778	·$0^4$015800	·$0^4$008843	·$0^4$004870	·$0^4$002639	*1·1*
1·2	·$0^4$14284	·$0^4$08577	·$0^4$05069	·$0^4$02948	·$0^4$01687	·$0^4$009498	·$0^4$005262	·$0^4$002868	·$0^4$001538	*1·2*
1·3	·$0^4$08888	·$0^4$05284	·$0^4$03091	·$0^4$01780	·$0^4$010081	·$0^4$005619	·$0^4$003081	·$0^4$001662	·$0^4$000882	*1·3*
1·4	·$0^4$05444	·$0^4$03204	·$0^4$018554	·$0^4$010573	·$0^4$005928	·$0^4$003271	·$0^4$001775+	·$0^4$000948	·$0^4$000498	*1·4*
1·5	·$0^4$032808	·$0^4$019114	·$0^4$010958	·$0^4$006181	·$0^4$003430	·$0^4$001873	·$0^4$001006	·$0^4$000532	·$0^4$000276	*1·5*
1·6	·$0^4$019459	·$0^4$011222	·$0^4$006368	·$0^4$003555+	·$0^4$001953	·$0^4$001055+	·$0^4$000561	·$0^4$000293	·$0^4$000151	*1·6*
1·7	·$0^4$011357	·$0^4$006483	·$0^4$003641	·$0^4$002012	·$0^4$001094	·$0^4$000585+	·$0^4$000308	·$0^4$000159	·$0^4$0000811	*1·7*
1·8	·$0^4$006522	·$0^4$003685−	·$0^4$002048	·$0^4$001120	·$0^4$000603	·$0^4$000319	·$0^4$0001661	·$0^4$0000851	·$0^4$0000429	*1·8*
1·9	·$0^4$003685−	·$0^4$002061	·$0^4$001134	·$0^4$000614	·$0^4$000327	·$0^4$0001712	·$0^4$0000882	·$0^4$0000447	·$0^4$0000223	*1·9*
2·0	·$0^4$002048	·$0^4$001134	·$0^4$0006174	·$0^4$0003307	·$0^4$0001743	·$0^4$0000903	·$0^4$0000461	·$0^4$0000231	·$0^4$0000114	*2·0*
2·1	·$0^4$001120	·$0^4$000614	·$0^4$0003307	·$0^4$0001753	·$0^4$0000914	·$0^4$0000469	·$0^4$0000237	·$0^4$0000117	·$0^4$0000058	*2·1*
2·2	·$0^4$000603	·$0^4$000327	·$0^4$0001743	·$0^4$0000914	·$0^4$0000472	·$0^4$0000239	·$0^4$0000119	·$0^4$0000059	·$0^4$0000028	*2·2*
2·3	·$0^4$000319	·$0^4$0001712	·$0^4$0000903	·$0^4$0000469	·$0^4$0000239	·$0^4$0000120	·$0^4$0000059	·$0^4$0000029	·$0^4$0000014	*2·3*
2·4	·$0^4$0001661	·$0^4$0000882	·$0^4$0000461	·$0^4$0000237	·$0^4$0000119	·$0^4$0000059	·$0^4$0000029	·$0^4$0000014	·$0^4$0000007	*2·4*
2·5	·$0^4$0000851	·$0^4$0000447	·$0^4$0000231	·$0^4$0000117	·$0^4$0000059	·$0^4$0000029	·$0^4$0000014	·$0^4$0000007	·$0^4$0000003	*2·5*
2·6	·$0^4$0000429	·$0^4$0000223	·$0^4$0000114	·$0^4$0000058	·$0^4$0000028	·$0^4$0000014	·$0^4$0000007	·$0^4$0000003	·$0^4$0000001	*2·6*

0^4 indicates that four zeros must be placed before the figures that follow.

k	*d/N* for $r = -\cdot 70$									k
	$h = 0{\cdot}0$	$h = 0{\cdot}1$	$h = 0{\cdot}2$	$h = 0{\cdot}3$	$h = 0{\cdot}4$	$h = 0{\cdot}5$	$h = 0{\cdot}6$	$h = 0{\cdot}7$	$h = 0{\cdot}8$	
0·0	·12659166	·10745519	·09004105+	·07445180	·06072555+	·04884035+	·03872180	·03025303	·02328595−	0·0
0·1	·10745519	·09052528	·07526022	·06172350+	·04991962	·03979999	·03127159	·02420725−	·01845654	0·1
0·2	·09004105+	·07526022	·06205979	·05046818	·04046128	·03196965+	·02488758	·01908328	·01440906	0·2
0·3	·07445180	·06172350+	·05046818	·04068415+	·03232451	·02530494	·01951290	·01481720	·01107722	0·3
0·4	·06072555+	·04991962	·04046128	·03232451	·02544562	·01973133	·01506763	·01132840	·00838350−	0·4
0·5	·04884035+	·03979999	·03196965+	·02530494	·01973133	·01515204	·01145612	·00852612	·00624477	0·5
0·6	·03872180	·03127159	·02488758	·01951290	·01506763	·01145612	·00857420	·00631560	·00457729	0·6
0·7	·03025303	·02420725−	·01908328	·01481720	·01132840	·00852612	·00631560	·00460324	·00330076	0·7
0·8	·02328595−	·01845654	·01440906	·01107722	·00838350−	·00624477	·00457729	·00330076	·00234126	0·8
0·9	·01765282	·01385649	·01071087	·00815108	·00610548	·00450030	·00326355−	·00232799	·00163320	0·9
1·0	·01317710	·01024119	·00783649	·00590236	·00437486	·00319039	·00228866	·00161471	·00112024	1·0
1·1	·00968298	·00744980	·00564202	·00420509	·00308372	·00222456	·00157836	·00110124	·00075544	1·1
1·2	·00700303	·00533269	·00399647	·00294702	·00213784	·00152536	·00107028	·00073838	·00050078	1·2
1·3	·00498381	·00375551	·00278464	·00203130	·00145745+	·00102839	·00071349	·00048666	·00032628	1·3
1·4	·00348941	·00260157	·00190826	·00137681	·00097694	·00068161	·00046755−	·00031525+	·00020892	1·4
1·5	·00240314	·00177243	·00128591	·00091753	·00064376	·00044408	·00030113	·00020070	·00013146	1·5
1·6	·00162768	·00118742	·00085197	·00060111	·00041698	·00028435+	·00019059	·00012555−	·00008127	1·6
1·7	·00108407	·00078212	·00055490	·00038709	·00026545+	·00017893	·00011854	·00007717	·00004936	1·7
1·8	·00070987	·00050643	·00035525−	·00024409	·00016607	·00011064	·00007243	·00004660	·00002945−	1·8
1·9	·00045695−	·00032232	·00022352	·00015237	·00010208	·00006721	·00004348	·00002764	·00001726	1·9
2·0	·00028912	·00020161	·00013821	·00009312	·00006166	·00004011	·00002564	·$0^4$16104	·$0^4$09935−	2·0
2·1	·00017978	·00012393	·00008397	·00005591	·00003658	·00002352	·$0^4$14854	·$0^4$09216	·$0^4$05616	2·1
2·2	·00010986	·00007485+	·00005012	·00003298	·00002132	·$0^4$13544	·$0^4$08451	·$0^4$05180	·$0^4$03118	2·2
2·3	·00006598	·00004442	·00002939	·00001911	·$0^4$12207	·$0^4$07660	·$0^4$04722	·$0^4$02859	·$0^4$01700	2·3
2·4	·00003891	·00002589	·00001693	·$0^4$10876	·$0^4$06864	·$0^4$04255+	·$0^4$02591	·$0^4$01549	·$0^4$00910	2·4
2·5	·00002255+	·00001483	·$0^4$09579	·$0^4$06080	·$0^4$03791	·$0^4$02321	·$0^4$01396	·$0^4$00825−	·$0^4$00478	2·5
2·6	·00001284	·$0^4$08340	·$0^4$05323	·$0^4$03338	·$0^4$02056	·$0^4$01243	·$0^4$00739	·$0^4$00431	·$0^4$00247	2·6

k	*d/N* for $r = -\cdot 70$									k
	$h = 0{\cdot}9$	$h = 1{\cdot}0$	$h = 1{\cdot}1$	$h = 1{\cdot}2$	$h = 1{\cdot}3$	$h = 1{\cdot}4$	$h = 1{\cdot}5$	$h = 1{\cdot}6$	$h = 1{\cdot}7$	
0·0	·01765282	·01317710	·00968298	·00700303	·00498381	·00348941	·00240314	·00162768	·00108407	0·0
0·1	·01385649	·01024119	·00744980	·00533269	·00375551	·00260157	·00177243	·00118742	·00078212	0·1
0·2	·01071087	·00783649	·00564202	·00399647	·00278464	·00190826	·00128591	·00085197	·00055490	0·2
0·3	·00815108	·00590236	·00420509	·00294702	·00203130	·00137681	·00091753	·00060111	·00038709	0·3
0·4	·00610548	·00437486	·00308372	·00213784	·00145745+	·00097694	·00064376	·00041698	·00026545+	0·4
0·5	·00450030	·00319039	·00222456	·00152536	·00102839	·00068161	·00044408	·00028435+	·00017893	0·5
0·6	·00326355	·00228866	·00157836	·00107028	·00071349	·00046755−	·00030113	·00019059	·00011854	0·6
0·7	·00232799	·00161471	·00110124	·00073838	·00048666	·00031525+	·00020070	·00012555−	·00007717	0·7
0·8	·00163320	·00112024	·00075544	·00050078	·00032628	·00020892	·00013146	·00008127	·00004936	0·8
0·9	·00112685−	·00076413	·00050946	·00033385+	·00021501	·00013607	·00008461	·00005169	·00003102	0·9
1·0	·00076413	·00051238	·00033770	·00021874	·00013923	·00008708	·00005350+	·00003230	·$0^4$19149	1·0
1·1	·00050946	·00033770	·00022000	·00014084	·00008859	·00005475+	·00003324	·$0^4$19824	·$0^4$11612	1·1
1·2	·00033385+	·00021874	·00014084	·00008910	·00005539	·00003382	·$0^4$20287	·$0^4$11952	·$0^4$06196	1·2
1·3	·00021501	·00013923	·00008859	·00005539	·00003402	·$0^4$20522	·$0^4$12161	·$0^4$07078	·$0^4$04046	1·3
1·4	·00013607	·00008708	·00005475+	·00003382	·$0^4$20522	·$0^4$12232	·$0^4$07160	·$0^4$04116	·$0^4$02324	1·4
1·5	·00008461	·00005350+	·00003324	·$0^4$20287	·$0^4$12161	·$0^4$07160	·$0^4$04141	·$0^4$02351	·$0^4$01311	1·5
1·6	·00005169	·00003230	·$0^4$19824	·$0^4$11952	·$0^4$07078	·$0^4$04116	·$0^4$02351	·$0^4$01319	·$0^4$00726	1·6
1·7	·00003102	·$0^4$19149	·$0^4$11612	·$0^4$06916	·$0^4$04046	·$0^4$02324	·$0^4$01311	·$0^4$00726	·$0^4$00395−	1·7
1·8	·$0^4$18285−	·$0^4$11152	·$0^4$06680	·$0^4$03930	·$0^4$02271	·$0^4$01288	·$0^4$00718	·$0^4$00393	·$0^4$00211	1·8
1·9	·$0^4$10587	·$0^4$06378	·$0^4$03774	·$0^4$02193	·$0^4$01252	·$0^4$00701	·$0^4$00386	·$0^4$00208	·$0^4$001105+	1·9
2·0	·$0^4$06020	·$0^4$03583	·$0^4$02094	·$0^4$01202	·$0^4$00677	·$0^4$00375−	·$0^4$00204	·$0^4$001086	·$0^4$000569	2·0
2·1	·$0^4$03362	·$0^4$01976	·$0^4$01141	·$0^4$00647	·$0^4$00360	·$0^4$00197	·$0^4$001055−	·$0^4$000556	·$0^4$000287	2·1
2·2	·$0^4$01844	·$0^4$01070	·$0^4$00610	·$0^4$00342	·$0^4$00188	·$0^4$001013	·$0^4$000537	·$0^4$000279	·$0^4$000142	2·2
2·3	·$0^4$00993	·$0^4$00569	·$0^4$00320	·$0^4$00177	·$0^4$000961	·$0^4$000512	·$0^4$000268	·$0^4$000137	·$0^4$000069	2·3
2·4	·$0^4$00525−	·$0^4$00297	·$0^4$00165+	·$0^4$000902	·$0^4$000483	·$0^4$000254	·$0^4$000131	·$0^4$000066	·$0^4$000033	2·4
2·5	·$0^4$00272	·$0^4$00152	·$0^4$000836	·$0^4$000451	·$0^4$000239	·$0^4$000124	·$0^4$000063	·$0^4$000032	·$0^4$000015+	2·5
2·6	·$0^4$00139	·$0^4$000766	·$0^4$000415+	·$0^4$000221	·$0^4$000115+	·$0^4$000059	·$0^4$000030	·$0^4$000015−	·$0^4$000007	2·6

0^4 indicates that four zeros must be placed before the figures that follow.

k	*d/N* for $r = -\cdot 70$									k
	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	
0·0	·00070987	·00045695−	·00028012	·00017978	·00010986	·00006596	·00003891	·00002255+	·00001284	0·0
0·1	·00050643	·00032232	·00020161	·00012393	·00007485+	·00004442	·00002589	·00001483	·0^{4}08340	0·1
0·2	·00035525−	·00022352	·00013821	·00008397	·00005012	·00002939	·00001693	·0^{4}09579	·0^{4}05323	0·2
0·3	·00024499	·00015237	·00009312	·00005591	·00003298	·00001911	·0^{4}10876	·0^{4}06080	·0^{4}03338	0·3
0·4	·00016607	·00010208	·00006166	·00003658	·00002132	·0^{4}12207	·0^{4}06864	·0^{4}03791	·0^{4}02056	0·4
0·5	·00011064	·00006721	·00004011	·00002352	·0^{4}13544	·0^{4}07660	·0^{4}04255+	·0^{4}02321	·0^{4}01243	0·5
0·6	·00007243	·00004348	·00002564	·0^{4}14854	·0^{4}08451	·0^{4}04722	·0^{4}02591	·0^{4}01396	·0^{4}00739	0·6
0·7	·00004660	·00002764	·000016104	·0^{4}09216	·0^{4}05180	·0^{4}02859	·0^{4}01549	·0^{4}00825−	·0^{4}00431	0·7
0·8	·00002945−	·0^{4}17259	·0^{4}09935−	·0^{4}05616	·0^{4}03118	·0^{4}01700	·0^{4}00910	·0^{4}00478	·0^{4}00247	0·8
0·9	·0^{4}18285−	·0^{4}10587	·0^{4}06020	·0^{4}03362	·0^{4}01844	·0^{4}00993	·0^{4}00525−	·0^{4}00272	·0^{4}00139	0·9
1·0	·0^{4}11152	·0^{4}06378	·0^{4}03583	·0^{4}01976	·0^{4}01070	·0^{4}00569	·0^{4}00297	·0^{4}00152	·0^{4}000768	1·0
1·1	·0^{4}06680	·0^{4}03774	·0^{4}02094	·0^{4}01141	·0^{4}00610	·0^{4}00320	·0^{4}00165+	·0^{4}000836	·0^{4}000415+	1·1
1·2	·0^{4}03930	·0^{4}02193	·0^{4}01202	·0^{4}00647	·0^{4}00342	·0^{4}00177	·0^{4}000902	·0^{4}000451	·0^{4}000221	1·2
1·3	·0^{4}02271	·0^{4}01252	·0^{4}00677	·0^{4}00360	·0^{4}00188	·0^{4}000961	·0^{4}000483	·0^{4}000239	·0^{4}000115+	1·3
1·4	·0^{4}01288	·0^{4}00701	·0^{4}00375−	·0^{4}00197	·0^{4}001013	·0^{4}000512	·0^{4}000254	·0^{4}000124	·0^{4}000059	1·4
1·5	·0^{4}00718	·0^{4}00386	·0^{4}00204	·0^{4}001055−	·0^{4}000537	·0^{4}000268	·0^{4}000131	·0^{4}000063	·0^{4}000030	1·5
1·6	·0^{4}00393	·0^{4}00208	·0^{4}001086	·0^{4}000556	·0^{4}000279	·0^{4}000137	·0^{4}000066	·0^{4}000032	·0^{4}000015−	1·6
1·7	·0^{4}00211	·0^{4}001105+	·0^{4}000569	·0^{4}000287	·0^{4}000142	·0^{4}000069	·0^{4}000033	·0^{4}000015+	·0^{4}000007	1·7
1·8	·0^{4}001111	·0^{4}000575+	·0^{4}000292	·0^{4}000146	·0^{4}000071	·0^{4}000034	·0^{4}000016	·0^{4}000007	·0^{4}000003	1·8
1·9	·0^{4}000575+	·0^{4}000294	·0^{4}000147	·0^{4}000072	·0^{4}000035+	·0^{4}000016	·0^{4}000008	·0^{4}000003	·0^{4}000002	1·9
2·0	·0^{4}000292	·0^{4}000147	·0^{4}000073	·0^{4}000035+	·0^{4}000017	·0^{4}000008	·0^{4}000004	·0^{4}000002	·0^{4}000001	2·0
2·1	·0^{4}000146	·0^{4}000072	·0^{4}000035+	·0^{4}000017	·0^{4}000008	·0^{4}000004	·0^{4}000002	·0^{4}000001		2·1
2·2	·0^{4}000071	·0^{4}000035+	·0^{4}000017	·0^{4}000008	·0^{4}000004	·0^{4}000002	·0^{4}000001			2·2
2·3	·0^{4}000034	·0^{4}000016	·0^{4}000008	·0^{4}000004	·0^{4}000002	·0^{4}000001				2·3
2·4	·0^{4}000016	·0^{4}000008	·0^{4}000004	·0^{4}000002	·0^{4}000001					2·4
2·5	·0^{4}000007	·0^{4}000003	·0^{4}000002	·0^{4}000001						2·5
2·6	·0^{4}000003	·0^{4}000002	·0^{4}000001							2·6

k	*d/N* for $r = -\cdot 75$									k
	$h = 0{\cdot}0$	$h = 0{\cdot}1$	$h = 0{\cdot}2$	$h = 0{\cdot}3$	$h = 0{\cdot}4$	$h = 0{\cdot}5$	$h = 0{\cdot}6$	$h = 0{\cdot}7$	$h = 0{\cdot}8$	
0·0	·11502673	·09601193	·07895514	·06393679	·05096068	·03996170	·03081764	·02336334	·01740584	0·0
0·1	·09601193	·07937492	·06462260	·05178732	·04083268	·03166336	·02413803	·01808359	·01330939	0·1
0·2	·07895514	·06462260	·05206585−	·04127527	·03218190	·02466872	·01858381	·01375392	·00999731	0·2
0·3	·06393679	·05178732	·04127527	·03235663	·02493843	·01889056	·01405849	·01027557	·00737442	0·3
0·4	·05096068	·04083268	·03218190	·02493843	·01899393	·01421330	·01044640	·00753871	·00534026	0·4
0·5	·03996170	·03166336	·02466872	·01889056	·01421330	·01050393	·00762221	·00542947	·00379547	0·5
0·6	·03081764	·02413803	·01858381	·01405849	·01044640	·00762221	·00545954	·00383772	·00264682	0·6
0·7	·02336334	·01808359	·01375392	·01027567	·00753871	·00542947	·00383772	·00266155−	·00181066	0·7
0·8	·01740584	·01330939	·00999731	·00737442	·00534026	·00379547	·00264682	·00181066	·00121481	0·8
0·9	·01273898	·00962023	·00713469	·00519483	·00371236	·00260314	·00179065−	·00120807	·00079919	0·9
1·0	·00915625+	·00682714	·00499785+	·00359100	·00253193	·00175128	·00118805+	·00079032	·00051544	1·0
1·1	·00646129	·00475554	·00343554	·00243550+	·00169384	·00115543	·00077289	·00050687	·00032585−	1·1
1·2	·00447528	·00325055+	·00231690	·00162018	·00111127	·00074745+	·00049291	·00031864	·00020188	1·2
1·3	·00304167	·00217976	·00153259	·00105695+	·00071483	·00047401	·00030812	·00019631	·00012256	1·3
1·4	·00202812	·00143371	·00099416	·00067606	·00045077	·00029464	·00018876	·00011851	·00007290	1·4
1·5	·00132639	·00092475−	·00063230	·00042391	·00027861	·00017948	·00011331	·00007009	·00004248−	1·5
1·6	·00085066	·00058481	·00039422	·00026052	·00016876	·00010713	·00006664	·00004061	·00002424	1·6
1·7	·00053489	·00036254	·00024090	·00015691	·00010016	·00006265−	·00003839	·00002305−	·00001355+	1·7
1·8	·00032970	·00022028	·00014426	·00009260	·00005824	·00003589	·00002166	·00001281	·00000742	1·8
1·9	·00019919	·00013116	·00008465+	·00005354	·00003317	·00002014	·00001197	·00000697	·00000398	1·9
2·0	·00011798	·00007653	·00004866	·00003032	·00001851	·00001107	·00000648	·0^{4}03716	·0^{4}02087	2·0
2·1	·00006841	·00004374	·00002740	·00001682	·00001011	·00000595+	·0^{4}03434	·0^{4}01939	·0^{4}01072	2·1
2·2	·00003888	·00002449	·00001512	·00000914	·00000541	·0^{4}03138	·0^{4}01782	·0^{4}00991	·0^{4}00539	2·2
2·3	·00002164	·00001343	·00000817	·00000486	·0^{4}02835+	·0^{4}01619	·0^{4}00905+	·0^{4}00496	·0^{4}00266	2·3
2·4	·00001180	·00000721	·00000432	·0^{4}02533	·0^{4}01454	·0^{4}00818	·0^{4}00450+	·0^{4}00243	·0^{4}001280	2·4
2·5	·00000630	·00000379	·0^{4}02237	·0^{4}01292	·0^{4}00730	·0^{4}00404	·0^{4}00219	·0^{4}001163	·0^{4}000604	2·5
2·6	·00000330	·0^{4}01954	·0^{4}01135−	·0^{4}00645+	·0^{4}00359	·0^{4}00196	·0^{4}001044	·0^{4}000545+	·0^{4}000279	2·6

0^4 indicates that four zeros must be placed before the figures that follow.

d/N for $r = -\cdot 75$

k	$h = 0{\cdot}9$	$h = 1{\cdot}0$	$h = 1{\cdot}1$	$h = 1{\cdot}2$	$h = 1{\cdot}3$	$h = 1{\cdot}4$	$h = 1{\cdot}5$	$h = 1{\cdot}6$	$h = 1{\cdot}7$	k
0·0	·01273898	·00915625+	·00646129	·00447528	·00304167	·00202812	·00132639	·00085066	·00053489	0·0
0·1	·00962023	·00682714	·00475554	·00325055+	·00217976	·00143371	·00092475-	·00058481	·00036254	0·1
0·2	·00713469	·00490785+	·00343554	·00231690	·00153259	·00099416	·00063230	·00039422	·00024090	0·2
0·3	·00519483	·00359109	·00243650+	·00162018	·00105695+	·00067606	·00042391	·00026052	·00015691	0·3
0·4	·00371236	·00253193	·00169384	·00111127	·00071483	·00045077	·00027861	·00016876	·00010016	0·4
0·5	·00260314	·00175128	·00115543	·00074745+	·00047401	·00029464	·00017948	·00010713	·00006265-	0·5
0·6	·00179065-	·00118805+	·00077289	·00049291	·00030812	·00018876	·00011331	·00006664	·00003839	0·6
0·7	·00120807	·00079032	·00050687	·00031864	·00019631	·00011851	·00007009	·00004061	·00002305-	0·7
0·8	·00079919	·00051544	·00032585-	·00020188	·00012256	·00007290	·00004248	·00002424	·00001355+	0·8
0·9	·00051832	·00032951	·00020530	·00012534	·00007497	·00004393	·00002522	·00001418	·00000780	0·9
1·0	·00032951	·00020645+	·00012675+	·00007625-	·00004493	·00002594	·00001466	·00000812	·0^{4}04401	1·0
1·1	·00020530	·00012675+	·00007668	·00004544	·00002638	·00001500-	·00000835-	·0^{4}04552	·0^{4}02430	1·1
1·2	·00012534	·00007625-	·00004544	·00002653	·00001517	·00000849	·0^{4}04656	·0^{4}02500-	·0^{4}01314	1·2
1·3	·00007497	·00004493	·00002638	·00001517	·00000854	·0^{4}04709	·0^{4}02542	·0^{4}01344	·0^{4}00695+	1·3
1·4	·00004393	·00002594	·00001500-	·00000849	·0^{4}04709	·0^{4}02557	·0^{4}01359	·0^{4}00707	·0^{4}00360	1·4
1·5	·00002522	·00001466	·00000835-	·0^{4}04656	·0^{4}02542	·0^{4}01359	·0^{4}00711	·0^{4}00364	·0^{4}001827	1·5
1·6	·00001418	·00000812	·0^{4}04552	·0^{4}02500	·0^{4}01344	·0^{4}00707	·0^{4}00364	·0^{4}001838	·0^{4}000907	1·6
1·7	·00000780	·0^{4}04401	·0^{4}02430	·0^{4}01314	·0^{4}00695+	·0^{4}00360	·0^{4}001827	·0^{4}000907	·0^{4}000441	1·7
1·8	·0^{4}04207	·0^{4}02336	·0^{4}01270	·0^{4}00676	·0^{4}00352	·0^{4}001797	·0^{4}000896	·0^{4}000438	·0^{4}000210	1·8
1·9	·0^{4}02220	·0^{4}01214	·0^{4}00650-	·0^{4}00340	·0^{4}001747	·0^{4}000877	·0^{4}000431	·0^{4}000207	·0^{4}000098	1·9
2·0	·0^{4}01147	·0^{4}00618	·0^{4}00325+	·0^{4}001679	·0^{4}000848	·0^{4}000419	·0^{4}000202	·0^{4}000096	·0^{4}000044	2·0
2·1	·0^{4}00580	·0^{4}00308	·0^{4}001596	·0^{4}000810	·0^{4}000403	·0^{4}000196	·0^{4}000094	·0^{4}000043	·0^{4}000020	2·1
2·2	·0^{4}00287	·0^{4}001499	·0^{4}000766	·0^{4}000385+	·0^{4}000187	·0^{4}000090	·0^{4}000041	·0^{4}000019	·0^{4}000009	2·2
2·3	·0^{4}001393	·0^{4}000715+	·0^{4}000359	·0^{4}000177	·0^{4}000085+	·0^{4}000040	·0^{4}000019	·0^{4}000008	·0^{4}000004	2·3
2·4	·0^{4}000661	·0^{4}000334	·0^{4}000166	·0^{4}000080	·0^{4}000038	·0^{4}000018	·0^{4}000008	·0^{4}000004	·0^{4}000002	2·4
2·5	·0^{4}000307	0^{4}000153	·0^{4}000074	·0^{4}000035+	·0^{4}000017	·0^{4}000008	·0^{4}000003	·0^{4}000002	·0^{4}000001	2·5
2·6	·0^{4}000139	·0^{4}000068	·0^{4}000033	·0^{4}000015+	·0^{4}000007	·0^{4}000003	·0^{4}000001	·0^{4}000001		2·6

d/N for $r = -\cdot 75$

k	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	k
0·0	·00032970	·00019919	·00011793	·00006841	·00003888	·00002164	·00001180	·00000630	·00000330	0·0
0·1	·00022028	·00013116	·00007653	·00004374	·00002449	·00001343	·00000721	·00000379	·0^{4}01954	0·1
0·2	·00014426	·00008465+	·00004866	·00002740	·00001512	·00000817	·00000432	·0^{4}02237	·0^{4}01135-	0·2
0·3	·00009260	·00005354	·00003032	·00001682	·00000014	·00000486	·0^{4}02533	·0^{4}01292	·0^{4}00645+	0·3
0·4	·00005824	·00003317	·00001851	·00001011	·00000541	·0^{4}02835+	·0^{4}01454	·0^{4}00730	·0^{4}00359	0·4
0·5	·00003589	·00002014	·00001107	·00000595+	·0^{4}03138	·0^{4}01619	·0^{4}00818	·0^{4}00404	·0^{4}00196	0·5
0·6	·00002166	·00001197	·00000648	·0^{4}03434	·0^{4}01782	·0^{4}00905+	·0^{4}00450+	·0^{4}00219	·0^{4}001044	0·6
0·7	·00001281	·00000697	·0^{4}03716	·0^{4}01939	·0^{4}00991	·0^{4}00496	·0^{4}00243	·0^{4}001163	·0^{4}000545+	0·7
0·8	·00000742	·0^{4}03976	·0^{4}02087	·0^{4}01072	·0^{4}00539	·0^{4}00266	·0^{4}001280	·0^{4}000604	·0^{4}000279	0·8
0·9	·0^{4}04207	·0^{4}02220	·0^{4}01147	·0^{4}00580	·0^{4}00287	·0^{4}001393	·0^{4}000661	·0^{4}000307	·0^{4}000139	0·9
1·0	·0^{4}02336	·0^{4}01214	·0^{4}00618	·0^{4}00308	·0^{4}001499	·0^{4}000715+	·0^{4}000334	·0^{4}000153	·0^{4}000068	1·0
1·1	·0^{4}01270	·0^{4}00650-	·000325+	·0^{4}001596	·0^{4}000766	·0^{4}000359	·0^{4}000166	·0^{4}000074	·0^{4}000033	1·1
1·2	·0^{4}00676	·0^{4}00340	·0^{4}001679	·0^{4}000810	·0^{4}000385+	·0^{4}000177	·0^{4}000080	·0^{4}000035+	·0^{4}000015+	1·2
1·3	·0^{4}00352	·0^{4}001747	·0^{4}000848	·0^{4}000403	·0^{4}000187	·0^{4}000085+	·0^{4}000038	·0^{4}000017	·0^{4}000007	1·3
1·4	·0^{4}001797	·0^{4}000877	·0^{4}000419	·0^{4}000196	·0^{4}000090	·0^{4}000040	·0^{4}000018	·0^{4}000008	·0^{4}000003	1·4
1·5	·0^{4}000896	·0^{4}000431	·0^{4}000202	·0^{4}000094	·0^{4}000041	·0^{4}000019	·0^{4}000008	·0^{4}000003	·0^{4}000001	1·5
1·6	·0^{4}000438	·0^{4}000207	·0^{4}000096	·0^{4}000043	·0^{4}000019	·0^{4}000008	·0^{4}000004	·0^{4}000002	·0^{4}000001	1·6
1·7	·0^{4}000210	·0^{4}000098	·0^{4}000044	·0^{4}000020	·0^{4}000009	·0^{4}000004	·0^{4}000002	·0^{4}000001		1·7
1·8	·0^{4}000098	·0^{4}000045-	·0^{4}000020	·0^{4}000009	·0^{4}000004	·0^{4}000002	·0^{4}000001			1·8
1·9	·0^{4}000045-	·0^{4}000020	·0^{4}000009	·0^{4}000004	·0^{4}000002	·0^{4}000001				1·9
2·0	·0^{4}000020	·0^{4}000009	·0^{4}000004	·0^{4}000002	·0^{4}000001					2·0
2·1	·0^{4}000009	·0^{4}000004	·0^{4}000002	·0^{4}000001						2·1
2·2	·0^{4}000004	·0^{4}000002	·0^{4}000001							2·2
2·3	·0^{4}000002	·0^{4}000001								2·3
2·4	·0^{4}000001									2·4
2·5										2·5
2·6										2·6

0^4 indicates that four zeros must be placed before the figures that follow.

A THEORY OF THE SAMPLING DISTRIBUTION OF STANDARD DEVIATIONS.

By T. KONDO.

Section I. Moment Coefficients of the Standard Deviations obtained in Sampling in terms of those of Variance.

(1) *Introduction.*

The standard deviation is one of the most important statistical constants, and with regard to it numerous researches have been made.

Suppose that from an infinite population, in which a character is measured by a variate x, samples of size N are drawn randomly and that this process is repeated indefinitely many times, then the standard deviation σ of the variate x will vary from sample to sample. I propose to consider here the distribution of σ in such cases. Concerning this problem several researches have already been made*, but some of them are only for a particular, not a general, parent distribution, and in others the degree of approximation in the results is not close enough for many purposes. I want here to deduce some general formulae for the sampling distribution of σ to a degree of approximation higher than that already obtained from a new point of view and by a different method of deduction.

Now any distribution law of a variate x can be defined by the moment coefficients for this distribution. If we can find the first four moment coefficients or, in the usual notation, μ_1' about a fixed origin and μ_2, μ_3, μ_4 about the mean, then we have as a rule enough information to define the distribution of frequency with sufficient accuracy for practical purposes.

The deduction of formulae for μ_1' and μ_r $(r=2, 3, 4)$ of σ in sampling is the primary object of this paper.

(A) *The first Method of Deduction.*

(2) Let $\phi(x)$ be the probability function of a continuous variate x, λ_s the sth semi-invariant of the distribution of x, and μ_r' the rth moment coefficient about a fixed origin; then the λ's are defined by the identity† with respect to ω

$$e^{\sum\limits_{r=1}^{\infty}\frac{\lambda_r\omega^r}{r!}}=\int_{-\infty}^{+\infty}\phi(x)\,e^{x\omega}\,dx \quad\text{.........................(1).}$$

* See "Student," *Biometrika*, Vol. vi. (March, 1908); K. Pearson, *Ibid.* Vol. xii. p. 277 (Nov. 1918); C. C. Craig, *Metron*, Vol. vii. No. 4 (Dec. 1928).

† Originally due to Thiele.

Expanding the right-hand side, we have

$$\int_{-\infty}^{+\infty} \phi(x) e^{x\omega} dx = \sum_{i=0}^{\infty} \frac{\omega^i}{i!} \int_{-\infty}^{+\infty} \phi(x) x^i dx$$
$$= \sum_{i=0}^{\infty} \frac{\mu_i'}{i!} \omega^i.$$

Equating the coefficient of the same powers of ω, we get the following well-known equations between the λ's and μ''s,

$$\mu_1' = \lambda_1, \quad \mu_2' = \lambda_2 + \lambda_1^2,$$
$$\mu_3' = \lambda_3 + 3\lambda_2\lambda_1 + \lambda_1^3,$$
$$\mu_4' = \lambda_4 + 4\lambda_3\lambda_1 + 3\lambda_2^2 + 6\lambda_2\lambda_1^2 + \lambda_1^4 \quad \text{..................(2)},$$

and so on.

Now let us choose the origin at the mean of x; then since $\mu_1' = \lambda_1 = 0$, we have

$$\mu_2 = \lambda_2, \quad \mu_3 = \lambda_3, \quad \mu_4 = \lambda_4 + 3\lambda_2^2,$$
$$\mu_5 = \lambda_5 + 10\lambda_3\lambda_2, \quad \mu_6 = \lambda_6 + 15\lambda_4\lambda_2 + 10\lambda_3^2 + 15\lambda_2^3,$$
$$\mu_7 = \lambda_7 + 21\lambda_5\lambda_2 + 35\lambda_4\lambda_3 + 105\lambda_3\lambda_2^2,$$
$$\mu_8 = \lambda_8 + 28\lambda_6\lambda_2 + 56\lambda_5\lambda_3 + 35\lambda_4^2 + 210\lambda_4\lambda_2^2 + 280\lambda_3^2\lambda_2 + 105\lambda_2^4,$$
$$\mu_9 = \lambda_9 + 36\lambda_7\lambda_2 + 84\lambda_6\lambda_3 + 126\lambda_5\lambda_4 + 378\lambda_5\lambda_2^2 + 1260\lambda_4\lambda_3\lambda_2 + 280\lambda_3^3 + 1260\lambda_3\lambda_2^3 \quad \text{......(3)},$$

and so on.

(3) Let us consider the case where x is the standard deviation σ; then

$$e^{\sum_{r=1}^{\infty} \frac{\lambda_r \omega^r}{r!}} = \int_{-\infty}^{+\infty} \phi(\sigma) e^{\sigma\omega} d\sigma \quad \text{..........................(4)}.$$

But the frequency of σ is the same as that of the variance μ_2. Therefore, if $\bar{\mu}_2$ be the mean of μ_2 in samples, y the deviation of μ_2 from $\bar{\mu}_2$, and $\Phi(y)$ the frequency function of μ_2, then since

$$\phi(\sigma) d\sigma = \Phi(y) dy,$$

we have
$$e^{\sum_{r=1}^{\infty} \frac{\lambda_r \omega^r}{r!}} = \int_{-\infty}^{+\infty} \Phi(y) e^{\omega\sqrt{\bar{\mu}_2 + y}} dy \quad \text{......................(5)}.$$

And
$$\int_{-\infty}^{+\infty} \Phi(y) e^{\omega\sqrt{\bar{\mu}_2 + y}} dy = \int_{-\infty}^{+\infty} \Phi(y) e^{\omega\sqrt{\bar{\mu}_2}\left(1 + \frac{y}{\bar{\mu}_2}\right)^{\frac{1}{2}}} dy$$
$$= \int_{-\infty}^{+\infty} \Phi(y) \left(\sum_{i=0}^{\infty} \frac{\omega^i (\bar{\mu}_2)^{i/2} \left(1 + \frac{y}{\bar{\mu}_2}\right)^{i/2}}{i!} \right) dy$$
$$= \sum_{i=0}^{\infty} \frac{\omega^i (\bar{\mu}_2)^{i/2}}{i!} \int_{-\infty}^{+\infty} \Phi(y) \left(1 + \frac{y}{\bar{\mu}_2}\right)^{i/2} dy.$$

Therefore if we write
$$a_i = \int_{-\infty}^{+\infty} \Phi(y) \left(1 + \frac{y}{\bar{\mu}_2}\right)^{i/2} dy \quad \text{..........................(6)},$$

then
$$e^{\sum_{r=1}^{\infty} \frac{\lambda_r \omega^r}{r!}} = \sum_{i=0}^{\infty} \frac{a_i (\bar{\mu}_2)^{i/2} \omega^i}{i!} \quad \text{..........................(7)}.$$

Differentiating the identity (7) with respect to ω and equating the coefficients of the same powers of ω, we have

$$\begin{aligned}
&a_1(\bar{\mu}_2)^{\frac{1}{2}} = \lambda_1, \quad a_2\bar{\mu}_2 = \lambda_2 + a_1\lambda_1(\bar{\mu}_2)^{\frac{1}{2}},\\
&a_3(\bar{\mu}_2)^{\frac{3}{2}} = \lambda_3 + 2a_1\lambda_2(\bar{\mu}_2)^{\frac{1}{2}} + a_2\lambda_1\bar{\mu}_2,\\
&a_4(\bar{\mu}_2)^{2} = \lambda_4 + 3a_1\lambda_3(\bar{\mu}_2)^{\frac{1}{2}} + 3a_2\lambda_2\bar{\mu}_2 + a_3\lambda_1(\bar{\mu}_2)^{\frac{3}{2}},\\
&\vdots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8).
\end{aligned}$$

We shall assume that it is justifiable to insert into the integral (6) the expansion of the binomial*. Then

$$\left(1+\frac{y}{\bar{\mu}_2}\right)^{i/2} = 1 + c_{i,1}\left(\frac{y}{\bar{\mu}_2}\right) + c_{i,2}\left(\frac{y}{\bar{\mu}_2}\right)^2 + \ldots + c_{i,r}\left(\frac{y}{\bar{\mu}_2}\right)^r + \ldots,$$

where $c_{i,r}$ is the coefficient of the $(r+1)$th term of $(1+x)^{i/2}$, or numerically

$$\begin{aligned}
&c_{1,2} = \frac{1}{8}, \quad c_{1,3} = \frac{1}{16}, \quad c_{1,4} = -\frac{5}{128}, \quad c_{1,5} = \frac{7}{256},\\
&c_{1,6} = -\frac{21}{1024}, \quad c_{1,7} = \frac{33}{2048}, \quad c_{1,8} = -\frac{429}{32768}, \ \ldots;\\
&c_{2,2} = c_{2,3} = \ldots = 0;\\
&c_{3,2} = \frac{3}{8}, \quad c_{3,3} = -\frac{1}{16}, \quad c_{3,4} = \frac{3}{128}, \quad c_{3,5} = -\frac{3}{256},\\
&c_{3,6} = \frac{7}{1024}, \quad c_{3,7} = -\frac{9}{2048}, \quad c_{3,8} = \frac{99}{32768}, \ \ldots;\\
&c_{4,2} = 1, \quad c_{4,3} = c_{4,4} = \ldots = 0; \quad \text{and so on} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9).
\end{aligned}$$

Now let ${}_2M_p$ be the pth moment coefficient about the mean for the distribution of μ_2 due to random sampling, and let us write

$$m_p = \frac{{}_2M_p}{2^p\bar{\mu}_2^{\,p}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10);$$

then

$$\begin{aligned}
a_i &= \int_{-\infty}^{+\infty} \Phi(y)\left(1+\frac{y}{\bar{\mu}_2}\right)^{i/2} dy\\
&= 1 + \sum_{r=1}^{\infty} \frac{c_{i,r}}{(\bar{\mu}_2)^r}\int_{-\infty}^{+\infty}\Phi(y)\, y^r\, dy\\
&= 1 + \sum_{r=1}^{\infty} \frac{c_{i,r}}{(\bar{\mu}_2)^r}\,{}_2M_r\\
&= 1 + \sum_{r=1}^{\infty} 2^r c_{i,r} m_r.
\end{aligned}$$

But ${}_2M_1 = 0$, consequently $m_1 = 0$. Therefore

$$\begin{aligned}
a_i &= 1 + \sum_{r=2}^{\infty} 2^r c_{i,r} m_r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)\\
&= 1 + 4c_{i,2}m_2 + 8c_{i,3}m_3 + 16c_{i,4}m_4 + \ldots \quad (i = 1, 2, 3, 4, \ldots).
\end{aligned}$$

* This condition is discussed later, see Art. (16).

If we eliminate the λ's from the equations (2) and (8), we get

$$\begin{aligned}
\mu_1' &= a_1\sqrt{\bar{\mu}_2},\\
\mu_2' &= a_2\bar{\mu}_2,\\
\mu_3' &= a_3(\bar{\mu}_2)^{\frac{3}{2}},\\
\mu_4' &= a_4(\bar{\mu}_2)^2 \quad \text{.................................}(12),
\end{aligned}$$

where the μ''s are the first four moment coefficients of σ about $\sigma=0$ in sampling.

Now, for simplicity, let us put

$$a_1=\alpha,\quad a_3=\alpha+\alpha';$$

then from the equation (11)

$$\alpha=1-\frac{m_2}{2}+\frac{m_3}{2}-\frac{5}{8}m_4+\frac{7}{8}m_5-\frac{21}{16}m_6+\frac{33}{16}m_7-\frac{429}{128}m_8+\ldots,$$

$$\alpha'=2m_2-m_3+m_4-\frac{5}{4}m_5+\frac{7}{4}m_6-\frac{21}{8}m_7+\frac{33}{8}m_8-\ldots \quad \text{..........}(13),$$

and since $a_2=1$, $a_4=1+4m_2$, from the equation (11), the equations (12) become

$$\mu_1'=\alpha\sqrt{\bar{\mu}_2},\quad \mu_2'=\bar{\mu}_2,$$

$$\mu_3'=(\alpha+\alpha')(\bar{\mu}_2)^{\frac{3}{2}},$$

and

$$\mu_4'=(1+4m_2)(\bar{\mu}_2)^2 \quad \text{...............................}(14).$$

But if $\tilde{\sigma}$ is the standard deviation of the parent distribution and N is the size of repeated samples, it is well known that

$$\bar{\mu}_2=\text{Mean }\mu_2=\frac{N-1}{N}\tilde{\sigma}^2.$$

Therefore the above equations for μ''s become

$$\mu_1'=\sqrt{\frac{N-1}{N}}\alpha\tilde{\sigma},\quad \mu_2'=\frac{N-1}{N}\tilde{\sigma}^2,$$

$$\mu_3'=\left(\frac{N-1}{N}\right)^{\frac{3}{2}}(\alpha+\alpha')\tilde{\sigma}^3,$$

and

$$\mu_4'=\left(\frac{N-1}{N}\right)^2(1+4m_2)\tilde{\sigma}^4 \quad \text{.............................}(14').$$

Now the mean of σ and the first three moment coefficients $\mu_r(\sigma)$ about the mean can easily be deduced from the equations (14) or (14′) by the well-known equations connecting μ_r''s or μ_r's, and the following equations are the results obtained*

$$\text{Mean }\sigma=\bar{\sigma}=\alpha\sqrt{\bar{\mu}_2}=\sqrt{\frac{N-1}{N}}\alpha\tilde{\sigma} \quad \text{..............................}(15\,a),$$

$$\mu_2(\sigma)=(1-\alpha^2)\bar{\mu}_2=\left(\frac{N-1}{N}\right)(1-\alpha^2)\tilde{\sigma}^2 \quad \text{.......................}(15\,b),$$

* Cf. Craig, *loc. cit.*

$$\mu_3(\sigma) = [\alpha' - 2\alpha(1-\alpha^2)]\sqrt{\bar{\mu}_2}^3$$
$$= \left(\frac{N-1}{N}\right)^{\frac{3}{2}}[\alpha' - 2\alpha(1-\alpha^2)]\,\tilde{\sigma}^3 \quad \text{.........................(15 c)},$$
$$\mu_4(\sigma) = [4(m_2 - \alpha\alpha') + (1-\alpha^2)(1+3\alpha^2)]\,\bar{\mu}_2^2$$
$$= \left(\frac{N-1}{N}\right)^2[4(m_2 - \alpha\alpha') + (1-\alpha^2)(1+3\alpha^2)]\,\tilde{\sigma}^4 \quad \text{.........(15 d)},$$

and consequently

$$\sigma_\sigma = \text{S.D. of } \sigma$$
$$= \sqrt{(1-\alpha^2)\,\bar{\mu}_2} = \sqrt{(1-\alpha^2)\frac{N-1}{N}}\,\tilde{\sigma} \quad \text{..................(16 a)},$$
$$\beta_1(\sigma) = \mu_3(\sigma)^2/\mu_2(\sigma)^3 = \frac{[\alpha' - 2\alpha(1-\alpha^2)]^2}{(1-\alpha^2)^3} \quad \text{..................(16 b)},$$
$$\beta_2(\sigma) = \mu_4(\sigma)/\mu_2(\sigma)^2 = \frac{1+3\alpha^2}{1-\alpha^2} + 4\,\frac{m_2 - \alpha\alpha'}{(1-\alpha^2)^2} \quad \text{........ ......(16 c)},$$

where $\bar{\mu}_2$ and ${}_2M_r$ are the mean of the variance μ_2 and the rth moment coefficient of μ_2 in sampling, further:

$$m_r = {}_2M_r/(2^r\bar{\mu}_2{}^r)$$

while α, α' are given by the equations (13).

The equations (15) and (16) give respectively the first four moment coefficients for the sampling distribution of σ, its standard deviation, also the β_1, and β_2 of σ in terms of m_2, α, α' and $\bar{\mu}_2$, or in terms of ${}_2M_r$. They will all be exact expressions provided that the expansion of $\left(1+\frac{y}{\bar{\mu}_2}\right)^{i/2}$ within the integral of (6) is justified, a condition which is discussed later*.

(B) *The second Method of Deduction.*

(4) We can deduce the equation (15) and consequently (16) also from another point of view.

Since
$$\sigma = \sqrt{\mu_2} = \sqrt{\bar{\mu}_2 + \delta\mu_2},$$
where $\delta\mu_2$ is the sampling deviation of μ_2 from its mean, if $\delta\sigma$ is the deviation of σ from $\sqrt{\bar{\mu}_2}$, not from the mean, then

$$\sigma = \sqrt{\bar{\mu}_2} + \delta\sigma = (\bar{\mu}_2)^{\frac{1}{2}}\sqrt{1 + \delta\mu_2/\bar{\mu}_2}\,;$$

therefore
$$\delta\sigma = \sqrt{\bar{\mu}_2}\,\{(1+\Delta)^{\frac{1}{2}} - 1\} \quad \text{.....................(17)},$$
where Δ stands for $\delta\mu_2/\bar{\mu}_2$.

Assuming $\left|\frac{\delta\mu_2}{\bar{\mu}_2}\right| < 1$ as before*, we have

$$\delta\sigma = \sqrt{\bar{\mu}_2}\,(a_1\Delta + a_2\Delta^2 + a_3\Delta^3 + \ldots),$$

where $a_r = c_{1,r}$ $(r = 1, 2, 3, \ldots)$ and is given by the equation (9) in Art. (3).

* See Art. (16).

Now let ν_r' be the rth moment coefficient of σ about $\sigma = \sqrt{\bar{\mu}_2}$, and let us use brackets [ ] for "mean in repeated samples"; then

$$\nu_1' = \text{Mean } \delta\sigma = \sqrt{\bar{\mu}_2}\left(\sum_{r=1}^{\infty} c_{1,r}\,[\Delta^r]\right)$$

$$= \sqrt{\bar{\mu}_2}\left(\sum_{r=1}^{\infty} \frac{c_{1,r}}{\bar{\mu}_2^{\,r}}\,[(\delta\mu_2)^r]\right)$$

$$= \sqrt{\bar{\mu}_2}\left(\sum_{r=2}^{\infty} 2^r c_{1,r}\,\frac{{}_2M_r}{2^r \bar{\mu}_2^{\,r}}\right)$$

$$= \sqrt{\bar{\mu}_2}\left(\sum_{r=2}^{\infty} 2^r c_{1,r} m_r\right),$$

therefore $$\nu_1' = \sqrt{\bar{\mu}_2}\,(\alpha - 1) \quad \text{.......................................(18 a).}$$

Also from the equation (17), if we find $(\delta\sigma)^r$ $(r = 2, 3, 4)$, and write as follows:

$$(\delta\sigma)^2 = \bar{\mu}_2\,\{b_2\Delta^2 + b_3\Delta^3 + b_4\Delta^4 + \ldots\},$$

$$(\delta\sigma)^3 = (\bar{\mu}_2)^{\frac{3}{2}}\,\{c_3\Delta^3 + c_4\Delta^4 + c_5\Delta^5 + \ldots\},$$

$$(\delta\sigma)^4 = (\bar{\mu}_2)^2\,\{d_4\Delta^4 + d_5\Delta^5 + d_6\Delta^6 + \ldots\},$$

then, after calculation, we get the following values of the coefficients:

$$b_2 = \frac{1}{4},\quad b_3 = -\frac{1}{8},\quad b_4 = \frac{5}{64},\quad b_5 = -\frac{7}{128},$$

$$b_6 = \frac{21}{512},\quad b_7 = -\frac{33}{1024},\quad b_8 = \frac{429}{16384},\quad \ldots;$$

$$c_3 = \frac{1}{8},\quad c_4 = -\frac{3}{32},\quad c_5 = \frac{9}{128},\quad c_6 = -\frac{7}{128},\quad c_7 = \frac{45}{1024},\quad c_8 = -\frac{297}{8192},\quad \ldots;$$

$$d_4 = \frac{1}{16},\quad d_5 = -\frac{1}{16},\quad d_6 = \frac{7}{128},\quad d_7 = -\frac{3}{64},\quad d_8 = \frac{165}{4096},\quad \ldots.$$

Now if we find the mean values of $(\delta\sigma)^2$, $(\delta\sigma)^3$ and $(\delta\sigma)^4$, we get

$$\nu_2' = \text{Mean } (\delta\sigma)^2 = \bar{\mu}_2 \sum_{r=2}^{\infty} (b_r\,[\Delta^r])$$

$$= \bar{\mu}_2\left(\sum_{r=2}^{\infty} 2^r b_r m_r\right) \quad \text{.......................................(18 b).}$$

Similarly $$\nu_3' = \text{Mean } (\delta\sigma)^3 = (\bar{\mu}_2)^{\frac{3}{2}}\left(\sum_{r=3}^{\infty} 2^r c_r m_r\right) \quad \text{..................(18 c).}$$

$$\nu_4' = \text{Mean } (\delta\sigma)^4 = (\bar{\mu}_2)^2\left(\sum_{r=4}^{\infty} 2^r d_r m_r\right) \quad \text{..................(18 d).}$$

But

$$\nu_2' = \bar{\mu}_2\left\{m_2 - m_3 + \frac{5}{4}m_4 - \frac{7}{4}m_5 + \frac{21}{8}m_6 - \frac{33}{8}m_7 + \frac{429}{64}m_8 - \ldots\right\}$$

$$= \bar{\mu}_2\left\{2 - 2\left(1 - \frac{m_2}{2} + \frac{m_3}{2} - \frac{5}{8}m_4 + \frac{7}{8}m_5 - \frac{21}{16}m_6 + \frac{33}{16}m_7 - \frac{429}{128}m_8 + \ldots\right)\right\}$$

$$= \bar{\mu}_2\,(2 - 2\alpha) \quad \text{...(18 b').}$$

Similarly we can transform the equations (18 c) and (18 d) into the following forms:

$$\nu_3' = (\bar{\mu}_2)^{\frac{3}{2}} \{\alpha' + 4(\alpha - 1)\} \qquad \text{.........(18 } c'),$$

$$\nu_4' = (\bar{\mu}_2)^2 \{4m_2 + 8(1 - \alpha) - 4\alpha'\} \qquad \text{.........(18 } d').$$

Hence we can find the mean σ and the first three moment coefficients about the mean in the sampling distribution of σ.

In fact from the equations (18 a), (18 b'), (18 c'), (18 d') and well-known formulae connecting ν''s and μ's, we get

$$\text{Mean } \sigma = (\bar{\mu}_2)^{\frac{1}{2}} + \nu_1' = \alpha\sqrt{\bar{\mu}_2},$$

$$\mu_2(\sigma) = \nu_2' - \nu_1'^2 = \bar{\mu}_2(1 - \alpha^2);$$

similarly

$$\mu_3(\sigma) = (\sqrt{\bar{\mu}_2})^{\frac{3}{2}} \{\alpha' + 4(\alpha - 1) - 3(\alpha - 1)(2 - 2\alpha) + 2(\alpha - 1)^3\}$$
$$= (\sqrt{\bar{\mu}_2})^{\frac{3}{2}} \{\alpha' - 2\alpha(1 - \alpha^2)\},$$

and

$$\mu_4(\sigma) = (\bar{\mu}_2)^2 \{4m_2 + 8(1 - \alpha) - 4\alpha' - 4(\alpha - 1)[\alpha' - 2\alpha(1 - \alpha^2)] - 6(\alpha - 1)^2(1 - \alpha^2) - (\alpha - 1)^4\}$$
$$= (\bar{\mu}_2)^2 \{4(m_2 - \alpha\alpha') + (1 - \alpha^2)(1 + 3\alpha^2)\}.$$

These equations are the same as (15 a), (15 b), (15 c) and (15 d) respectively, already obtained in Art. (3).

Section II. Moment Coefficients of σ in terms of the Constants of the Sampled Population.

(5) Now if repeated random samples of size N be drawn from an infinite population which is specified by the standard deviation $\tilde{\sigma}$, and the constants $\tilde{\beta}_1, \tilde{\beta}_2, \ldots \tilde{\beta}_r, \ldots$, where

$$\beta_{2r-2} = \frac{\mu_{2r}}{(\mu_2)^r}, \quad \beta_{2r-1} = \frac{\mu_{2r+1}\mu_3}{(\mu_2)^{r+2}},$$

then it is well known* that the first four moment coefficients of the sampling distribution of the variance are given by

$${}_2M_1' = \text{Mean } \mu_2 = \frac{N-1}{N}\tilde{\sigma}^2,$$

$${}_2M_2 = \frac{(N-1)^2}{N^3}\left(\tilde{\beta}_2 - 3 + \frac{2N}{N-1}\right)\tilde{\sigma}^4,$$

$${}_2M_3 = \frac{1}{N^2}\left(k_{32} - \frac{k_{33}}{N} + \frac{k_{34}}{N^2} - \frac{k_{35}}{N^3}\right)\tilde{\sigma}^6,$$

$${}_2M_4 = \frac{1}{N^2}\left(k_{42} + \frac{k_{43}}{N} - \frac{k_{44}}{N^2} + \frac{k_{45}}{N^3} - \frac{k_{46}}{N^4} + \frac{k_{47}}{N^5}\right)\tilde{\sigma}^8 \qquad \text{............(19)},$$

* See A. A. Tchouproff, *Biometrika*, Vol. XII, pp. 193—4; A. E. R. Church, *Ibid.* Vol. XVII, pp. 79—83.

where $k_{32}=\tilde{\beta}_4-3\tilde{\beta}_2-6\tilde{\beta}_1+2,\qquad k_{33}=3\tilde{\beta}_4-21\tilde{\beta}_2-18\tilde{\beta}_1+26$

$$k_{34}=3\tilde{\beta}_4-33\tilde{\beta}_2-22\tilde{\beta}_1+54,\quad k_{35}=\tilde{\beta}_4-15\tilde{\beta}_2-10\tilde{\beta}_1+30;$$

$$k_{42}=3(\tilde{\beta}_2-1)^2,$$

$$k_{43}=\tilde{\beta}_6-4\tilde{\beta}_4-24\tilde{\beta}_3-15\tilde{\beta}_2^2+48\tilde{\beta}_2+96\tilde{\beta}_1-30,$$

$$k_{44}=4\tilde{\beta}_6-40\tilde{\beta}_4-96\tilde{\beta}_3-54\tilde{\beta}_2^2+336\tilde{\beta}_2+528\tilde{\beta}_1-306,$$

$$k_{45}=6\tilde{\beta}_6-96\tilde{\beta}_4-176\tilde{\beta}_3-102\tilde{\beta}_2^2+924\tilde{\beta}_2+1232\tilde{\beta}_1-1044,$$

$$k_{46}=4\tilde{\beta}_6-88\tilde{\beta}_4-160\tilde{\beta}_3-95\tilde{\beta}_2^2+1050\tilde{\beta}_2+1360\tilde{\beta}_1-1395,$$

$$k_{47}=\tilde{\beta}_6-28\tilde{\beta}_4-56\tilde{\beta}_3-35\tilde{\beta}_2^2+420\tilde{\beta}_2+560\tilde{\beta}_1-630.$$

Therefore if the approximation is good enough up to ${}_2M_4$, we can express μ_1' and μ_2, μ_3 and μ_4 at once in terms of $\tilde{\sigma}$ and the $\tilde{\beta}$'s, i.e. in terms of the constants of the parent distribution.

Suppose further that we neglect terms of higher order than N^{-2}, then

$$\alpha=1-\frac{m_2}{2}+\frac{m_3}{2}-\frac{5}{8}m_4,\quad \alpha'=2m_2-m_3+m_4,$$

where

$$m_2=\frac{{}_2M_2}{4\bar{\mu}_2^2}=\frac{1}{4N}\left\{\tilde{\beta}_2-3+\frac{2N}{N-1}\right\}$$

$$=\frac{1}{2N}\left(\frac{\tilde{\beta}_2-1}{2}+\frac{1}{2N}\right),$$

$$m_3=\frac{1}{8N^2}\{\tilde{\beta}_4-3\tilde{\beta}_2-6\tilde{\beta}_1+2\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20),$$

$$m_4=\frac{3(\tilde{\beta}_2-1)^2}{16N^2}\text{ and }\bar{\mu}_2=\frac{N-1}{N}\tilde{\sigma}^2.$$

Substituting these values of α, α', $\bar{\mu}_2$ and the m's into the general and fundamental equations (15) and (16), after transformation and simplification, we get

$$\text{Mean }\sigma=\tilde{\sigma}\left\{1-\frac{\tilde{\beta}_2+3}{8N}+\frac{8\tilde{\beta}_4-15\tilde{\beta}_2^2+14\tilde{\beta}_2-48\tilde{\beta}_1-55}{128N^2}\right\}\ldots\ldots(21\,a),$$

$$\mu_2(\sigma)=\frac{\tilde{\sigma}^2}{4N}\left\{\tilde{\beta}_2-1-\frac{4\tilde{\beta}_4-7\tilde{\beta}_2^2+10\tilde{\beta}_2-24\tilde{\beta}_1-23}{8N}\right\}\ \ldots\ldots\ldots(21\,b),$$

$$\mu_3(\sigma)=\frac{\tilde{\sigma}^3}{16N^2}\{2\tilde{\beta}_4-3\tilde{\beta}_2^2-12\tilde{\beta}_1+1\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(21\,c),$$

and

$$\mu_4(\sigma)=\frac{3}{16N^2}(\tilde{\beta}_2-1)^2\tilde{\sigma}^4\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(21\,d);$$

consequently

$$\sigma_\sigma=\frac{1}{2}\sqrt{\frac{\tilde{\beta}_2-1}{N}}\left\{1-\frac{4\tilde{\beta}_4-7\tilde{\beta}_2^2+10\tilde{\beta}_2-24\tilde{\beta}_1-23}{16(\tilde{\beta}_2-1)N}\right\}\tilde{\sigma}\ \ldots\ldots(21\,e).$$

In the cases where the parent distribution is normal; since

$$\tilde{\beta}_1 = 0, \quad \tilde{\beta}_2 = 3 \quad \text{and} \quad \tilde{\beta}_4 = 15,$$

we have

$$\text{Mean } \sigma = \left(1 - \frac{3}{4N} - \frac{7}{32N^2}\right)\tilde{\sigma},$$

$$\mu_2(\sigma) = \frac{1}{2N}\left(1 - \frac{1}{4N}\right)\tilde{\sigma}^2,$$

$$\mu_3(\sigma) = \frac{1}{4N^2}\tilde{\sigma}^3, \quad \mu_4(\sigma) = \frac{3}{4N^2}\tilde{\sigma}^4 \quad \text{..............(22 a)},$$

and

$$\sigma_\sigma = \frac{1}{\sqrt{2N}}\left(1 - \frac{1}{8N}\right)\tilde{\sigma},$$

$$\beta_1(\sigma) = \frac{1}{2N}, \quad \beta_2(\sigma) = 3\left\{1 + \frac{1}{2N}\right\} \quad \text{..................(22 b)}.$$

Some of the equations (21) and (22) have been already deduced by Prof. K. Pearson*.

Now we must notice that the above approximate formulae have been obtained by neglecting terms of the general equations (15) and (16) in two different ways. Firstly we neglected the moment coefficients ${}_2M_r$ for $r > 4$, and secondly we neglected those terms of order higher than N^{-2}.

But such a double method of approximation cannot be carried through correctly unless we know the order of ${}_2M_r$.

(6) Now if λ_r be the semi-invariant for the sampling distribution of the variance μ_2, and N the size of the repeated random samples, in the semi-invariant theory, it is known that (i) λ_r $(r = 2, 3, 4, \ldots)$ is independent of the origin, and (ii) λ_r $(r = 2, 3, 4, \ldots)$ is of order $r-1$ in N^{-1}, and from the general equations (3), we have

$$ {}_2M_2 = \lambda_2, \qquad {}_2M_3 = \lambda_3,$$

$$ {}_2M_4 = \lambda_4 + 3\lambda_2^2, \quad {}_2M_5 = \lambda_5 + 10\lambda_3\lambda_2,$$

$$ {}_2M_6 = \lambda_6 + 15\lambda_4\lambda_2 + 10\lambda_3^2 + 15\lambda_2^3, \text{ and so on } \quad \text{.........(23)}.$$

Thus we can find the order of the coefficient ${}_2M_r$.

For instance, if we can assume that the approximation is good enough only up to the order N^{-2}, since ${}_2M_r$ $(r \geqq 5)$ is of order N^{-3} or higher, we can neglect these ${}_2M_r$ and at the same time we can neglect those terms of m_r $(r = 2, 3, 4)$ of order higher than N^{-2}.

This is the case treated in the foregoing article as an example.

Secondly, if we can assume that the approximation is good enough when we include terms up to the order N^{-3} only, we may neglect ${}_2M_r$ $(r = 7, 8, 9, 10, \ldots)$, and at the same time those terms of m_r $(r \leqslant 6)$ of order higher than N^{-3}.

* See *Biometrika*, Vol. XII, p. 277 (Nov. 1918). See also Craig, *loc. cit.*

This is the case treated by Dr C. C. Craig. His results are given as the semi-invariants of the sampling distribution of σ in terms of those of the parent distribution, but his final results correspond to those which will be obtained by neglecting the highest order terms in N^{-1} of my general approximate formulae, given in Art. (9).

The formulae, obtained in these two cases, give us good estimates of the mean σ, $\mu_2(\sigma)$ and σ_σ if N be not very small, and also of $\mu_3(\sigma)$ and $\beta_1(\sigma)$ if N be large, but we cannot get good estimates of $\mu_4(\sigma)$ and $\beta_2(\sigma)$ unless N be very large.

Thirdly, if it be necessary to proceed to a further approximation and include terms up to the order N^{-4}, we can neglect ${}_2M_r$ $(r \geqq 9)$, for they are of order N^{-5} or higher, and at the same time we can neglect those terms of m_r $(r=2, 3, \ldots, 8)$ of order N^{-5} or higher. Such an approximation is necessary for the calculation of $\beta_2(\sigma)$, and with such approximations I shall now deal.

In this case we have

$$\begin{aligned}
&{}_2M_2=\lambda_2, \quad {}_2M_3=\lambda_3, \quad {}_2M_4=\lambda_4+3\lambda_2^2,\\
&{}_2M_5=\lambda_5+10\lambda_3\lambda_2,\\
&{}_2M_6=15\lambda_4\lambda_2+10\lambda_3^2+15\lambda_2^3 \text{ (approximately)},\\
&{}_2M_7=105\lambda_2^2\lambda_3 \qquad (\quad " \quad),\\
\text{and}\quad &{}_2M_8=105\lambda_2^4 \qquad (\quad " \quad) \quad \ldots\ldots\ldots\ldots(24).
\end{aligned}$$

And also
$$\alpha=1-\frac{m_2}{2}+\frac{m_3}{2}-\frac{5}{8}m_4+\frac{7}{8}m_5-\frac{21}{16}m_6+\frac{33}{16}m_7-\frac{429}{128}m_8 \quad \ldots\ldots(25\,a),$$

$$\alpha'=2m_2-m_3+m_4-\frac{5}{4}m_5+\frac{7}{4}m_6-\frac{21}{8}m_7+\frac{33}{8}m_8 \quad \ldots\ldots\ldots\ldots\ldots(25\,b).$$

Now the equation (24) suggests that to get this degree of approximation we have at first to find ${}_2M_6$ or the corresponding semi-invariant λ_6.

The values of ${}_2M_6$ or λ_6 for the sampling distribution of μ_2 were not known until recently, but they have now been found by R. A. Fisher and published with many other valuable results in an important paper*.

(7) R. A. Fisher's "cumulative moment function" κ_r is the same as the usual semi-invariant λ_r, introduced by Thiele, but his "k-function" is a new function defined as follows:

$$k_1=\frac{1}{N}s_1=\text{Mean } x,$$

$$k_2=\frac{N}{N-1}\mu_2=\frac{1}{N-1}\left(s_2-\frac{s_1^2}{N}\right),$$

$$\begin{aligned}
k_3&=\frac{N^2}{(N-1)(N-2)}\mu_3\\
&=\frac{N^2}{(N-1)(N-2)}\left(s_3-\frac{3s_2s_1}{N}+\frac{2s_1^3}{N^2}\right),
\end{aligned}$$

and so on,

* R. A. Fisher, *Proceedings of the London Mathematical Society*, Ser. 2, Vol. xxx, Part 3 (Dec. 1929).

where $s_r = S(x^r)$ is the power sum of order r for any variate x, and

$$\mu_r = \frac{1}{N} S(x - \text{Mean } x)^r$$

as usual.

Fisher uses $\kappa(2^5)$ to denote the fifth cumulative moment function of k_2, due to sampling, and his expression for $\kappa(2^5)$ is as follows:

$$\begin{aligned}\kappa(2^5) = {} & \frac{\kappa_{10}}{N^4} + \frac{40\kappa_8\kappa_2}{N^3(N-1)} + \frac{80(N-2)\kappa_7\kappa_3}{N^3(N-1)^2} + \frac{40(5N^2-12N+9)\kappa_6\kappa_4}{N^3(N-1)^3} \\ & + \frac{16(N-2)(6N^2-12N+17)\kappa_5^2}{N^3(N-1)^4} + \frac{480\kappa_6\kappa_2^2}{N^2(N-1)^2} + \frac{1280(N-2)\kappa_5\kappa_3\kappa_2}{N^2(N-1)^3} \\ & + \frac{320(4N^2-9N+6)\kappa_4^2\kappa_2}{N^2(N-1)^4} + \frac{480(2N^2-7N+6)\kappa_4\kappa_3^2}{N^2(N-1)^4} + \frac{1920\kappa_4\kappa_2^3}{N(N-1)^3} \\ & + \frac{1920(N-2)\kappa_3^2\kappa_2^2}{N(N-1)^4} + \frac{384\kappa_2^5}{(N-1)^4},\end{aligned}$$

where κ_s is the sth cumulative function of the parent distribution. In this equation if we neglect terms of order N^{-5} or higher, we have

$$\begin{aligned}\kappa(2^5) &= \frac{1}{N^4}\{\kappa_{10} + 40\kappa_8\kappa_2 + 80\kappa_7\kappa_3 + 200\kappa_6\kappa_4 + 96\kappa_5^2 + 480\kappa_6\kappa_2^2 + 1280\kappa_5\kappa_3\kappa_2 \\ &\qquad + 1280\kappa_4^2\kappa_2 + 960\kappa_4\kappa_3^2 + 1920\kappa_4\kappa_2^3 + 1920\kappa_3^2\kappa_2^2 + 384\kappa_2^5\} \\ &= \frac{4}{N^4}\tau_0\tilde{\sigma}^{10}, \text{ say} \qquad \text{(25 c)}.\end{aligned}$$

If we transform the above equation into terms of the $\tilde{\beta}$'s of the parent distribution, we get

$$\begin{aligned}4\tau_0 = {} & \tilde{\beta}_8 - 5\tilde{\beta}_6 - 40\tilde{\beta}_5 - 10\tilde{\beta}_4(\tilde{\beta}_2 - 2) - 30\tilde{\beta}_3(\tilde{\beta}_3/\tilde{\beta}_1 - 16) \\ & + 30\tilde{\beta}_2(\tilde{\beta}_2 + 12\tilde{\beta}_1 - 2) - 1560\tilde{\beta}_1 + 24 \qquad \text{(26)}.\end{aligned}$$

(8) Now, by the definition of κ_r, $\kappa(2^5)$ is the coefficient of $\frac{\omega^5}{5!}$ in the following identity

$$\sum_{r=1}^{\infty} \kappa_r\left(\frac{\omega^r}{r!}\right) = \log \int_{-\infty}^{+\infty} \phi(k_2)\, e^{\omega k_2}\, dk_2.$$

And since $k_2 = \frac{N}{N-1}\mu_2$, and consequently the frequency of k_2 is the same as that of μ_2, we have

$$\sum_{r=1}^{\infty} \kappa_r\left(\frac{\omega^r}{r!}\right) = \log \int_{-\infty}^{+\infty} e^{\omega\left(\frac{N}{N-1}\right)\mu_2}\, \phi(\mu_2)\, d\mu_2;$$

therefore $$\lambda_r(\mu_2) = \frac{(N-1)^r}{N^r}\kappa(2^r),$$

and in particular $$\lambda_5(\mu_2) = \frac{(N-1)^5}{N^5}\kappa(2^5) \qquad \text{(27)}.$$

But $$\lambda_5(\mu_2) = {}_2M_5 - 10\,.\,{}_2M_3\,.\,{}_2M_2.$$

Therefore $$ {}_sM_5 = 10 \,.\, {}_sM_2 \,.\, {}_sM_3 + \frac{(N-1)^5}{N^5}\,\kappa\,(2^5) $$

$$ = 10 \,.\, {}_sM_2 \,.\, {}_sM_3 + \frac{4\tau_0}{N^4}\,\tilde{\sigma}^{10} \text{ (approximately)} \quad \text{......(28)}. $$

Now we can find the expressions for m's in terms of $\tilde{\beta}$'s which are necessary for our present purposes, and consequently can find the expressions for the mean σ and $\mu_r(\sigma)$ $(r=2, 3, 4)$ up to the order N^{-4}.

(9) Let us next proceed to find expressions for all the m's up to the order N^{-4}, which are necessary for our present purposes.

Since $$ {}_sM_2 = \frac{(N-1)^2}{N^3}\left(\tilde{\beta}_2 - 3 + \frac{2N}{N-1}\right)\tilde{\sigma}^4, \quad \bar{\mu}_2 = \frac{N-1}{N}\,\tilde{\sigma}^2 \text{ ....(29)}, $$

$$ m_2 = \frac{{}_sM_2}{4\bar{\mu}_2^{\,2}} = \frac{1}{2N}\left(\frac{\tilde{\beta}_2 - 1}{2} + \frac{1}{N-1}\right) $$

$$ = \frac{1}{2N}\left(p + \frac{1}{N} + \frac{1}{N^2} + \frac{1}{N^3}\right) \text{ (approximately) } \text{...(30 }a\text{)}, $$

where $$ p = \tfrac{1}{2}(\tilde{\beta}_2 - 1) \text{ ....(31)}. $$

Similarly from the equations (24) we have

$$ m_3 = \frac{{}_sM_3}{8\bar{\mu}_2^{\,3}} = \frac{1}{8N^2}\left\{q - 6p + \frac{24p-8}{N} + \frac{4(6p-\tilde{\beta}_1)}{N^2}\right\} \text{ (approximately) (30 }b\text{)}, $$

where $$ q = \tilde{\beta}_4 - 6\tilde{\beta}_1 - 1 \text{ ....(32)}, $$

$$ m_4 = \frac{{}_sM_4}{16\bar{\mu}_2^{\,4}} = \frac{3}{4N^2}\left\{p^2 + \frac{s_2}{N} + \frac{2q + 8p^2 - 26p + 5}{N^2}\right\} \text{ (approximately) (30 }c\text{)}, $$

where $$ s_2 = \tfrac{1}{12}\{\tilde{\beta}_6 - 24\tilde{\beta}_3 + 72\tilde{\beta}_1 - 4q - 12p^2 + 36p - 1\}, $$

$$ m_5 = \frac{{}_sM_5}{32\bar{\mu}_2^{\,5}} = \frac{\lambda_5}{32\bar{\mu}_2^{\,5}} + 10\,\frac{{}_sM_2 \,.\, {}_sM_3}{32\bar{\mu}_2^{\,5}} $$

$$ = \frac{1}{8N^3}\left\{5p(q-6p) + \frac{1}{N}(120p^2 - 70p + 5q + \tau_0\right\} \text{ (approximately)} $$

$$ \text{.........(30 }d\text{)}, $$

$$ m_6 = \frac{5}{8N^3}\left\{3p^3 + \frac{1}{4N}(36ps_2 + q^2 - 12pq)\right\} \text{ (approximately) ....(30 }e\text{)}, $$

$$ m_7 = \frac{105p^2(q-6p)}{32N^4} \text{ (approximately) ....(30 }f\text{)}, $$

and $$ m_8 = \frac{105p^4}{16N^4} \text{ (approximately) ....(30 }g\text{)}. $$

If we substitute these values of the m's into the expressions (25 a) for α, and (25 b) for α', after simplification, we have

$$ \alpha = 1 - \frac{p}{4N} - \frac{\tau_2}{32N^2} - \frac{\tau_3}{128N^3} - \frac{\tau_4}{2048N^4} \text{ ....(33 }a\text{)}, $$

and $$ \alpha' = \frac{1}{N}\left(p + \frac{\tau_2'}{8N} + \frac{\tau_3'}{32N^2} + \frac{\tau_4'}{256N^3}\right) \text{ ....(33 }b\text{)}, $$

where

$$\tau_2 = 15p^2 + 12p - 2q + 8$$
$$= \tfrac{1}{4}(-8\tilde{\beta}_4 + 15\tilde{\beta}_2^2 - 6\tilde{\beta}_2 + 48\tilde{\beta}_1 + 31),$$
$$\tau_3 = 60s_3 + 315p^3 - 70p(q - 6p) - 64(3p - 1) + 32$$
$$= \tfrac{1}{8}(40\tilde{\beta}_6 - 280\tilde{\beta}_4\tilde{\beta}_2 + 120\tilde{\beta}_4 - 960\tilde{\beta}_3 + 1680\tilde{\beta}_2\tilde{\beta}_1 + 315\tilde{\beta}_2^3 - 225\tilde{\beta}_2^2 - 263\tilde{\beta}_2 + 2160\tilde{\beta}_1 + 1061),$$

$$\tau_4 = 1260p(\tilde{\beta}_6 - 24\tilde{\beta}_3 + 72\tilde{\beta}_1) - 224\tau_0 + 45045p^4 - 13860p^2q + 68040p^3 + 26160p^2 - 10080pq + 420q^2 - 13612p + 800q + 512\tilde{\beta}_1 + 5312,$$
$$= \tfrac{1}{16}\{-896(\tilde{\beta}_8 - 40\tilde{\beta}_5) + 1120\tilde{\beta}_6(9\tilde{\beta}_2 - 5) + 26880\tilde{\beta}_3(\tilde{\beta}_3/\tilde{\beta}_1 - 9\tilde{\beta}_2 - 7)$$
$$+ 45045\tilde{\beta}_2^4 - 44100\tilde{\beta}_2^3 + 10080\tilde{\beta}_1(33\tilde{\beta}_2^2 + 22\tilde{\beta}_2 + 24) - 4770\tilde{\beta}_2^2 - 66596\tilde{\beta}_2$$
$$+ 532832\tilde{\beta}_1 - 38827\} \quad \ldots\ldots(34\,a);$$

and

$$\tau_2' = 6p^2 + 6p - q + 8$$
$$= \tfrac{1}{2}(3\tilde{\beta}_2^2 - 2\tilde{\beta}_4 + 12\tilde{\beta}_1 + 15),$$
$$\tau_3' = 2\tilde{\beta}_6 - 48\tilde{\beta}_3 + 144\tilde{\beta}_1 + 105p^3 + 126p^2 - 25pq - 24p - 8q + 62$$
$$= \tfrac{1}{8}(16\tilde{\beta}_6 - 384\tilde{\beta}_3 + 36\tilde{\beta}_4 - 100\tilde{\beta}_4\tilde{\beta}_2 + 105\tilde{\beta}_2^3 - 63\tilde{\beta}_2^2 + 600\tilde{\beta}_2\tilde{\beta}_1 - 185\tilde{\beta}_2 + 936\tilde{\beta}_1 + 70),$$

$$\tau_4' = 210p(\tilde{\beta}_6 - 24\tilde{\beta}_3 + 72\tilde{\beta}_1) - 40\tau_0 + 6930p^4 + 10710p^3 - 2205p^2q + 4296p^2 - 1680pq + 70q^2 - 3170p + 184q + 128\tilde{\beta}_1 + 1216$$
$$= \tfrac{1}{8}\{-80(\tilde{\beta}_8 - 40\tilde{\beta}_5) + 40\tilde{\beta}_6(21\tilde{\beta}_2 - 11)$$
$$+ 2\tilde{\beta}_4(280\tilde{\beta}_4 - 2205\tilde{\beta}_2^2 + 1450\tilde{\beta}_2 - 3360\tilde{\beta}_1 + 531)$$
$$+ 480\tilde{\beta}_3(5\tilde{\beta}_3/\tilde{\beta}_1 - 42\tilde{\beta}_2 - 38) + 180\tilde{\beta}_1(147\tilde{\beta}_2^2 + 106\tilde{\beta}_2 + 112\tilde{\beta}_1)$$
$$+ 3465\tilde{\beta}_2^4 - 3150\tilde{\beta}_2^3 - 738\tilde{\beta}_2^2 - 8894\tilde{\beta}_2 + 49372\tilde{\beta}_1 + 18613\} \quad \ldots\ldots(34\,b).$$

Substituting these values of α, α' and m_2 into the general equations for the Mean σ, $\mu_2(\sigma)$, $\mu_3(\sigma)$ and $\mu_4(\sigma)$, and neglecting terms of order N^{-5} or higher, after long transformations and simplifications I have obtained the following results:

$$\bar{\sigma} = \left\{1 - \frac{p+2}{4N} - \frac{\tau_2 - 4p + 4}{32N^2} - \frac{\tau_3 - 2\tau_2 - 4p + 8}{128N^3} - \frac{\tau_4 - 8(\tau_3 + \tau_2) - 32p + 80}{2048N^4}\right\}\tilde{\sigma} \quad \ldots\ldots(35\,a),$$

$$\mu_2(\sigma) = \frac{1}{2N}\left\{p + \frac{\tau_2 - p^2 - 8p}{8N} + \frac{\tau_3 - p\tau_2 - 4(\tau_2 - p^2)}{32N^2} + \frac{\tau_4 - 4(p+4)\tau_3 - \tau_2(\tau_2 - 16p)}{512N^3}\right\}\tilde{\sigma}^2 \quad \ldots\ldots(35\,b),$$

$$\mu_3(\sigma) = \frac{1}{8N^2}\left\{\tau_2' - \tau_2 + 3p^2 + \frac{1}{4N}[\tau_3' - \tau_3 + 3p\tau_2 - 6(\tau_2' - \tau_2) - p^2(p+18)]\right.$$

$$+ \frac{1}{64N^2}[2\tau_4' - \tau_4 + 12p\tau_3 - 24(\tau_3' - \tau_3) + 3\tau_2^2 - 6p^2\tau_2 + 24(\tau_2' - \tau_2)$$

$$\left. - 72p\tau_2 + 24p^2(p+3)]\right\}\tilde{\sigma}^3 \quad \text{.........(35 c)},$$

$$\mu_4(\sigma) = \frac{1}{4N^2}\left\{\tau_2 - 2\tau_2' + 8 + \frac{1}{4N}[\tau_3 - 2\tau_3' - 2p(\tau_2 - \tau_2') - 8(\tau_2 - 2\tau_2') + 3p^3 - 32]\right.$$

$$+ \frac{1}{64N^2}[\tau_4 - 4\tau_4' - 8p(\tau_3 - \tau_3') - 32(\tau_3 - 2\tau_3') + 2\tau_2(2\tau_2' - 2\tau_2 + 9p^2)$$

$$\left. + 64p(\tau_2 - \tau_2') + 64(\tau_2 - 2\tau_2') - 3p^3(p+32)]\right\}\tilde{\sigma}^4 \text{.........(35 d)},$$

where $p = \frac{1}{2}(\tilde{\beta}_2 - 1)$, and the τ's, τ''s are given in (34 a) and (34 b).

To find σ_σ, $\beta_1(\sigma)$ and $\beta_2(\sigma)$, in particular numerical cases, it is most convenient to use the general equations (16) directly after the calculation of m_2, α and α'.

(10) Now let us consider the special cases where the approximation is adequate up to the orders N^{-2} and N^{-3}.

In the first case, neglecting terms of orders N^{-3} and N^{-4} in the general approximate equations (35), we get

$$\text{Mean } \sigma = \tilde{\sigma}\left(1 - \frac{p+2}{4N} - \frac{\tilde{\tau}_2 - 4p + 4}{32N^2}\right),$$

$$\mu_2(\sigma) = \frac{\tilde{\sigma}^2}{2N}\left(p + \frac{\tau_2 - p^2 - 8p}{8N}\right),$$

$$\mu_3(\sigma) = \frac{\tilde{\sigma}^3}{8N^2}(\tau_2' - \tau_2 + 3p^2),$$

and

$$\mu_4(\sigma) = \frac{\tilde{\sigma}^4}{4N^2}(\tau_2 - 2\tau_2' + 8) \text{ ....(36)}.$$

But

$$-\tfrac{1}{4}(p+2) = -\tfrac{1}{8}(\tilde{\beta}_2 + 3),$$

$$\tau_2 - 4p + 4 = -\tfrac{1}{4}(8\tilde{\beta}_4 - 15\tilde{\beta}_2^2 + 14\tilde{\beta}_2 - 48\tilde{\beta}_1 - 55),$$

$$\tau_2 - p^2 - 8p = -\tfrac{1}{2}(4\tilde{\beta}_4 - 7\tilde{\beta}_2^2 + 10\tilde{\beta}_2 - 24\tilde{\beta}_1 - 23),$$

$$\tau_2' - \tau_2 + 3p^2 = \tfrac{1}{2}(2\tilde{\beta}_4 - 3\tilde{\beta}_2^2 - 12\tilde{\beta}_1 + 1),$$

and

$$\tau_2 - 2\tau_2' + 8 = \tfrac{3}{4}(\tilde{\beta}_2 - 1)^2 \text{ ....(37)}.$$

If we substitute these expressions in the equations (36), we get a set of equations for the mean σ, $\mu_2(\sigma)$, $\mu_3(\sigma)$ and $\mu_4(\sigma)$ which are the same as the equations (21), as we should expect.

In the second case, neglecting terms of order N^{-4} in (35), we have

$$\text{Mean}\ \sigma = \tilde{\sigma}\left\{1 - \frac{p+2}{4N} - \frac{\tau_2 - 4p + 4}{32N^2} - \frac{\tau_3 - 2\tau_2 - 4p + 8}{128N^3}\right\},$$

$$\mu_2(\sigma) = \frac{\tilde{\sigma}^2}{2N}\left\{p + \frac{\tau_2 - p^2 - 8p}{8N} + \frac{\tau_3 - p\tau_2 - 4(\tau_2 - p^2)}{32N^2}\right\},$$

$$\mu_3(\sigma) = \frac{\tilde{\sigma}^3}{8N^2}\left\{\tau_2' - \tau_2 + 3p^2 + \frac{\tau_3' - \tau_3 + 3p\tau_2 - 6(\tau_2' - \tau_2) - p^2(p+18)}{4N}\right\},$$

$$\mu_4(\sigma) = \frac{\tilde{\sigma}^4}{4N^2}\left\{\tau_2 - 2\tau_2' + 8 + \frac{\tau_3 - 2\tau_3' - 2p(\tau_2 - \tau_2') - 8(\tau_2 - 2\tau_2') + 3p^2 - 32}{4N}\right\} \qquad \text{.........(38).}$$

But

$$\tau_3 - 2\tau_2 - 4p + 8 = \tfrac{1}{8}\{40\tilde{\beta}_6 - 8\tilde{\beta}_4(35\tilde{\beta}_2 - 19) - 960\tilde{\beta}_3 + 315\tilde{\beta}_2^3 + 1680\tilde{\beta}_2\tilde{\beta}_1 - 285\tilde{\beta}_2^2 - 255\tilde{\beta}_2 + 1968\tilde{\beta}_1 + 1017\},$$

$$\tau_3 - p\tau_2 - 4(\tau_2 - p^2) = \tfrac{1}{2}\{10\tilde{\beta}_6 - 4\tilde{\beta}_4(17\tilde{\beta}_2 - 11) - 240\tilde{\beta}_3 + 75\tilde{\beta}_2^3 - 79\tilde{\beta}_2^2 - 67\tilde{\beta}_2 + 408\tilde{\beta}_2\tilde{\beta}_1 + 456\tilde{\beta}_1 + 213\},$$

$$\tau_3' - \tau_3 + 3p\tau_2 - 6(\tau_2' - \tau_2) - p^2(p+18) = \tfrac{1}{8}\{-24\tilde{\beta}_6 + 12\tilde{\beta}_4(13\tilde{\beta}_2 - 9) + 576\tilde{\beta}_3 - 166\tilde{\beta}_2^3 + 174\tilde{\beta}_2^2 - 936\tilde{\beta}_2\tilde{\beta}_1 + 186\tilde{\beta}_2 - 1080\tilde{\beta}_1 - 474\},$$

$$\tau_3 - 2\tau_3' - 2p(\tau_2 - \tau_2') - 8(\tau_2 - 2\tau_2') + 3p^2 - 32 = \tfrac{1}{8}\{8\tilde{\beta}_6 - 8\tilde{\beta}_4(9\tilde{\beta}_2 - 5) - 192\tilde{\beta}_3 + 432\tilde{\beta}_2\tilde{\beta}_1 + 90\tilde{\beta}_2^3 - 126\tilde{\beta}_2^2 + 198\tilde{\beta}_2 + 336\tilde{\beta}_1 - 138\} \qquad \text{......(39).}$$

Therefore, from the equations (37), (38) and (39), we have

$$\text{Mean}\ \sigma = \tilde{\sigma}\Big\{1 - \frac{1}{8N}(\tilde{\beta}_2 + 3) + \frac{1}{128N^2}(8\tilde{\beta}_4 - 15\tilde{\beta}_2^2 + 14\tilde{\beta}_2 - 48\tilde{\beta}_1 - 55) - \frac{1}{1024N^3}[40\tilde{\beta}_6 - 8\tilde{\beta}_4(35\tilde{\beta}_2 - 19) - 960\tilde{\beta}_3 + 315\tilde{\beta}_2^3 + 1680\tilde{\beta}_2\tilde{\beta}_1 - 285\tilde{\beta}_2^2 - 255\tilde{\beta}_2 + 1968\tilde{\beta}_1 + 1017]\Big\},$$

$$\mu_2(\sigma) = \frac{\tilde{\sigma}^2}{4N}\Big\{\tilde{\beta}_2 - 1 - \frac{1}{8N}(4\tilde{\beta}_4 - 7\tilde{\beta}_2^2 + 10\tilde{\beta}_2 - 24\tilde{\beta}_1 - 23) + \frac{1}{32N^2}[10\tilde{\beta}_6 - 4\tilde{\beta}_4(17\tilde{\beta}_2 - 11) - 240\tilde{\beta}_3 + 75\tilde{\beta}_2^3 - 79\tilde{\beta}_2^2 - 67\tilde{\beta}_2 + 408\tilde{\beta}_2\tilde{\beta}_1 + 456\tilde{\beta}_1 + 213]\Big\},$$

$$\mu_3(\sigma) = \frac{\tilde{\sigma}^3}{16N^2}\Big\{2\tilde{\beta}_4 - 3\tilde{\beta}_2^2 - 12\tilde{\beta}_1 + 1 - \frac{1}{16N}[24\tilde{\beta}_6 - 12\tilde{\beta}_4(13\tilde{\beta}_2 - 9) - 576\tilde{\beta}_3 + 166\tilde{\beta}_2^3 - 174\tilde{\beta}_2^2 + 936\tilde{\beta}_2\tilde{\beta}_1 - 186\tilde{\beta}_2 + 1080\tilde{\beta}_1 + 474]\Big\},$$

and
$$\mu_4(\sigma) = \frac{3\tilde{\sigma}^4}{16N^2}\Big\{(\tilde{\beta}_2 - 1)^2 + \frac{1}{8N}[8\tilde{\beta}_6 - 8\tilde{\beta}_4(9\tilde{\beta}_2 - 5) - 192\tilde{\beta}_3 + 432\tilde{\beta}_2\tilde{\beta}_1 + 90\tilde{\beta}_2^3 - 126\tilde{\beta}_2^2 + 198\tilde{\beta}_2 + 336\tilde{\beta}_1 - 138]\Big\} \qquad \text{......(40).}$$

These special formulae (40) correspond, when expressed in terms of semi-invariants, with Dr Craig's formulae for $\lambda_r(\sigma)$ $(r = 1, 2, 3, 4)$*.

* See Craig, *Metron*, Vol. VII. No. 4, p. 56, also *Biometrika*, Vol. XXI. pp. 287—293.

(11) Now let us consider the case where the parent distribution is normal.

In this case $\tilde{\beta}_1 = \tilde{\beta}_3 = \tilde{\beta}_5 = \ldots = 0,$

$$\tilde{\beta}_2 = 3, \quad \tilde{\beta}_4 = 15, \quad \tilde{\beta}_6 = 105,$$

and $\tilde{\beta}_8 = 945.$

Consequently we have, from the equations (25) and (30),

$$m_2 = \frac{1}{2N}\left(1 + \frac{1}{N} + \frac{1}{N^2} + \frac{1}{N^3}\right), \quad m_3 = \frac{1}{N^2}\left(1 + \frac{2}{N} + \frac{3}{N^2}\right),$$

$$m_4 = \frac{3}{4N^2}\left(1 + \frac{6}{N} + \frac{15}{N^2}\right), \qquad m_5 = \frac{5}{8N^3},$$

$$m_6 = \frac{5}{8N^3}\left(3 + \frac{61}{N}\right), \qquad m_7 = \frac{105}{4N^4},$$

$$m_8 = \frac{105}{16N^4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(41),$$

and also

$$\alpha = 1 - \frac{1}{4N} - \frac{7}{32N^2} - \frac{19}{128N^3} - \frac{101}{2048N^4},$$

$$\alpha' = \frac{1}{N}\left(1 + \frac{3}{4N} + \frac{17}{32N^2} + \frac{49}{128N^3}\right) \quad \ldots\ldots\ldots\ldots\ldots(42).$$

Consequently, from the equations (14′), we have

$$\text{Mean } \sigma = \tilde{\sigma}\left(1 - \frac{3}{4N} - \frac{7}{32N^2} - \frac{9}{128N^3} - \frac{59}{2048N^4}\right) \quad \ldots\ldots\ldots(43\,a),$$

$$\mu_2(\sigma) = \frac{\tilde{\sigma}^2}{2N}\left(1 - \frac{1}{4N} - \frac{3}{8N^2} - \frac{27}{64N^3}\right) \ldots\ldots\ldots\ldots\ldots\ldots(43\,b),$$

$$\mu_3(\sigma) = \frac{\tilde{\sigma}^3}{4N^2}\left(1 + \frac{3}{4N} + \frac{11}{32N^2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(43\,c),$$

$$\mu_4(\sigma) = \frac{3\tilde{\sigma}^4}{4N^2}\left(1 - \frac{1}{2N} - \frac{7}{16N^2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(43\,d).$$

If we find the values of p, τ's and τ''s in this case we have:

$$p = 1, \quad \tau_2 = 7, \quad \tau_3 = 19, \quad \tau_4 = 101, \quad \tau_2' = 6, \quad \tau_3' = 17 \text{ and } \tau_4' = 98.$$

Substituting these values into the general approximate formulae (35), we can also deduce the same equations as (43) directly.

Now, from the equations (43), if we find the expressions for σ_σ, $\beta_1(\sigma)$ and $\beta_2(\sigma)$, we get

$$\sigma_\sigma = \frac{\tilde{\sigma}}{\sqrt{2N}}\left(1 - \frac{1}{4N} - \frac{3}{8N^2} - \frac{27}{64N^3}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots(44\,a)$$

$$= \frac{\tilde{\sigma}}{\sqrt{2N}}\left(1 - \frac{1}{8N} - \frac{25}{128N^2} - \frac{241}{1024N^3}\right),$$

$$\beta_1(\sigma) = \frac{1}{2N}\left(1 + \frac{9}{4N} + \frac{31}{8N^2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(44\,b),$$

and

$$\beta_2(\sigma) = 3 + \frac{3}{4N^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(44\,c).$$

(12) *Special cases (continued).*

Finally let us consider the case where the parent distribution is of the form of Pearson's Type III curve, i.e.

$$y = y_0\left(1+\frac{x}{a}\right)^{\gamma a} e^{-\gamma x} \qquad \text{.........................(45)},$$

which is a skew distribution, and also the case of the exponential function (known as Pearson's Type X curve),

$$y = y_0 e^{-\frac{x}{s}} \qquad \text{.........................(46)},$$

which is a special case of the distribution (45).

In these cases $\qquad \tilde\beta_2 = \frac{3}{2}(\tilde\beta_1+2), \quad \tilde\beta_1 = \frac{4}{\gamma a+1}.$

and generally $\qquad \tilde\beta_{2m} = \frac{2m+1}{2}\{\tilde\beta_{2m-1}+2\tilde\beta_{2m-2}\},$

$$\tilde\beta_{2m+1} = (m+1)\{\tilde\beta_1\tilde\beta_{2m}+2\tilde\beta_{2m-1}\} \qquad \text{.........................(47)}.$$

From the equations (47) we get

$$\tilde\beta_3 = 3\tilde\beta_1^2+10\tilde\beta_1, \quad \tilde\beta_4 = \frac{5}{2}(3\tilde\beta_1^2+13\tilde\beta_1+6),$$

$$\tilde\beta_5 = \frac{3}{2}(15\tilde\beta_1^3+77\tilde\beta_1^2+70\tilde\beta_1),$$

$$\tilde\beta_6 = \frac{7}{4}(45\tilde\beta_1^3+261\tilde\beta_1^2+340\tilde\beta_1+60),$$

$$\tilde\beta_7 = 315\tilde\beta_1^4+2007\tilde\beta_1^3+3304\tilde\beta_1^2+1260\tilde\beta_1,$$

$$\tilde\beta_8 = \frac{9}{4}(630\tilde\beta_1^4+4829\tilde\beta_1^3+8435\tilde\beta_1^2+4900\tilde\beta_1+420) \qquad \text{.........(48)}.$$

On substituting these values for $\tilde\beta_m$ into the expressions for p, and the τ's and τ''s, we get, after simplification,

$$p = \frac{1}{4}(3\tilde\beta_1+4), \quad \tau_2 = -\frac{1}{16}(105\tilde\beta_1^2+344\tilde\beta_1-112),$$

$$\tau_2' = -\frac{1}{8}(33\tilde\beta_1^2+104\tilde\beta_1-48),$$

$$\tau_3 = \frac{1}{64}(8505\tilde\beta_1^3+3,7860\tilde\beta_1^2+1,6304\tilde\beta_1+1216),$$

$$\tau_3' = \frac{1}{64}(3915\tilde\beta_1^3+1,7484\tilde\beta_1^2+7952\tilde\beta_1+1088),$$

$$\tau_4 = -\frac{1}{256}(654,2235\tilde\beta_1^4+3489,9984\tilde\beta_1^3+3571,9200\tilde\beta_1^2+313,7792\tilde\beta_1-2,5856),$$

and $\tau_4' = -\frac{1}{256}(126,5670\tilde\beta_1^4+680,7360\tilde\beta_1^3+716,1536\tilde\beta_1^2+71,4752\tilde\beta_1-2,5088)$

$$\text{.........(49)}.$$

Now, from these equations and the equations (35), we can deduce easily

$$\text{Mean } \sigma = \tilde{\sigma}\left\{1 - \frac{3\tilde{\beta}_1 + 12}{16N} + \frac{105\tilde{\beta}_1^2 + 392\tilde{\beta}_1 - 112}{512N^2}\right.$$
$$- \frac{8505\tilde{\beta}_1^3 + 3{,}8700\tilde{\beta}_1^2 + 1{,}8864\tilde{\beta}_1 + 576}{8192N^3}$$
$$\left. + \frac{654{,}2235\tilde{\beta}_1^4 + 3517{,}2144\tilde{\beta}_1^3 + 3691{,}7280\tilde{\beta}_1^2 + 362{,}1632\tilde{\beta}_1 + 1{,}5104}{52{,}4288N^4}\right\},$$

$$\mu_2(\sigma) = \frac{\tilde{\sigma}^2}{8N}\left\{3\tilde{\beta}_1 + 4 - \frac{57\tilde{\beta}_1^2 + 232\tilde{\beta}_1 + 16}{16N} + \frac{2205\tilde{\beta}_1^3 + 1{,}0284\tilde{\beta}_1^2 + 5808\tilde{\beta}_1 - 192}{128N^2}\right.$$
$$\left. - \frac{83{,}1915\tilde{\beta}_1^4 + 451{,}5888\tilde{\beta}_1^3 + 489{,}1424\tilde{\beta}_1^2 + 55{,}5776\tilde{\beta}_1 + 6912}{4096N^3}\right\},$$

$$\mu_3(\sigma) = \frac{\tilde{\sigma}^3}{64N^2}\left\{33\tilde{\beta}_1^2 + 104\tilde{\beta}_1 + 16 - \frac{2781\tilde{\beta}_1^3 + 1{,}3212\tilde{\beta}_1^2 + 8304\tilde{\beta}_1 - 192}{16N}\right.$$
$$\left. + \frac{108{,}8955\tilde{\beta}_1^4 + 596{,}0208\tilde{\beta}_1^3 + 663{,}7088\tilde{\beta}_1^2 + 87{,}8848\tilde{\beta}_1 + 2816}{512N^2}\right\},$$

$$\mu_4(\sigma) = \frac{\tilde{\sigma}^4}{64N^2}\left\{27\tilde{\beta}_1^2 + 72\tilde{\beta}_1 + 48 + \frac{495\tilde{\beta}_1^3 + 1740\tilde{\beta}_1^2 - 240\tilde{\beta}_1 - 192}{8N}\right.$$
$$\left. - \frac{162{,}3348\tilde{\beta}_1^4 + 864{,}6336\tilde{\beta}_1^3 + 878{,}0160\tilde{\beta}_1^2 + 67{,}8912\tilde{\beta}_1 + 2{,}1504}{1024N^2}\right\}$$

.........(50).

These results correspond to those given by Craig*, except that in each formula I have proceeded to one degree higher approximation.

Further, if the parent distribution (45) reduce to the particular form

$$y = y_0 e^{-\frac{x}{\tilde{\sigma}}},$$

then the $\tilde{\beta}$'s become

$$\tilde{\beta}_1 = 4,\quad \tilde{\beta}_2 = 9,\quad \tilde{\beta}_3 = 88,\quad \tilde{\beta}_4 = 265,\quad \tilde{\beta}_5 = 3708,\quad \tilde{\beta}_6 = 1{,}4833,$$
$$\tilde{\beta}_7 = 26{,}6992,\quad \tilde{\beta}_8 = 133{,}4961;$$

consequently

$$p = 4,\quad \tau_2 = -184,\quad \tau_2' = -112,\quad \tau_3 = 1{,}9008,\quad \tau_3' = 8800,$$
$$\tau_4 = -1754{,}8608,\quad \tau_4' = -342{,}6176,$$

and, from the equations (50), we have

$$\text{Mean } \sigma = \left(1 - \frac{3}{2N} + \frac{49}{8N^2} - \frac{2421}{16N^3} + \frac{110{,}6208}{128N^4}\right)\tilde{\sigma},$$
$$\mu_2(\sigma) = \frac{1}{N}\left(2 - \frac{29}{2N} + \frac{321}{N^2} - \frac{14{,}2207}{8N^3}\right)\tilde{\sigma}^2,$$
$$\mu_3(\sigma) = \frac{1}{N^2}\left(15 - \frac{825}{2N} + \frac{18{,}7973}{8N^2}\right)\tilde{\sigma}^3,$$
$$\mu_4(\sigma) = \frac{1}{N^2}\left(12 + \frac{114}{N} - \frac{6{,}7881}{4N^2}\right)\tilde{\sigma}^4 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(51);$$

which give us the Mean σ, $\mu_r(\sigma)$ ($r = 2, 3, 4$) in sampling when the parent distribution is of the form of Pearson's Type X curve.

* *Biometrika*, Vol. XXI. p. 292, equation (17).

SECTION III. DEGREE OF APPROXIMATION. NUMERICAL VERIFICATION.

(13) If the parent distribution be normal and infinite, it is well known that the sampling distribution of σ is theoretically given by

$$y = A\sigma^{n-1} e^{-\frac{n\sigma^2}{2\tilde{\sigma}^2}} \quad \text{.................................(52)},$$

where n is the size of repeated random samples, i.e.

$n = N$ in foregoing articles,

and if M be the total number of samples

$$A = \frac{M}{(n-3)(n-5)\dots 3.1}\sqrt{\frac{2}{\pi}}\left(\frac{\tilde{\sigma}}{\sqrt{n}}\right)^{-(n-1)} \text{ if } n \text{ is even,}$$

or
$$\frac{M}{(n-3)(n-5)\dots 4.2}\left(\frac{\tilde{\sigma}}{\sqrt{n}}\right)^{-(n-1)} \text{ if } n \text{ is odd.}$$

We may now compare the moments calculated from the equations (43) with the true moments obtained from the distribution law (52).

If μ_m' is the mth moment coefficient of (52) about $\sigma = 0$, then

$$\begin{aligned}
M\mu_m' &= A\int_0^\infty \sigma^{m+n-2} e^{-\frac{n\sigma^2}{2\tilde{\sigma}^2}} d\sigma \\
&= -A\left(\frac{\tilde{\sigma}^2}{n}\right)\left\{\left[\sigma^{n+m-3} e^{-\frac{n\sigma^2}{2\tilde{\sigma}^2}}\right]_0^\infty - (n+m-3)\int_0^\infty \sigma^{m+n-4} e^{-\frac{n\sigma^2}{2\tilde{\sigma}^2}} d\sigma\right\} \\
&= A(m+n-3)\left(\frac{\tilde{\sigma}^2}{n}\right)\int_0^\infty \sigma^{m+n-4} e^{-\frac{n\sigma^2}{2\tilde{\sigma}^2}} d\sigma \\
&= A(n+m-3)(n+m-5)\left(\frac{\tilde{\sigma}^2}{n}\right)^2\int_0^\infty \sigma^{m+n-6} e^{-\frac{n\sigma^2}{2\tilde{\sigma}^2}} d\sigma.
\end{aligned}$$

Similarly, after integrating by parts k times, we have

$$\mu_m' = \frac{(n+m-3)(n+m-5)\dots(n+m-2k-1)}{(n-3)(n-5)\dots 3.1\begin{cases}\sqrt{\pi/2} \\ \text{or } 4.2\end{cases}}\left(\frac{\tilde{\sigma}}{\sqrt{n}}\right)^{2k-n+1}$$
$$\times \int_0^\infty \sigma^{n+m-2(k+1)} e^{-\frac{n\sigma^2}{2\tilde{\sigma}^2}} d\sigma \quad \text{.........(53)},$$

according as n is even or odd.

From this general formula we can easily deduce

$$\mu_1' = \frac{(n-2)(n-4)\dots 3.1}{(n-3)(n-5)\dots 4.2}\sqrt{\frac{\pi}{2n}}\,\tilde{\sigma} \text{ if } n \text{ is odd,}$$

or
$$\frac{(n-2)(n-4)\dots 4.2}{(n-3)(n-5)\dots 3.1\sqrt{\pi/2}}\,\tilde{\sigma} \text{ if } n \text{ is even} \quad \text{..................(54 a)},$$

$$\mu_2' = \left(1 - \frac{1}{n}\right)\tilde{\sigma}^2, \quad \mu_4' = \left(1 - \frac{1}{n^2}\right)\tilde{\sigma}^4 \quad \text{..................(54 b)},$$

and
$$\mu_3' = \tilde{\sigma}^2\mu_1' \quad \text{......................................(55)},$$

which correspond to the special case of my formulae (12) when the parent distribution is normal, i.e.

$$\mu_1' = a_1\sqrt{\bar{\mu}_2} = \left(\frac{n-1}{n}\right)^{\frac{1}{2}} \alpha\tilde{\sigma} \quad \text{..............................(56 } a\text{)},$$

$$\mu_2' = a_2\bar{\mu}_2 = \left(1-\frac{1}{n}\right)\tilde{\sigma}^2 \quad \text{..............................(56 } b\text{)},$$

$$\mu_3' = a_3(\bar{\mu}_2)^{\frac{3}{2}} = \left(\frac{n-1}{n}\right)^{\frac{3}{2}} (\alpha+\alpha')\tilde{\sigma}^3 \quad \text{..................(56 } c\text{)},$$

$$\mu_4' = a_4(\bar{\mu}_2)^2 = \left(\frac{n-1}{n}\right)^2 (1+4m_2)\tilde{\sigma}^4 \text{..................(56 } d\text{)},$$

where $$m_2 = \frac{1}{2(n-1)},$$

and $$\alpha = 1 - \frac{1}{4n} - \frac{7}{32n^2} - \frac{19}{128n^3} - \frac{101}{2048n^4} - \dots,$$

$$\alpha' = \frac{1}{n}\left(1 + \frac{3}{4n} + \frac{17}{32n^2} + \frac{49}{128n^3} + \dots\right);$$

as we found already in Art. (11).

Now, since the equation (56 d) can be transformed into

$$\mu_4' = \left(1 - \frac{1}{n^2}\right)\tilde{\sigma}^4,$$

we can see that my formulae for μ_2' and μ_4', when the parent distribution is normal, are identically equal to (54 b) respectively, and we have only to examine the accuracy of μ_1' and μ_3' in (56).

Now if we insert the expressions for α and α' into equations (56 a) and (56 c), retaining terms up to the order n^{-4}, we have

$$\mu_1' = \left(1 - \frac{3}{4n} - \frac{7}{32n^2} - \frac{9}{128n^3} + \frac{59}{2048n^4}\right)\tilde{\sigma},$$

and $$\mu_3' = \left(1-\frac{1}{n}\right)^{\frac{3}{2}}\left(1 + \frac{3}{4n} + \frac{17}{32n^2} + \frac{49}{128n^3} + \frac{683}{2048n^4}\right)\tilde{\sigma}^3$$

$$= \left(1 - \frac{3}{4n} - \frac{7}{32n^2} - \frac{9}{128n^3} + \frac{59}{2048n^4}\right)\tilde{\sigma}^3 \text{........................(57)},$$

therefore $$\mu_3' = \tilde{\sigma}^2\mu_1'.$$

Accordingly, as far as terms of order n^{-4}, my formulae also lead to the relation (55), holding between the first and third moment coefficients of the distribution law (52) about $\sigma = 0$. Thus the accuracy of my formulae, for the special case of normal parent distribution, depends mainly upon the accuracy of the formula for μ_1'.

Now, from the equation (54 a), we have

$$\mu_1' = \frac{2^{n-2}\left[\left(\frac{n}{2}-1\right)!\right]^2}{(n-2)!}\sqrt{\frac{2}{n\pi}}\,\tilde{\sigma} \text{ if } n \text{ is even,}$$

or

$$\frac{(n-2)(n-3)!}{2^{n-3}\left[\left(\frac{n-3}{2}\right)!\right]^2}\sqrt{\frac{\pi}{2n}}\,\tilde{\sigma} \text{ if } n \text{ is odd} \quad \ldots\ldots\ldots\ldots\ldots\ldots(58).$$

If we apply to these expressions Stirling's formula

$$n! = n^n e^{-n}\sqrt{2\pi n}\left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{5,1840n^3} - \frac{571}{248,8320n^4} + \ldots\right),$$

after calculation and simplification we obtain

$$\mu_1' = \tilde{\sigma}\left(1 - \frac{3}{4n} - \frac{7}{32n^2} - \frac{9}{128n^3} + \frac{59}{2048n^4} + \ldots\right),$$

which shows us how far my formula for μ_1' is accurate.

(14) Finally let us examine the accuracy of the formulae (43) for the moments of σ, and (44) for σ_σ, $\beta_1(\sigma)$ and $\beta_2(\sigma)$, numerically for various values of n, the size of sample.

Now if we write the equation (54 a) as follows:

$$\text{Mean } \sigma = \mu_1' = u\tilde{\sigma},$$

then

$$\mu_2(\sigma) = \left(1 - \frac{1}{n} - u^2\right)\tilde{\sigma}^2,$$

$$\sigma_\sigma = \sqrt{\left(1 - \frac{1}{n} - u^2\right)}\,.\,\tilde{\sigma},$$

which correspond to my formulae (43 a), (43 b) and (44 a).

The numbers in Tables I, II and III are the values of $\bar{\sigma}$, $\mu_2(\sigma)$ and σ_σ respectively, obtained from the equations above and my corresponding formulae.

For instance, in Table I the "accurate values" are those of the Mean σ, given by equation (54 a), and the values in the third column are for $\bar{\sigma}$ given by the formula (43 a).

The values in the fourth and fifth columns are for $\bar{\sigma}$, given also by the formula (43 a) when terms of order n^{-4} and terms of order n^{-3} respectively are neglected, i.e. by

$$\bar{\sigma} = \left(1 - \frac{3}{4n} - \frac{7}{32n^2} - \frac{9}{128n^3}\right)\tilde{\sigma},$$

and

$$\bar{\sigma} = \left(1 - \frac{3}{4n} - \frac{7}{32n^2}\right)\tilde{\sigma}.$$

The second of these equations is a special case of the formula (21 a) in Art. (5) when the parent distribution is normal, and the first is the special case which corresponds to Dr Craig's formula for $\lambda_1(\sigma)$ in terms of semi-invariants.

Comparing the values, given by my formulae, with the corresponding accurate values in these tables, we can see that my formulae up to the order n^{-4} are very accurate, even when $n < 10$.

TABLE I.

Values of $\dfrac{\bar{\sigma}}{\bar{\bar{\sigma}}}$.

n size of sample	Accurate values	Approximations, given by my formulae, including terms of order		
		$1/n^4$	$1/n^3$	$1/n^2$
5	·840 7487	·840 7336	·840 6875	·841 2500
7	·888 2029	·888 1999	·888 1879	·888 3929
9	·913 8749	·913 8740	·913 8696	·913 9601
10	·922 7456	·922 7451	·922 7422	·922 8125
15	·949 0076	·949 0075	·949 0069	·949 0278
20	·961 9445	·961 9445	·961 9443	·961 9531
25	·969 6456	·969 6456	·969 6455	·969 6500
30	·974 7544	·974 7544	·974 7543	·974 7569
50	·984 9119	·984 9119	·984 9119	·984 9125
75	·989 9609	·989 9609	·989 9609	·989 9611
100	·992 4781	·992 4781	·992 4781	·992 4781
150	·994 9903	·994 9903	·994 9903	·994 9903
200	·996 2445	·996 2445	·996 2445	·996 2445

TABLE II.

Values of $\dfrac{\mu_2(\sigma)}{\bar{\bar{\sigma}}^2}$.

n size of sample	Accurate values	Approximations, given by my formulae, including terms of order		
		$1/n^4$	$1/n^3$	$1/n^2$
5	·093 1417	·093 1625	·093 5000	·095 0000
7	·068 2385	·068 2431	·068 3309	·068 8776
9	·053 7216	·053 7230	·053 7551	·054 0123
10	·048 5405	·048 5414	·048 5625	·048 7500
15	·032 7179	·032 7181	·032 7222	·032 7778
20	·024 6627	·024 6627	·024 6641	·024 6875
25	·019 7875	·019 7875	·019 7880	·019 8000
30	·016 5206	·016 5206	·016 5208	·016 5278
50	·009 9485	·009 9485	·009 9485	·009 9500
75	·006 6440	·006 6440	·006 6440	·006 6444
100	·004 9873	·004 9873	·004 9873	·004 9875
150	·003 3277	·003 3277	·003 3277	·003 3278
200	·002 4969	·002 4969	·002 4969	·002 4969

TABLE III.

Values of $\sigma_\sigma : \bar{\sigma}$.

n size of sample	Accurate values	Approximations, given by my formulae, including terms of order		
		$1/n^{\frac{3}{2}}$	$1/n^{\frac{5}{2}}$	$1/n^{\frac{7}{2}}$
5	·3052	·3053	·3059	·3083
7	·2612	·2612	·2614	·2625
9	·2318	·2318	·2319	·2324
10	·2203	·2203	·2204	·2208
15	·1809	·1809	·1809	·1811
20	·1570	·1570	·1570	·1571
25	·1407	·1407	·1407	·1407
30	·1285	·1285	·1285	·1286
50	·0997	·0997	·0997	·0998
75	·0815	·0815	·0815	·0815
100	·0706	·0706	·0706	·0706
150	·0577	·0577	·0577	·0577
200	·0500	·0500	·0500	·0500

For the calculation of $\bar{\sigma}$, $\mu_2(\sigma)$ and σ_σ, the simple formulae

$$\left\{\begin{aligned} \bar{\sigma} &= \left(1 - \frac{3}{4n} - \frac{7}{32n^2}\right)\tilde{\sigma}, \\ \mu_2(\sigma) &= \frac{1}{2n}\left(1 - \frac{1}{4n}\right)\tilde{\sigma}^2, \\ \sigma_\sigma &= \frac{1}{\sqrt{2n}}\left(1 - \frac{1}{8n}\right)\tilde{\sigma} \end{aligned}\right. \qquad \text{.............................(22 } bis\text{)},$$

and also

$$\left\{\begin{aligned} \bar{\sigma} &= \left(1 - \frac{3}{4n} - \frac{7}{32n^2} - \frac{9}{128n^3}\right)\tilde{\sigma}, \\ \mu_2(\sigma) &= \frac{1}{2n}\left(1 - \frac{1}{4n} - \frac{3}{8n^2}\right)\tilde{\sigma}^2, \\ \sigma_\sigma &= \frac{1}{\sqrt{2n}}\left(1 - \frac{1}{8n} - \frac{25}{128n^2}\right)\tilde{\sigma} \end{aligned}\right. \qquad \text{........................(59)},$$

give us good estimates when n is not very small, as we can see in the tables, and we may therefore use these simple formulae in the calculation of $\bar{\sigma}$, $\mu_2(\sigma)$ and σ_σ, provided n is not small.

Next, let us examine the values of $\mu_3(\sigma)$ and $\mu_4(\sigma)$. From the equations (54), (55) and well-known relations between μ_r' and μ_r, moment coefficients about the mean, we can deduce

$$\mu_3(\sigma) = \frac{\bar{\sigma}}{n}\left\{1 - \frac{2n\mu_2(\sigma)}{\tilde{\sigma}^2}\right\}\tilde{\sigma}^2,$$

and $$\mu_4(\sigma) = \left\{\frac{5}{4n^2} - \frac{1}{2n}\left(4 - \frac{3}{n}\right)\left(1 - \frac{2n\mu_2(\sigma)}{\tilde{\sigma}^2}\right) - \frac{3}{4n^2}\left(1 - \frac{2n\mu_2(\sigma)}{\tilde{\sigma}^2}\right)^2\right\}\tilde{\sigma}^4 \text{...(60)*},$$

* These non-approximate equations were first deduced by K. Pearson; see *Biometrika*, Vol. x. p. 526. They were used to table the accurate values of $\beta_1(\sigma)$, $\beta_2(\sigma)$ for various sized samples.

which correspond to my formulae (43) for $\mu_3(\sigma)$ and $\mu_4(\sigma)$, i.e.

$$\mu_3(\sigma)=\frac{\tilde{\sigma}^3}{4n^2}\left(1+\frac{3}{4n}+\frac{11}{32n^2}\right),$$

$$\mu_4(\sigma)=\frac{3\tilde{\sigma}^4}{4n^2}\left(1-\frac{1}{2n}-\frac{7}{16n^2}\right) \quad \text{.....................(43 \textit{bis})},$$

as already found in Art. (11).

I have calculated the values of $\mu_3(\sigma)$ and $\mu_4(\sigma)$ given by (60), and also those given by (43 *bis*), and they are shown in the second and the third columns of Tables IV and V.

We can obtain also two degrees of approximation more, as before, according to whether we neglect terms of the equations (43 *bis*), of order $\frac{1}{n^4}$, or also those of order $\frac{1}{n^3}$. These values are given in the fourth and five columns of Tables IV and V.

It will be seen that the complete formulae (43 *bis*) give us very good approximations, even for quite small samples, while the approximation which neglects the terms in n^{-4} will often be quite adequate.

But the simplest formulae

$$\mu_3(\sigma)=\frac{1}{4n^2}\tilde{\sigma}^3$$

and

$$\mu_4(\sigma)=\frac{3}{4n^2}\tilde{\sigma}^4$$

do not give good results unless n is very large, and when it is the distribution of σ has become practically a normal curve.

TABLE IV.

Values of $\mu_3(\sigma)/\tilde{\sigma}^3$.

n size of sample	Accurate values	Approximations, given by my formulae, including terms of order		
		$1/n^4$	$1/n^3$	$1/n^2$
5	·011 5323	·011 6375	·011 5000	·010 0000
7	·005 6669	·005 6845	·005 6487	·005 1020
9	·003 3521	·003 3567	·003 3436	·003 0864
10	·002 6934	·002 6961	·002 6875	·002 5000
15	·001 1680	·001 1684	·001 1667	·001 1111
20	·000 6489	·000 6490	·000 6484	·000 6250
25	·000 4122	·000 4122	·000 4120	·000 4000
30	·000 2848	·000 2848	·000 2847	·000 2778
50	·000 1015	·000 1015	·000 1015	·000 1000
75	·000 0449	·000 0449	·000 0449	·000 0444
100	·000 0252	·000 0252	·000 0252	·000 0250
150	·000 0112	·000 0112	·000 0112	·000 0111
200	·000 0063	·000 0063	·000 0063	·000 0063

TABLE V.

Values of $\mu_4(\sigma)/\tilde{\sigma}^4$.

n size of sample	Accurate values	Approximations, given by my formulae, including terms of order		
		$1/n^4$	$1/n^3$	$1/n^2$
5	·026 541	·026 475	·027 000	·030 000
7	·014 086	·014 076	·014 213	·015 306
9	·008 607	·008 605	·008 745	·009 259
10	·007 094	·007 092	·007 125	·007 500
15	·003 210	·003 210	·003 222	·003 333
20	·001 826	·001 826	·001 828	·001 875
25	·001 175	·001 175	·001 176	·001 200
30	·000 819	·000 819	·000 819	·000 833
50	·000 297	·000 297	·000 297	·000 300
75	·000 132	·000 132	·000 132	·000 133
100	·000 075	·000 075	·000 075	·000 075
150	·000 033	·000 033	·000 033	·000 033
200	·000 019	·000 019	·000 019	·000 019

Finally, let us examine the values of $\beta_1(\sigma)$ and $\beta_2(\sigma)$. From the equations (59) and (60) Prof. K. Pearson deduced

$$\beta_1(\sigma)=\frac{\tilde{\sigma}^2\tilde{\sigma}^4}{n^2\mu_2(\sigma)^3}\left(1-\frac{2n\mu_2(\sigma)}{\tilde{\sigma}^2}\right)^2,$$

and $$\beta_2(\sigma)=\left(\frac{\tilde{\sigma}^2}{2n\mu_2(\sigma)}\right)^2\left\{5-2n\left(4-\frac{3}{n}\right)\left(1-\frac{2n\mu_2(\sigma)}{\tilde{\sigma}^2}\right)-3\left(1-\frac{2n\mu_2(\sigma)}{\tilde{\sigma}^2}\right)^2\right\}$$

$$\text{.........}(61),$$

which give the values of the second columns of Tables VI and VII

My corresponding formulae are

$$\beta_1(\sigma)=\frac{1}{2n}\left(1+\frac{9}{4n}+\frac{31}{8n^2}\right),\quad \beta_2(\sigma)=3+\frac{3}{4n^2}\quad \text{.........}(44\,bis),$$

as already found in Art. (11).

These expressions for β's lead to the figures in the third columns of Tables VI and VII, while the fourth and fifth columns result from the equations (44 *bis*), neglecting (i) the terms of the highest order in n^{-1}, and (ii) also the next highest order terms.

It will be seen, from these numerical values, that if n is not very small the accuracy of my full approximation for $\beta_1(\sigma)$ and $\beta_2(\sigma)$, as far as four decimal places, is very satisfactory. But the other less accurate approximations are not so and, especially, are quite inadequate for $\beta_2(\sigma)$ when n is small.

TABLE VI.

Values of $\beta_1(\sigma)$.

n size of sample	Accurate values	Approximations, given by my formulae, including terms of order $1/n^3$	$1/n^2$	$1/n$
5	·1646	·1605	·1450	·1000
7	·1011	·1000	·0944	·0714
9	·0725	·0721	·0694	·0556
10	·0634	·0632	·0613	·0500
15	·0390	·0389	·0383	·0333
20	·0281	·0281	·0278	·0250
25	·0219	·0219	·0218	·0200
30	·0180	·0180	·0179	·0167
50	·0105	·0105	·0105	·0100
75	·0069	·0069	·0069	·0067
100	·0051	·0051	·0051	·0050
150	·0038	·0038	·0038	·0033
200	·0028	·0028	·0028	·0025

TABLE VII.

Values of $\beta_2(\sigma)$.

n size of sample	Accurate values	Approximations, given by my formulae, including terms of order $1/n^2$	$1/n$	$(1/n)^0$	From formula (62)
5	3·0593	3·0300	3·0000	3·0000	3·0488
7	3·0251	3·0153	3·0000	3·0000	3·0221
9	3·0136	3·0093	3·0000	3·0000	3·0125
10	3·0106	3·0075	3·0000	3·0000	3·0098
15	3·0042	3·0033	3·0000	3·0000	3·0040
20	3·0022	3·0019	3·0000	3·0000	3·0022
25	3·0014	3·0012	3·0000	3·0000	3·0014
30	3·0009	3·0008	3·0000	3·0000	3·0009
50	3·0003	3·0003	3·0000	3·0000	3·0003
75	3·0001	3·0001	3·0000	3·0000	3·0001
100	3·0001	3·0001	3·0000	3·0000	3·0001
150	3·0000	3·0000	3·0000	3·0000	3·0000
200	3·0000	3·0000	3·0000	3·0000	3·0000

[*Note.*] From our approximation we cannot find the term in n^{-3} for $\beta_2(\sigma)$ exactly, but if in expanding $(\mu_2(\sigma))^{-2}$ when using (43 *b*) to calculate β_2, we retain one more term in our calculation, we get

$$\beta_2(\sigma) = 3 + \frac{3}{4n^2}\left(1 + \frac{25}{8n}\right) \qquad (62).$$

If we use this equation for the calculation of $\beta_2(\sigma)$, we get more accurate values of $\beta_2(\sigma)$, as indicated in the sixth column of Table VII, when the parent distribution is normal. But, from these results only, we cannot assert that this incomplete term in n^{-3} of $\beta_2(\sigma)$ should always be retained*.

(15) The numerical examination we have carried out in the foregoing article gives us some idea of the accuracy of the general approximate formulae (35), for if in the special case when the parent distribution is normal we get good results, we may anticipate a somewhat like adequacy in the more general result although the convergency is usually less satisfactory in non-normal cases.

Moreover, I may remark that calculations by the formulae (43) and (45) are all much simpler than those by the formulae deduced from the distribution law

$$y = A\sigma^{n-1}e^{-\frac{n\sigma^2}{2\bar{\sigma}^2}}.$$

Accordingly formulae (43) and (45) are themselves of value not only as special cases of the general formulae (35), but also for their adaptability to simple calculations. Thus they may be useful even in the case when the sampled population is normal but the Table† of the constants of the σ distribution is not at hand.

(16) The general equation (15) was obtained subject to only one condition, namely that the expansion of $\left(1+\frac{y}{\bar{\mu}_2}\right)^{\frac{1}{2}}$ under the integral in a series of ascending powers of y was justifiable. This would be true if

$$\frac{|y|}{\bar{\mu}_2} < 1 \quad \text{or} \quad \frac{|\mu_2 - \bar{\mu}_2|}{\bar{\mu}_2} < 1 \quad \text{......................(63).}$$

Let us consider more carefully what this implies. Since $\mu_2 \geqq 0$ and $\bar{\mu}_2 = \bar{\sigma}^2(N-1)/N$, the condition (63) may be written as follows:

$$\frac{N-1}{N}(1-\epsilon)\,\bar{\sigma}^2 \leqslant \mu_2 \leqslant \frac{N-1}{N}(1+\epsilon)\,\bar{\sigma}^2 \quad \text{..................(63'),}$$

where ϵ is a positive quantity, less than 1, but $1-\epsilon$ may be as small as we please. This means that our expansion would be justified if we exclude values of μ_2 whose deviations from the mean, $\bar{\mu}_2 = \frac{N-1}{N}\bar{\sigma}^2$, are numerically greater than

$$\epsilon\bar{\mu}_2 = \epsilon\frac{N-1}{N}\bar{\sigma}^2.$$

Suppose l_1 and l_2 are the lower and upper limits of $y = \mu_2 - \bar{\mu}_2$ in sampling, l_1 will presumably be $-\bar{\mu}_2$ while l_2 may for certain theoretical distributions at any

* [To calculate the true term of $\beta_2(\sigma)$ in n^{-3} requires us to find μ_1' to the *fifth* order term, or another term, that in n^{-5}, in Stirling's Theorem. A better approximation than (62) is:

$$\beta_2(\sigma) = 3\left(1+\frac{1}{4n^2}+\frac{1}{n^3}\right). \text{ Ed.]}$$

† *Biometrika*, Vol. x, p. 528.

rate take the value ∞, but practically l_2 is finite; then in the equation (11), $a_i\,(i=1,2,3,4)$ may be written

$$a_i=\int_{l_1}^{-\epsilon\bar{\mu}_2}\Phi(y)\left(1+\frac{y}{\bar{\mu}_2}\right)^{i/2}dy+\int_{-\epsilon\bar{\mu}_2}^{\epsilon\bar{\mu}_2}\Phi(y)\left(1+\frac{y}{\bar{\mu}_2}\right)^{i/2}dy+\int_{\epsilon\bar{\mu}_2}^{l_2}\Phi(y)\left(1+\frac{y}{\bar{\mu}_2}\right)^{i/2}dy$$
$$=f_1+f_2+f_3,\text{ say} \quad\ldots\ldots\ldots(64),$$

and, with the restriction (63), we retain only the second integral f_2 for a_i.

Thus we have first to consider the convergence of f_2.

Now $$f_2=1+\sum_{r=1}^{\infty}\frac{C_{i.r}}{\bar{\mu}_2^{\,r}}\int_{-\epsilon\bar{\mu}_2}^{\epsilon\bar{\mu}_2}\Phi(y)\,y^r\,dy \quad\ldots\ldots\ldots(65)$$

and $\Phi(y)$ is always positive and in most cases limited.

Hence if M be the upper limit of Φ, then

$$\frac{1}{\bar{\mu}_2^{\,r}}\int_{-\epsilon\bar{\mu}_2}^{\epsilon\bar{\mu}_2}\Phi(y)\,y^r\,dy\leqslant\frac{1}{\bar{\mu}_2^{\,r}}\int_{-\epsilon\bar{\mu}_2}^{\epsilon\bar{\mu}_2}My^r\,dy\leqslant\frac{2M\bar{\mu}_2}{r+1}\,\epsilon^{r+1} \quad\ldots\ldots\ldots(66),$$

and therefore the first expression in (66) decreases as r increases. Further $C_{i.r}\ (r\leqslant 4)$ is less than 1 for finite values of r, becomes alternatively positive and negative and is such that $\lim_{r\to\infty} C_{i.(r+1)}/C_{i.r}=1$. Hence we can see that the series (65) for f_2 is convergent.

Secondly, if l_1, l_2 lie within the interval $(-\epsilon\bar{\mu}_2, \epsilon\bar{\mu}_2)$, then since $\Phi(y)=0$ always in the intervals $(-\epsilon\bar{\mu}_2, l_1)$ and $(\epsilon\bar{\mu}_2, l_2)$,

$$f_1=f_3=0,$$

and the equation (11) becomes quite exact. If l_1 is out of the interval $(-\epsilon\bar{\mu}_2, \epsilon\bar{\mu}_2)$, as $1-\epsilon$ may be as small as we please, l_1 must be $-\bar{\mu}_2$ and

$$f_1=\int_{-\bar{\mu}_2}^{-\epsilon\bar{\mu}_2}\Phi(y)\,y^r dy\to 0 \text{ as } \epsilon\to 1.$$

Therefore we can say that f_1 is always negligible as compared with f_2.

Now ${}_2M_r$ being the rth moment coefficient about the mean of μ_2,

$${}_2M_r=\int_{l_1}^{l_2}\Phi(y)\,y^r\,dy=\int_{l_1}^{-\epsilon\bar{\mu}_2}\Phi(y)\,y^r\,dy+\int_{-\epsilon\bar{\mu}_2}^{\epsilon\bar{\mu}_2}\Phi(y)\,y^r\,dy+\int_{\epsilon\bar{\mu}_2}^{l_2}\Phi(y)\,y^r\,dy$$
$$=s_r+\int_{-\epsilon\bar{\mu}_2}^{\epsilon\bar{\mu}_2}\Phi(y)\,y^r\,dy+s_r',\text{ say} \quad\ldots\ldots\ldots(67).$$

But we can easily show that s_r is negligible compared with the second integral as before.

Accordingly $${}_2M_r=\int_{-\epsilon\bar{\mu}_2}^{\epsilon\bar{\mu}_2}\Phi(y)\,y^r\,dy+s_r' \quad\ldots\ldots\ldots(67').$$

It will be seen therefore that the justification of my equation (11), expressing $a_i\,(i=1,2,3,4)$ in terms of m_r or ${}_2M_r$, is based upon the assumption that *the integrals f_3 in (64) and s_r' in (67) are negligible compared with f_2 and ${}_2M_r$ respectively.*

If $l_2 \leqslant \epsilon\bar{\mu}_2$, this assumption becomes needless. Generally speaking the question whether the condition be satisfied or not must depend for its answer upon the form of $\Phi(y)$, but to obtain the general form of Φ, for which s_r' and f_s are negligible, is not an easy problem. We may note, however, that in usual cases for $|y| > \epsilon\bar{\mu}_2$, $\Phi(y)$ is very small, and since

$$\sigma_y = \frac{N-1}{N^{\frac{3}{2}}}\sqrt{\bar{\beta}_2 - 3 + \frac{2N}{N-1}}\,\bar{\sigma}^2 \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty,$$

Φ will tend steadily to zero as the sample size is increased. The results, given in Art. (14), suggest that at any rate for populations in the neighbourhood of the normal, where we have a direct check, the formulae I have developed on this assumption provide good approximations to the moments of the standard deviation.

A STUDY OF SEVENTY-ONE NINTH DYNASTY EGYPTIAN SKULLS FROM SEDMENT.

By T. L. WOO, Ph.D.

1. *Source of the Material.* In the winter 1920—21 the British School of Archaeology in Egypt, directed by Sir Flinders Petrie, excavated a large cemetery at Gebel Sedment. This site is 70 miles south of Cairo and it overlooks the Fayum. A report on the excavations has been published*. Graves of various dynasties ranging from the 1st to the 19th were examined. The majority of the interments belong to the 9th dynasty and 71 well-preserved skulls of that date were kindly presented to the Biometric Laboratory. These form the subject of the present paper. As far as is known the crania preserved were not selected in any way except that preference was given to complete specimens. The grave furniture did not suggest that any foreign elements were present in the population represented.

2. *The Nature of the Series and Remarks on Individual Crania.* The series from Sedment consists of 71 skulls of which the majority are well preserved and almost complete. There are 62 more or less complete mandibles. Several specimens which were damaged in transportation have been repaired. In general appearance the skulls do not differ markedly from those belonging to other dynastic Egyptian series and they appear to belong to a single racial type. There is no reason to believe that the sample is more heterogeneous than normal ones representing a single cemetery and a restricted period of time. Only one skull (No. 28†) possesses Negroid features.

There are 69 fully adult specimens. One (No. 20) is juvenile and no use was made of its measurements. Another (No. 64) is not quite adult as the basal suture is open, although all the 3rd molars are fully erupted: its measurements were used in computing the ♂ means. The series was sexed anatomically by Dr Morant and he distinguished 40 males and 30 females. These sexes could be compared with ones given in the archaeological report in 28 cases: there is agreement in 25 and the evidence of the skull alone was accepted in the other 3 cases (Nos. 12, 45 and 67). Several ♂ specimens are exceptionally massive and muscular (especially Nos. 7 and 39) and one ♀ is unusually small and feeble.

* Sir Flinders Petrie and Guy Brunton: *Sedment. British School of Archaeology in Egypt and Egyptian Research Account.* 27th year, 1921. 2 Vols. London, 1924.

† The skull numbers given in the text are the serial ones which were marked on the specimens in the Biometric Laboratory. The corresponding grave numbers are given in the tables of individual measurements (Appendix I).

Remarks on individual crania are given in the tables of measurements (Appendix I). The condition of the coronal, sagittal and lambdoid sutures was examined and, unless otherwise stated, it may be assumed that these three at least are all open. In order to obtain a rough estimate of the age constitution of the sample, division was made into the three groups for which frequencies are given in the table below. The divisions between them are not precise but a clear sexual difference is indicated. It is known that sutural closing normally begins at an earlier age for male than for female skulls*, so this is no evidence of a sexual difference between the mean ages at death. Individuals at different adult stages of development are evidently represented.

Condition of coronal, sagittal and lambdoid sutures	Males	Females
All open	6 (15·0 %)	16 (53·3 %)
Beginning to close or partly closed	19 (47·5 %)	10 (33·3 %)
All closed or nearly closed ...	15 (37·5 %)	4 (13·3 %)

Thinning of the calvarial walls due to age was noted in the case of 1 ♂ skull (No. 71). In the present series the majority of the skulls show the sutures closing in a definite order, the sagittal being first, the coronal second and the lambdoid last. A few specimens show the coronal closing before the sagittal. The lambdoid suture frequently closes before the coronal in the case of Western European crania. Three specimens have a complete metopic suture, contact being made between the right frontal and left parietal bones in 2 cases (Nos. 22 ♂ and 48 ♂) and between the left frontal and right parietal in the other (No. 67 ♀). Faint traces of the frontal suture were also found on 2 other skulls (Nos. 19 ♂ and 52 ♂). The frequency of this anomaly (4·3 %) is almost exactly the same as for the Badari Egyptians† (5·1 %) but not a single instance of it was found among 47 skulls of the 1st Dynasty‡. Metopism is more frequently met with in European series, as several have provided percentages ranging between 7 and 10.

It is commonly found that one or more wormian bones are present in the lambdoid suture in considerably more than 50 % of the specimens forming an unselected European series. Judging from the small samples which have been examined, these supernumerary bones are less frequently met with among Egyptian crania. Of the 40 ♂ skulls from Sedment, 15 have wormians, and of the 30 ♀ skulls there are also 15 with one or more wormian bones. The following ossicles were found in other positions, the sexes being pooled: 7 of the lambda, 8 in

* See J. Frédéric: "Untersuchungen über die normale Obliteration der Schädelnähte." *Zeitschrift für Morphologie und Anthropologie*, Band IX. (1906), S. 373—456. See especially S. 439 and S. 442—448.

† B. Stoessiger: "A Study of the Badarian Crania recently excavated by the British School of Archaeology in Egypt." *Biometrika*, Vol. XIX. (1927), pp. 110—150.

‡ G. M. Morant: "A Study of Egyptian Craniology from Prehistoric to Roman Times." *Ibid.* Vol. XVII. (1925), pp. 1—52.

the sagittal, 2 in the coronal suture, 3 at the right asterion and 1 at both asteria. There are no ossicles of the bregma. Inter-parietal bones are also lacking and only 2 skulls (Nos. 59 and 65) show traces of the transverse occipital suture. Traces of the suture between the ex- and supra-occipital bones were also found in 2 cases (Nos. 30 and 49). There is one ♀ specimen with fronto-temporal articulation on the right side (No. 51) among 67 which could be examined for this feature. Epipteric bones appear to be unusually common: 9 cases of one or more epipteric bones on one or both sides were found among 32 ♂ specimens and 12 cases among 25 ♀ specimens. One ♂? skull (No. 63) has a large protruding ossicle above the right parieto-mastoid suture (see Plate V B).

In general the teeth are in a remarkably good state of preservation. A large proportion of the skulls, including ageing and aged specimens, was found with dentition complete and free from disease. Most of the teeth are markedly worn. A few cases were noted of adult skulls with no 3rd molars in one or both jaws. Three carious molars were the only diseased teeth found.

The frequency with which palate bridges occur was not observed, as experience has shown that they are often broken before the skull is examined in the laboratory and it is not possible then to tell whether the original bridge was complete or not. One unusual case of multiple bridges (No. 65) may be noted

The existence of single or double precondyles on the basi-occipital was recorded. We found 5 cases of double and 3 of single precondyles.

There is one case of marked calvarial asymmetry (No. 26) and one of asymmetry of the facial skeleton (No. 58). The sizes of the jugular foramina were compared: *JR* denotes that the right is the greater, *JL* that the left is the greater and *J=* that no difference can be discerned between the sizes of the two. The following frequencies are given:

	Male	Female
JR	24	14
J=	3	7
JL	8	6

Every cranial series which has been examined in this way has shown that the right jugular foramen tends to be larger than the left. The question of calvarial asymmetry is discussed in the section on contour measurements below.

All cases of tympanic perforation have been recorded. There are 11 with perforation of the plate on both sides and 6 others with perforation on one side or the other. The percentage frequencies appear to be unusually high.

Few uncommon anomalies of particular interest were observed. One skull (No. 33) has a small exostosis on the supra-occipital, another (No. 10) has an excrescence behind the left condyle. One ♂ specimen (No. 65) has a markedly retreating frontal bone, a prominent superciliary ridge and also a basi-occipital incisure on the left side (see Plate V A).

3. Comparison with other Series by the Method of the Coefficient of Racial Likeness.

Measurements of the series were determined according to the technique used by previous workers in this Laboratory and the usual index letters are employed*. The ♂ and ♀ mean measurements are in Tables I and II respectively and the individual measurements in Appendix I. As the sample is a small one, all the standard deviations were not calculated, but comparison is made below between male values for six characters and the corresponding standard deviations which have been given for the series of 900 ♂ Egyptian skulls of the 26th—30th Dynasties†.

	L	B	H'	$100\,B/L$	$100\,H'/L$	$100\,B/H'$
9th Dynasty Egyptians from Sedment	5·85 ± ·44	3·98 ± ·30	4·36 ± ·39	2·71 ± ·20	2·99 ± ·23	3·55 ± ·27
26th—30th Dynasty Egyptians from Gizeh	5·72 ± ·09	4·76 ± ·08	5·03 ± ·08	2·68 ± ·06	2·94 ± ·05	4·30 ± ·06

This comparison suggests that the Sedment series is not more variable than the Gizeh one and the latter is known to be more homogeneous than most cranial series available.

Professor Karl Pearson's method of the coefficient of racial likeness was used to determine the racial affinities of our sample. This is defined for practical purposes to be:

$$\frac{1}{m} S \left[\frac{(M_s - M'_s)^2}{\sigma_s^2} \times \frac{n_s n'_s}{n_s + n'_s}\right] - 1 + \frac{1}{m} \pm \cdot 67449 \sqrt{\frac{2}{m}},$$

where M_s is the mean based on n_s skulls for the first series, M'_s and n'_s are the corresponding constants for the second series and m characters are compared. The σ's of the long 26th—30th Dynasty Egyptian series were used throughout‡ and the coefficients were calculated for the 31 characters usually employed, or for as many of them as were available. The coefficient may be written:

$$\frac{1}{m} S(\alpha) - 1 + \frac{1}{m} \pm \cdot 67449 \sqrt{\frac{2}{m}},$$

where
$$\alpha = \frac{(M_s - M'_s)^2}{\sigma_s^2} \times \frac{n_s n'_s}{n_s + n'_s}.$$

The mean number of skulls available for the characters used is denoted by $\bar{n}_s$ in the case of the first sample and $\bar{n}'_s$ in the case of the second sample and these "sizes" of the samples are usually unequal and may be of very different orders. The values of the coefficients are largely determined by the "sizes" of the

* Definitions of the measurements will be found in any recent volume of *Biometrika*.

† Karl Pearson and Adelaide G. Davin: "On the Biometric Constants of the Human Skull." *Biometrika*, Vol. XVI. (1924), pp. 328—363.

‡ *Ibid.* Vol. XVI. (1924), pp. 358 and 359.

TABLE I.

Mean Male Measurements of the Sedment and related Series.

Characters	9th Dynasty Egyptians: Sedment (Woo)	4th and 5th Dynasty Egyptians: Deshasheh and Medum (Thomson and MacIver)	Modern Cretans* (v. Luschan)	18th—20th Dynasty Egyptians: Thebes (Stahr)	Middle Dynastic Egyptians: El-Kubanieh North (Toldt)	Modern Egyptians*: Cairo (Schmidt)	Modern Abyssinians: Tigre district (Sergi)
C	1426·9 (35)	—	—	1451·0 (50)	1355·0 (27)	1348·0 (47)	—
F	179·6 (40)	183·0 (30)	—	—	—	—	—
L	181·9 (40)	184·9 (54)	180·2 (53)	183·5 (57)	182·2 (37)	176·1 (47)	183·7 (65)
B	138·3 (40)	139·3 (54)	139·4 (53)	137·3 (56)	134·2 (37)	137·6 (40)	136·3 (65)
B'	92·6 (40)	—	96·1 (52)	94·9 (50)	90·4 (37)	94·3 (47)	94·6 (65)
H	138·2 (38)	—	—	—	—	136·0 (46)	—
H'	137·4 (38)	136·0 (50)	137·5 (52)	133·7 (54)	135·9 (35)	134·6 (46)	133·5 (64)
OH	115·3 (39)	—	—	114·1 (56)	116·1 (36)	—	112·6 (65)
LB	100·8 (37)	101·8 (47)	101·5 (40)	101·9 (56)	101·6 (34)	100·9 (46)	99·8 (64)
Q'	314·9 (39)	—	—	313·5 (52)	—	312·0 (46)	309·8 (65)
S	373·8 (36)	—	—	372·1 (51)	372·7 (33)	364·8 (46)	372·7 (58)
S_1	129·0 (39)	—	—	125·7 (54)	126·0 (36)	—	130·2 (65)
S_2	129·9 (40)	—	—	129·8 (50)	130·9 (33)	—	128·9 (65)
S_3	115·0 (37)	—	—	114·7 (54)	115·1 (30)	—	113·7 (57)
S_1'	113·3 (39)	—	—	111·9 (50)	—	—	112·0 (65)
S_2'	115·6 (40)	—	—	116·1 (52)	—	—	114·9 (65)
S_3'	98·6 (37)	—	—	96·1 (54)	97·3 (30)	95·4 (46)	95·8 (57)
U	511·7 (40)	—	—	512·6 (54)	505·6 (35)	501·3 (46)	513·0 (65)
$G'H$	71·5 (38)	71·3 (19)	69·1 (49)	70·1 (56)	70·1 (29)	69·8 (46)	69·1 (65)
GB	93·9 (37)	—	—	95·5 (54)	94·0 (29)	94·8 (43)	93·1 (57)
J	127·2 (29)	127·1 (16)	129·7 (51)	128·6 (53)	125·5 (26)	127·9 (44)	126·0 (42)
NH'	51·4 (38)	52·0 (44)	50·7 (49)	50·8 (56)	49·8 (37)	50·2 (46)	48·9 (64)
NB	24·5 (39)	25·1 (43)	24·2 (49)	25·3 (53)	24·8 (36)	24·8 (45)	25·0 (63)
DC	21·2 (32)	—	21·6 (48)	21·1 (52)	23·4 (34)	—	22·0 (66)
SC	9·5 (37)	—	—	10·8 (52)	—	—	9·5 (64)
O_1'	(R) 39·7 (35)	—	—	38·3 (59)	38·1 (35)‡	—	39·4 (65)
O_2	(R) 33·3 (40)	—	—	32·8 (53)	32·3 (35)	32·5 (46)	33·2 (65)
G_1'	46·7 (36)	—	—	—	46·4 (25)	46·6 (41)	—
G_2	40·1 (31)	—	—	40·5 (46)	—	40·3 (39)	39·0 (61)
GL	96·0 (37)	96·3 (42)	93·3 (32)	96·1 (53)	96·5 (25)	97·5 (45)	95·2 (64)
fml	35·9 (37)	—	—	35·2 (55)	34·9 (32)	35·0 (46)	36·1 (57)
fmb	30·1 (38)	—	—	29·7 (54)	29·2 (32)	29·0 (46)	29·5 (59)
$100\,B/L$	76·1 (40)	{75·4 (50)}†	77·5 (53)	74·8 (54)	73·7 (37)	{78·1 (46)}	74·3 (65)
$100\,H'/L$	75·5 (38)	{72·6 (50)}	{76·3 (52)}	72·8 (52)	74·7 (35)	{76·4 (46)}	72·7 (64)
$100\,B/H'$	100·9 (38)	{102·4 (50)}	{101·4 (52)}	{102·7 (54)}	{98·7 (35)}	{102·3 (46)}	{102·1 (64)}
$100(B-H')/L$	+0·6 (38)	{+1·8 (50)}	{+1·1 (52)}	{+2·0 (54)}	{−0·9 (35)}	{+1·7 (46)}	{+1·5 (64)}
$100\,G'H/GB$	76·3 (36)	—	—	73·9 (49)	{73·9 (29)}	73·7 (41)	73·9 (57)
$100\,NB/NH'$	47·7 (38)	{48·2 (44)}	47·7 (49)	50·5 (49)	49·9 (36)	49·4 (45)	51·1 (63)
$100\,O_2/O_1'$	(R) 84·1 (35)	—	—	86·5 (53)	84·9 (35)	—	84·3 (66)
$100\,G_2/G_1'$	85·6 (30)	—	—	—	—	87·0 (36)	—
$100\,fmb/fml$	84·1 (37)	—	—	{84·4 (54)}	84·0 (31)	83·0 (46)	{81·7 (57)}
$Oc.\ I.$	64·0 (37)	—	—	60·7 (54)	61·9 (30)	—	62·0 (57)
$N\angle$	65°·1 (36)	{64°·6 (19)}	{63°·3 (32)}	{64°·5 (53)}	{65°·2 (25)}	{66°·7 (45)}	{65°·4 (64)}
$A\angle$	72°·4 (36)	{73°·4 (19)}	{74°·6 (32)}	{74°·0 (53)}	{73°·4 (25)}	{72°·3 (45)}	{73°·2 (64)}
$B\angle$	42°·5 (36)	{42°·0 (19)}	{41°·9 (32)}	{41°·5 (53)}	{41°·4 (25)}	{41°·0 (45)}	{41°·4 (64)}
θ_1	30°·6 (36)	—	—	{30°·3 (53)}	{29°·2 (25)}	—	{30°·4 (64)}
θ_2	11°·9 (36)	—	—	{11°·2 (53)}	{12°·2 (25)}	—	{11°·0 (64)}
$P\angle$	84°·4 (37)	—	—	85°·2 (55)	85°·6 (28)	—	84°·2 (65)

* The means of these series calculated by Dr Morant have not been previously published.

† The indices and angles in curled brackets were obtained from the means of the component lengths instead of from individual values.

‡ O_1=40·8 (36).

TABLE II.

Mean Female Measurements of the Sedment and related Series.*

Characters	9th Dynasty Egyptians: Sedment (Woo)	18th—20th Dynasty Egyptians†: Thebes (Stahr)	Modern Egyptians†: Cairo (Schmidt)	Modern Abyssinians†: Tigre district (Sergi)	6th–12th Dynasty Egyptians†: Denderah (Thomson and MacIver)	12th–15th Dynasty Egyptians†: Hou & Abydos (Thomson and MacIver)	18th Dynasty Egyptians†: Abydos (Thomson and MacIver)
C	1262·5 (25)	1284·8 (43)	1211·4 (26)	1318·4 (23)	—	—	—
F	173·1 (30)	—	—	—	174·6 (149)	174·9 (88)	176·8 (67)
L	172·9 (30)	176·2 (48)	171·2 (26)	175·1 (24)	174·8 (149)	175·3 (88)	177·5 (67)
B	133·5 (30)	133·2 (44)	131·3 (26)	129·4 (24)	130·3 (147)	131·4 (88)	131·5 (67)
B'	87·4 (30)	90·4 (48)	91·7 (26)	89·8 (24)	—	—	—
H	131·7 (30)	—	128·8 (26)	—	—	—	—
H'	130·6 (30)	127·0 (43)	126·7 (26)	126·5 (22)	129·5 (147)	127·7 (88)	128·1 (66)
OH	110·4 (30)	109·2 (45)	—	107·2 (23)	—	—	—
LB	91·7 (30)	97·1 (46)	96·0 (26)	94·9 (22)	96·3 (141)	95·0 (88)	95·8 (65)
Q'	301·5 (30)	299·5 (44)	296·1 (26)	295·8 (23)	—	—	—
S	367·1 (27)	354·0 (41)	351·6 (26)	355·4 (22)	—	—	—
S_1	123·2 (30)	121·0 (42)	—	123·0 (24)	—	—	—
S_2	124·7 (30)	124·3 (42)	—	122·5 (24)	—	—	—
S_3	111·1 (27)	110·0 (45)	—	110·0 (22)	—	—	—
S_1'	107·8 (30)	106·5 (42)	—	106·7 (24)	—	—	—
S_2'	110·8 (30)	111·2 (42)	—	110·0 (24)	—	—	—
S_3'	95·4 (27)	93·8 (45)	91·4 (26)	93·0 (22)	—	—	—
U	490·4 (30)	494·2 (46)	484·9 (26)	489·3 (24)	—	—	—
$G'H$	67·1 (29)	66·3 (46)	63·5 (26)	65·5 (24)	66·4 (136)	67·3 (88)	67·5 (66)
GB	89·4 (28)	91·7 (43)	92·9 (26)	91·0 (20)	—	—	—
J	117·3 (25)	121·2 (43)	120·6 (26)	116·3 (13)	118,3 (121)	118·8 (82)	121·4 (64)
NH'	48·4 (29)	48·0 (45)	45·3 (26)	47·5 (24)	48·5 (136)	48·4 (87)	49·2 (67)
NB	23·6 (29)	24·3 (43)	24·6 (26)	24·5 (24)	24·9 (137)	24·4 (87)	25·1 (67)
DC	19·7 (24)	21·1 (42)	—	20·5 (23)	—	—	—
SC	10·0 (29)	10·6 (39)	—	9·3 (23)	—	—	—
$O_1\,(R)$	40·4 (29)	—	—	—	—	—	—
$O_1\,(L)$	39·8 (29)	—	—	—	—	—	—
$O_1'\,(R)$	37·4 (26)	37·1 (45)	—	38·5 (20)	—	—	—
$O_2\,(R)$	32·7 (30)	33·0 (46)	32·5 (26)	32·7 (23)	—	—	—
$O_2\,(L)$	32·7 (29)	—	—	—	—	—	—
G_1	47·4 (23)	—	—	—	—	—	—
G_1'	43·5 (24)	—	46·0 (24)	—	—	—	—
G_2	67·9 (24)	48·0 (39)	38·7 (24)	38·0 (23)	—	—	—
GL	89·7 (29)	92·3 (44)	95·0 (26)	91·1 (22)	92·4 (138)	90·6 (88)	91·7 (65)
fml	33·7 (27)	34·1 (43)	33·5 (26)	34·2 (21)	—	—	—
fmb	28·4 (28)	28·7 (45)	27·0 (26)	27·9 (21)	—	—	—
$100\,B/L$	77·2 (30)	75·5 (44)	{76·7 (26)}	74·0 (24)	{74·5 (147)}	{75·0 (88)}	{75·8 (67)}
$100\,H'/L$	75·5 (30)	72·6 (39)	{74·0 (26)}	72·5 (22)	{74·1 (147)}	{72·8 (88)}	{72·2 (66)}
$100\,B/H'$	102·3 (30)	{104·1 (43)}‡	{103·6 (26)}	{102·3 (22)}	{100·6 (147)}	{102·9 (88)}	{106·0 (66)}
$100\,(B-H')/L$	+1·7 (30)	{+3·0 (39)}	{+2·7 (26)}	{+1·7 (22)}	{+0·5 (147)}	{+2·1 (88)}	{+3·6 (66)}
$100\,G'H/GB$	74·9 (28)	71·9 (37)	68·5 (26)	71·8 (20)	—	—	—
$100\,NB/NH'$	48·9 (29)	50·3 (40)	54·6 (26)	51·7 (24)	{51·3 (137)}	{50·4 (87)}	{51·0 (67)}
$100\,O_2/O_1\,(R)$	81·2 (29)	—	—	—	—	—	—
$100\,O_2/O_1\,(L)$	82·2 (29)	—	—	—	—	—	—
$100\,O_2/O_1'\,(R)$	87·3 (26)	88·7 (45)	—	85·3 (20)	—	—	—
$100\,G_2/G_1$	80·2 (18)	—	—	—	—	—	—
$100\,G_2/G_1'$	87·7 (19)	—	84·4 (24)	—	—	—	—
$100\,fmb/fml$	84·4 (25)	{84·2 (43)}	81·0 (26)	{81·6 (21)}	—	—	—
$Oc.\ I.$	63·5 (27)	63·4 (45)	—	61·7 (22)	—	—	—
$N\angle$	64°·5 (29)	{65°·7 (44)}	{69°·6 (26)}	{66°·2 (22)}	{66°·1 (136)}	{64°·9 (88)}	{65°·3 (65)}
$A\angle$	72°·9 (29)	{73°·2 (44)}	{71°·5 (26)}	{72°·4 (22)}	{72°·7 (136)}	{72°·4 (88)}	{72°·5 (65)}
$B\angle$	42°·6 (29)	{41°·1 (44)}	{38°·9 (26)}	{41°·4 (22)}	{41°·2 (136)}	{42°·7 (88)}	{42°·2 (65)}
θ_1	30°·8 (28)	{28°·9 (44)}	—	{28°·9 (22)}	—	—	—
θ_2	11°·6 (28)	{12°·2 (44)}	—	{12°·5 (22)}	—	—	—
$P\angle$	84°·4 (28)	85°·4 (45)	—	84°·9 (22)	—	—	—

* The ♀ mean measurements of the El-Kubanieh North and South series compared in Table III are given in *Biometrika*, Vol. xix. (1927), Table II facing p. 117.

† The means of these series have not been previously published.

‡ The indices and angles in curled brackets were obtained from the means of the component lengths instead of from individual values.

samples which happen to be available, and some method is needed to eliminate this factor as we wish to obtain, as far as possible, a measure of the absolute divergence of the types compared which does not depend on the numbers of skulls. The correction needed has been given by Professor Pearson in a recent paper*.

* "Note on Standardisation of Method of using the Coefficient of Racial Likeness." *Biometrika*, Vol. xx^B. (1928), pp. 376—378. [I am by no means satisfied with this so-called correction or reduction, but was urged to it by the criticisms of Dr Morant and Professor Mahalanobis as possibly the best *at present* available. The coefficient of racial likeness as originally proposed by me was a measure by which we might *roughly* judge the likelihood that two cranial series were samples of the same population. I, personally, should lay no weight on any "crude" coefficient of the order three or less obtained from not more than 50 crania in each series as indicating a racial difference. Crania from two adjacent London burial places of the same period will give coefficients of this order. Such coefficients may easily arise from the differences in personal equation of two measurers, or from differences of class or environment of the two populations under consideration. We have also to remember the neglected correlation terms of our expression for the C.R.L. It is not alone the error of random sampling which has to be weighed.

Suppose three series A, B, C, then if the C.R.L. of A and C be much larger than that of A and B we are justified in supposing that *to the extent of the information derivable from the given data* (and beyond that we cannot get) C is racially more remote from A, than B is. To this Dr Morant and Professor Mahalanobis reply: "Yes, but the C.R.L. increases when the numbers of one or both series of crania increase, and therefore it is dependent on the numbers used, as well as on racial divergence." To illustrate this I take the 18 cases of Mr Woo's Table III and rank them (a) in average number of crania measured, (b) in order of crude coefficients, (c) in order of reduced or corrected coefficients. If ρ be the correlation of ranks we have:

$$\rho_{ab} = \cdot 7882, \quad \rho_{ac} = \cdot 2289,$$

and accordingly for the variates approximately:

$$r_{ab} = \cdot 7972, \quad r_{ac} = \cdot 2389.$$

The reduction has therefore very sensibly reduced the correlation between the numbers of crania used and the magnitude of the coefficient. Thus far the argument in favour of the reduction seems valid, but we have to consider how the changes in rank which lessen the correlation between the C.R.L.'s and the numbers of crania used have been achieved.

The series with the lowest numbers are: 18th Dynasty Egyptians (23·9), Bronze Age Spanish (24·2), Middle Dynastic Egyptians (33·1), 1st Dynasty Egyptians (33·6) and 4th—5th Dynasty Egyptians (39·9), giving the "crude" coefficients 2·74, 4·58, 2·74, 6·24 and 1·95 respectively. These occur in the ranking of the crude coefficients in the 3rd, 9th, 2nd, 12th and 1st places. In ranking of the reduced coefficients they occupy the 8th, 15th, 3rd, 17th and 1st places respectively. There is thus for the series with the fewest crania an average change of 3·4 places in the ranking. If we take the five series with the largest numbers of crania, i.e. 26th—30th Dynasty Egyptians (685·1), Modern Portuguese (494·0), 17th and 18th century Maltese (439·4), the 6th—12th Dynasty Egyptians (168·6), and the Naqada series (66·1), their ranks in the crude coefficients are 17th, 16th, 18th, 14th and 15th and these obtain with the reduced coefficients the ranks 14th, 10th, 16th, 9th and 18th or an average loss of 3·8 places. If we take the 6th—9th ranks in numbers, we find without regard to sign an average change of *one* place only, and if we take the 10th to the 13th as the next highest series of numbers the average displacement in rank is again *one*. Hence it would appear that, not only as we should expect are the series with lowest and highest numbers most changed in magnitude of coefficient by the correction, but it is these series which are also most changed in their ranking. It is, however, in the values for the high numbers of crania in the crude coefficients that we are most sure of the order of significance, and it is the coefficients with the least frequency in the data, of which we are most uncertain, which are brought into high places in the series. The average number of crania compared with the Sedment series is here about 147. Mr Woo reduces to 100. I chose 75 because it seemed a fair average of the series then before me and it is desirable to raise the reduced coefficients as little as possible above the crude. The present note is only one of warning. What counsel can we draw from it? I think it is: Be moderate in the emphasis laid on the order of "reduced" coefficients, and lay no, or very little, emphasis on series with coefficients

The coefficient is first calculated by the formula above, this being known as the crude coefficient, and it is then reduced to the value it would have if all the means of both series in the comparison were based on 100 individuals. This reduced value is:

$$50 \times \frac{\bar{n}_s + \bar{n}'_s}{\bar{n}_s \times \bar{n}'_s}\left[\frac{1}{m} S(\alpha) - 1 + \frac{1}{m} \pm \cdot 67449 \sqrt{\frac{2}{m}}\right].$$

Both crude and reduced coefficients are given below for all the comparisons made.

In a paper published in this journal in 1925 Dr Morant gave the mean measurements of a considerable number of early Egyptian series calculated from the data provided by a number of different anthropologists*. He suggested that two extreme forms of skull could be distinguished—known respectively as the Upper and Lower Egyptian types—but these extremes could be linked together by a number of intermediate types so that all the material then available formed a fairly continuous series. A close resemblance could be found between every sample and one, or more, of the other samples. A map given (*loc. cit.* p. 3) showed the distribution of the sites from which the series were obtained. The majority of them are in Upper Egypt and these range from Early Predynastic to Roman times. There are only four from Lower Egypt, ranging from the 4th Dynasty to the Ptolemaic Period. Sedment is more southerly than any other of the sites in Lower Egypt, but it is nearly 130 miles from the most northerly of the Upper Egyptian sites (Qau). I endeavoured to find the closest relationships, i.e. the lowest reduced coefficients of racial likeness, between the Sedment series and any others available, whether Egyptian or not. In dealing with European and Egyptian types it has been repeatedly observed that the most significant differences are almost always found for the length, breadth and height of the calvaria and the three major indices derived from these chords. Facial measurements are generally far less differentiated. By taking this fact into account, it is possible to select at once the series which will certainly not provide low coefficients of racial likeness. The following arbitrarily fixed ranges were suggested by Dr Morant and they were derived from his study of the inter-relationships of 39 European and 2 Egyptian cranial series†.

	L	B	H'	$100\,B/L$	$100\,H'/L$	$100\,B/H'$
Sedment ♂ Mean Ranges	181·9 175·4—188·4	138·3 131·8—144·8	137·4 132·9—141·9	76·1 71·6—80·6	75·5 72·0—79·0	100·9 95·9—105·9

below three, or on those containing fewer than 40 to 50 crania. Above all it is desirable on the earliest possible opportunity to take numerous samples of 20 to 60 individuals from a long series of skulls like that of the 26th—30th Dynasty Egyptians, and find their coefficients of racial likeness with each other and with the entire parent population. It would be a long task, but would throw much light on present difficulties. EDITOR.]

* "A Study of Egyptian Craniology from Prehistoric to Roman Times." *Biometrika*, Vol. XVII. (1925), pp. 1—52.

† "A Preliminary Classification of European Races based on Cranial Measurements." *Ibid.* Vol. XX^B. (1928), pp. 301—375.

TABLE III.

Coefficients of Racial Likeness with the Male and Female Series from Sedment.

Races compared	9th Dynasty Egyptians (Sedment)									
	Crude Coefficients						Reduced Coefficients			
	Males ($\bar{n}_s=37{\cdot}1$)			Females ($\bar{n}_s=28{\cdot}0$)			Males		Females	
	$\bar{n}'_s$	All Characters	Indices and Angles	$\bar{n}'_s$	All Characters	Indices and Angles	All Characters	Indices and Angles	All Characters	Indices and Angles
th and 5th Dynasty Egyptians: Deshasheh and Medum (Thomson and MacIver) ...	39·9	1·95 ± ·25 (14)	3·64 ± ·39 (6)	—	—	—	5·07	9·47	—	—
Modern Cretans (von Luschan)	47·9	2·84 ± ·25 (15)	2·72 ± ·39 (6)	—	—	—	6·79	6·51	—	—
Middle Dynastic Egyptians: El-Kubanieh North (Toldt)	33·1	2·74 ± ·18 (28)	2·72 ± ·29 (11)	18·6	0·92 ± ·18 (28)	3·11 ± ·29 (11)	7·83	7·77	4·10	13·87
8th—20th Dynasty Egyptians: Thebes (Stahr)	52·8	3·64 ± ·18 (29)	6·18 ± ·29 (11)	43·7	2·69 ± ·18 (29)	3·34 ± ·29 (11)	8·38	14·18	7·88	9·79
Modern Egyptians: Cairo (Schmidt) ...	44·9	3·41 ± ·18 (27)	2·62 ± ·32 (9)	25·8	7·21 ± ·18 (27)	11·10 ± ·32 (9)	8·39	6·45	26·85	41·33
Modern Abyssinians: Tigre District (Sergi)	61·8	4·52 ± ·18 (28)	5·68 ± ·29 (11)	22·1	3·03 ± ·18 (28)	4·60 ± ·29 (11)	9·17	12·25	12·27	18·62
Early and Middle Dynastic Egyptians: El-Kubanieh South (Toldt)	63·5	4·35 ± ·18 (28)	5·49 ± ·29 (11)	42·6	4·03 ± ·18 (28)	6·23 ± ·29 (11)	9·29	11·72	11·88	18·36
8th Dynasty Egyptians: Thebes? (Schultz and Oetteking)	23·9	2·74 ± ·17 (31)	4·05 ± ·28 (12)	—	—	—	9·43	13·93	—	—
th—12th Dynasty Egyptians: Denderah (Thomson and MacIver)	168·6	7·45 ± ·25 (14)	8·69 ± ·39 (6)	140·4	5·74 ± ·24 (14)	8·08 ± ·39 (6)	12·25	14·29	12·29	17·31
Modern Portuguese (Ferraz de Macedo) ...	494·0	8·93 ± ·22 (18)	16·36 ± ·43 (5)	—	—	—	12·94	23·70	—	—
8th Dynasty Egyptians: Abydos (Thomson and MacIver)	50·0	5·59 ± ·25 (14)	7·57 ± ·39 (6)	66.1	7·24 ± ·25 (14)	7·61 ± ·39 (6)	13·12	17·77	18·40	19·35
anary Islands Modern (Lajard)	50·0	5·62 ± ·22 (18)	3·85 ± ·43 (5)	50·0	8·14 ± ·22 (18)	4·51 ± ·43 (5)	13·19	9·04	22·68	12·56
2th—15th Dynasty Egyptians: Hou and Abydos (Thomson and MacIver) ...	65·9	6·29 ± ·25 (14)	8·55 ± ·39 (6)	87·4	3·76 ± ·25 (14)	6·82 ± ·39 (6)	13·25	18·01	8·87	16·08
6th—30th Dynasty Egyptians: Gizeh (Davin and Pearson)	885·4	10·11 ± ·17 (31)	15·72 ± ·28 (12)	567·3	5·83 ± ·17 (31)	4·45 ± ·28 (12)	14·20	22·07	10·92	8·34
ronze Age Spanish (Jacques)	24·2	4·58 ± ·21 (21)	6·89 ± ·39 (6)	33·8	10·11 ± ·21 (21)	12·36 ± ·39 (6)	15·64	23·52	33·01	40·36
7th and 18th century Maltese (Buxton) ...	439·4	11·11 ± ·21 (20)	6·34 ± ·34 (8)	—	—	—	16·24	9·27	—	—
t Dynasty Egyptians, Private Tombs: Abydos (Thomson and MacIver) ...	33·6	6·24 ± ·25 (14)	8·27 ± ·39 (6)	55·1	10·44 ± ·25 (14)	13·84 ± ·39 (6)	17·70	23·45	27·35	36·26
redynastic Egyptians: Naqada *A* and *Q* series (Fawcett)	66·1	8·46 ± ·17 (31)	12·50 ± ·28 (12)	109·5	7·92 ± ·19 (25)	12·75 ± ·32 (9)	17·80	26·30	17·76	28·59

If the ♂ mean for any series falls outside the range indicated for any one, or more, of the 6 characters then it will be safe to assume that there is no close relationship between it and the Sedment series. Comparison was made in this way with about 80 ♂ European and 24 ♂ ancient and modern Egyptian types. All except 23 could be excluded on account of the aberrance of one or other of the 6 measurements compared, and it was only thought necessary to calculate the coefficients of racial likeness with ten predynastic and dynastic Egyptian, three modern Egyptian, eight European, one Abyssinian and one Canary Island series. The reduced values of these twenty-three coefficients range from 5·07 to 42·72 and the eighteen values less than 20 are arranged in order in Table III. We may feel confident that comparison is made there with all the series available which are most closely related to the Sedment one. All the coefficients which could be calculated with the ♀ Sedment means are also in Table III. Male means of the most closely related series are in Table I and female means in Table II*. Comparisons are made in Table III between the ♂ Sedment series and 18 others. The coefficients of racial likeness are given for the standard set of 31 characters, or as many of them as

* The comparative material used in the present paper was derived from the following sources:

(i) Thomson, A. and Randall-MacIver, D.: *The Ancient Races of the Thebaid*. Oxford (1905). Use is made of five series for which individual measurements are given in this work. The ♂ means of these and of all the other ancient Egyptian and the Abyssinian series are given in *Biometrika*, Vol. XVII. (1925), pp. 14—36.

(ii) Stahr, Hermann: *Die Rassenfrage im antiken Aegypten. Kraniologische Untersuchungen an Mumienköpfen aus Theben*. Leipzig (1907).

(iii) Toldt, C.: "Anthropologische Untersuchung der menschlichen Ueberreste aus den altägyptischen Gräberfeldern von El-Kubanieh." *Denkschriften der Akademie der Wissenschaften in Wien, Math.-naturwiss. Klasse*, Band XCVI. (1919), S. 593—672.

(iv) Schmidt, Emil: *Die anthropologischen Sammlungen Deutschlands*. Leipzig Catalogue (1887), also published with *Archiv für Anthropologie*, Band XVII.

(v) Sergi, Sergio: *Crania Habessinica. Contributo all' Antropologia dell' Africa Orientale*. Rome (1912).

(vi) Oetteking, B.: "Kraniologische Studien an Altägypten." *Archiv für Anthropologie*, Band XXXVI. (1909), S. 1—90. The sexes of the skulls used in computing the ♂ means are given by Schultz in *ibid.* Band XLIV. (1918), S. 72—79.

(vii) Pearson, Karl and Davin, Adelaide G.: "On the Biometric Constants of the Human Skull." *Biometrika*, Vol. XVI. (1924), pp. 328—363.

(viii) Fawcett, Cicely D.: "A Second Study of the Variation and Correlation of the Human Skull, with special Reference to the Naqada Crania." *Ibid.* Vol. I. (1902), pp. 408—467.

(ix) v. Luschan, Felix: "Beiträge zur Anthropologie von Kreta." *Zeitschrift für Ethnologie*, Jahrgang XIV. (1913), S. 307—393.

(x) Lajard: "La race Ibère, Crânes des Canaries et des Açores." *Bulletins de la Société d'Anthropologie de Paris*, XV^e série, tome III. (1892), pp. 294—326.

(xi) de Macedo, Francisco Ferraz: *Crime et Criminel. Essai de synthétique d'observations anatomiques, physiologiques, pathologiques et psychiques sur les délinquants et morts selon la méthode et les procédés anthropologiques les plus rigoureux*. Lisbon, 1892.

(xii) Buxton, L. H. Dudley: "The Ethnology of Malta and Gozo." *Journal of the Royal Anthropological Institute*, Vol. LII. (1922), pp. 164—211.

(xiii) Siret, H. and L.: *Les premiers âges du métal dans le sud-est de l'Espagne*. Antwerp (1887). The skull measurements are given in the section on *Ethnologie* (pp. 267—396) by V. Jacques. The means are for the pooled Argar, Gerúndia and Puerto-Blanco series. The ♂ means have been given already in *Biometrika*, Vol. XX^B. (1928), pp. 374 and 375.

could be used in a particular comparison, and for the 12, or fewer, indices and angles which form part of the total 31. The orders given by the two kinds of coefficients are not in good agreement and this may be supposed due to the fact that the values for indices and angles are not all based on the same number of characters. It is known that the average contributions of the individual characters to the coefficients are far from being constant and the omission of a few measurements may have a marked effect if the total number compared is small. Female data for 13 of the 18 series are available and the corresponding ♂ and ♀ reduced coefficients are in fair agreement with one another. The Sedment and several of the other series are short ones and the errors of random sampling are probably large enough to account for most of the sexual differences of this kind observed. The most marked divergence is found in the comparisons with the modern Egyptian series from Cairo, the ♂ reduced coefficient being 8·39 and the ♀ 26·85. The disagreement in this case is almost certainly due to the fact that the ♂ and ♀ samples from Cairo do not represent the same racial population*. The mean indices and angles for the two sexes (cf. Tables I and II) have differences which are too large to be attributed to random sampling. The ♂ and ♀ mean indices and angles of the Sedment series are in close agreement and there is every reason to believe that they represent the same race. The relationships of that type will be measured most accurately by the ♂ reduced coefficients of racial likeness for all characters given in Table III. It is extremely satisfactory to find that the closest connection is with the 4th and 5th dynasty series from Deshasheh and Medum, since Sedment is closer to these two than to any of the other Egyptian sites represented, and the time interval between the series is also small. The reduced coefficient of 5·07 indicates a very close degree of resemblance and it will be of interest to compare it with the closest connections which have been found for other Egyptian series. In Morant's paper all the lowest crude coefficients of racial likeness are given for 23 closely related ♂ series, omitting the Aeneolithic Egyptian one which is of an aberrant type. These are all Egyptian of periods ranging from early predynastic to Roman times except in the Abyssinian series (Tigre district) which is modern. The coefficients were reduced and the distribution of the lowest values for each series is given below.

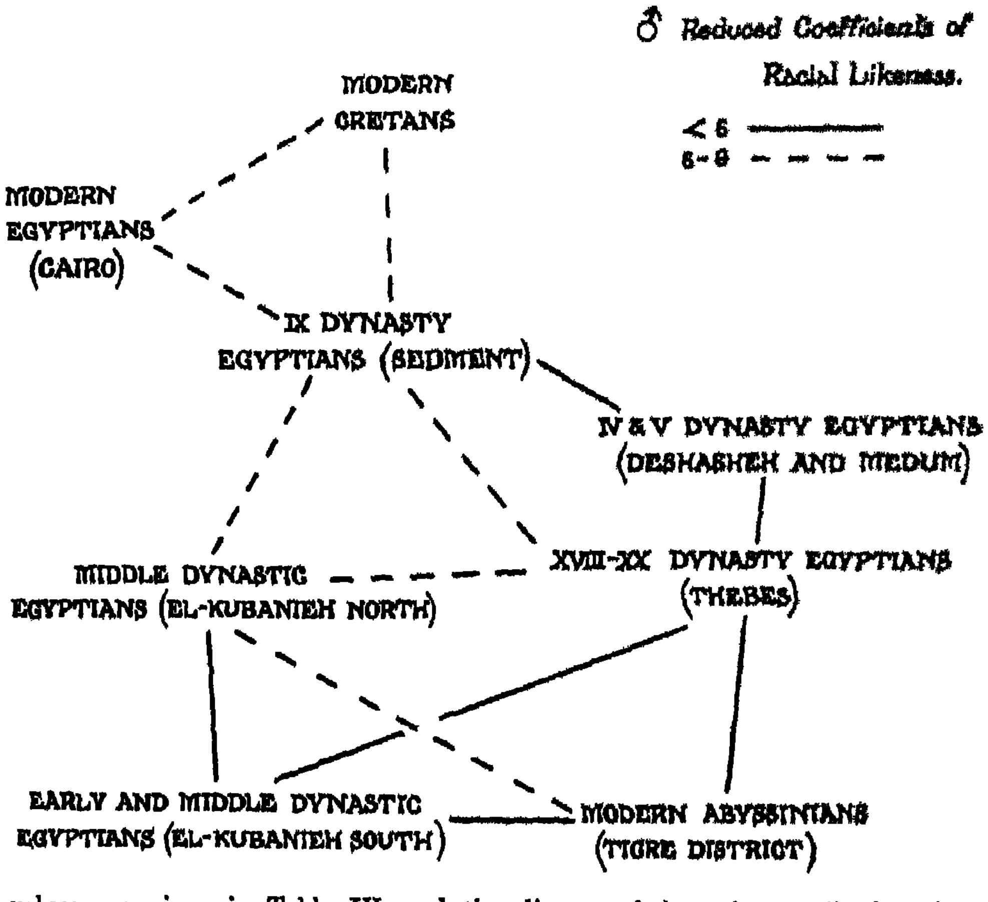
	−0·5—0·5	*0·5—1·5*	*1·5—2·5*	*2·5—3·5*	*3·5—4·5*
Lowest Reduced Coefficients	2	10	7	3	1

The lowest ♂ reduced coefficient in Table III is 5·07, so the Sedment series must be supposed less typically Egyptian than any of the 23 previously described. The fact that its second closest connection is with Modern Cretans again suggests foreign admixture. Seven series of the Egyptian type, of which one is a modern

* The ♂ and ♀ indices and angles of this series are in bad agreement. Judging from the nasal index and angle the ♀ series is predominantly of Negroid origin; though the same is not true of the ♂ series.

series from Cairo and another is of modern Abyssinians, follow, then the Portuguese, and other Egyptian, North African and European types. The only European populations represented are ones from the Mediterranean area. The feature of the relationships observed which is most unexpected is the fact that the Sedment sample bears a closer resemblance to modern Cretans than to all except one of the available Egyptian types. To throw further light on the point the coefficients of racial likeness were computed between all pairs of the seven series which were found to be most closely related to the one from Sedment. Crude and reduced

RELATIONSHIPS OF SERIES RESEMBLING THE SEDMENT SERIES.



values are given in Table IV, and the diagram below shows all the closest connections. The arrangement given by these means corresponds closely with the geographical distribution of the types since Sedment, Deshasheh and Medum are in Lower Egypt, while El-Kubanieh is south of Thebes. The Sedment series appears to be distinguished from all other early Egyptian ones by bearing a peculiarly close resemblance to both modern Cretans and modern Egyptians from Cairo. The earlier and later populations of Lower Egypt are represented by the 4th and 5th dynasty series from Deshasheh and Medum and by the 26th—30th dynasty series from Gizeh. Neither of these is so closely related to the modern

TABLE IV.

Coefficients of Racial Likeness between Series closely related to the Sedment Series (Male and Female).

				4th and 5th Dynasty Egyptians: Deshasheh and Medum	Modern Cretans	18th—20th Dynasty Egyptians: Thebes	Middle Dynastic Egyptians: El-Kubanieh North	Modern Egyptians: Cairo	Modern Abyssinians: Tigre District	Early and Middle Dynastic Egyptians: El-Kubanieh South
Crude Coefficients	4th and 5th Dynasty Egyptians: Deshasheh and Medum	♂ ♀	39·9 —	— —	6·25 ± ·25 (14) —	1·16 ± ·25 (14) —	4·94 ± ·25 (14) —	9·80 ± ·25 (14) —	5·05 ± ·25 (14) —	5·96 ± ·25 (14) —
	Modern Cretans	♂ ♀	47·9 —	6·25 ± ·25 (14) —	— —	7·72 ± ·25 (15) —	10·11 ± ·25 (15) —	4·00 ± ·25 (15) —	13·13 ± ·25 (15) —	16·33 ± ·25 (15) —
	18th—20th Dynasty Egyptians: Thebes	♂ ♀	52·8 43·7	1·16 ± ·25 (14) —	7·72 ± ·25 (15) —	— —	3·27 ± ·18 (27) 1·14 ± ·18 (27)	6·76 ± ·19 (25) 5·96 ± ·19 (25)	2·58 ± ·18 (28) 1·96 ± ·18 (28)	3·18 ± ·18 (27) 2·59 ± ·18 (27)
	Middle Dynastic Egyptians: El-Kubanieh North	♂ ♀	33·1 18·6	4·94 ± ·25 (14) —	10·11 ± ·25 (15) —	3·27 ± ·18 (27) 1·14 ± ·18 (27)	— —	5·37 ± ·19 (24) 4·69 ± ·19 (24)	3·16 ± ·18 (27) 0·99 ± ·19 (26)	1·64 ± ·18 (28) 0·56 ± ·18 (28)
	Modern Egyptians: Cairo	♂ ♀	44·9 25·8	9·80 ± ·25 (14) —	4·00 ± ·25 (15) —	6·76 ± ·19 (25) 5·96 ± ·19 (25)	5·37 ± ·19 (24) 4·69 ± ·19 (24)	— —	8·45 ± ·19 (24) 2·64 ± ·19 (24)	6·20 ± ·19 (24) 5·24 ± ·19 (24)
	Modern Abyssinians: Tigre District	♂ ♀	61·8 22·1	5·05 ± ·25 (14) —	13·13 ± ·25 (15) —	2·58 ± ·18 (28) 1·96 ± ·18 (28)	3·16 ± ·18 (27) 0·99 ± ·19 (26)	8·45 ± ·19 (24) 2·64 ± ·19 (24)	— —	1·91 ± ·20 (23) 1·91 ± ·19 (26)
	Early and Middle Dynastic Egyptians: El-Kubanieh South	♂ ♀	63·5 42·6	5·96 ± ·25 (14) —	16·33 ± ·25 (15) —	3·18 ± ·18 (27) 2·59 ± ·18 (27)	1·64 ± ·18 (28) 0·56 ± ·18 (28)	6·20 ± ·19 (24) 5·24 ± ·19 (24)	1·91 ± ·20 (23) 1·91 ± ·19 (26)	— —
Reduced Coefficients	4th and 5th Dynasty Egyptians: Deshasheh and Medum	♂ ♀		— —	14·46 —	2·54 —	11·56 —	23·12 —	10·41 —	12·24 —
	Modern Cretans	♂ ♀		14·46 —	— —	15·25 —	25·69 —	8·55 —	24·08 —	29·86 —
	18th—20th Dynasty Egyptians: Thebes	♂ ♀		2·54 —	15·25 —	— —	8·05 4·35	13·82 18·36	4·50 7·17	5·48 4·67
	Middle Dynastic Egyptians: El-Kubanieh North	♂ ♀		11·56 —	25·69 —	8·05 4·35	— —	14·15 21·73	7·48 4·90	3·77 2·17
	Modern Egyptians: Cairo	♂ ♀		23·12 —	8·55 —	13·82 18·36	14·15 21·73	— —	16·15 11·07	11·68 16·24
	Modern Abyssinians: Tigre District	♂ ♀		10·41 —	24·08 —	4·50 7·17	7·48 4·90	16·15 11·07	— —	3·05 6·57
	Early and Middle Dynastic Egyptians: El-Kubanieh South	♂ ♀		12·24 —	29·86 —	5·48 4·67	3·77 2·17	11·68 16·24	3·05 6·57	— —

Cretans. The ranges of the coefficients on which the diagram above is based are for ♂ readings. There are no adequate ♀ means for modern Cretans and the 4th and 5th Dynasty Egyptians from Deshasheh and Medum. The ♀ coefficients between the remaining six series furnish a scheme of relationship which is closely similar to the one already considered except in one particular; the Sedment and modern series from Cairo have ♀ means which are much further removed than the corresponding ♂ means.

TABLE V.

Values of "a" between the Male and Female Sedment and other Series.

Races	4th and 5th Dynasty Egyptians: Deshasheh and Medum (Thomson and MacIver)	Modern Cretans (von Luschan)	18th—20th Dynasty Egyptians: Thebes (Stahr)	Middle Dynastic Egyptians: El-Kubanieh North (Toldt)	Mean a for 18 Series (No. of a's)	18th—20th Dynasty Egyptians: Thebes (Stahr)	Middle Dynastic Egyptians: El Kubanieh North (Toldt)	Mean a for 13 Series (No. of a's)
Sex	♂	♂	♂	♂	♂	♀	♀	♀
$\bar{n}$	39·9	47·9	52·8	33·1	·	43·7	18·6	—
100 H'/L	21·01	1·62	18·53	1·35	17·85 (18)	17·37	3·90	16·14 (13)
Oc. I.	—	—	22·35	4·50	16·28 (7)	2·38	7·73	6·44 (5)
100 B/L	1·51	6·22	5·40	15·42	15·37 (18)	7·81	12·02	15·03 (13)
Q'	—	—	·45	—	12·90 (6)	·91	—	4·07 (5)
O_1' (or O_1)	—	—	15·44	16·07	11·95 (9)	·61	·00	5·03 (7)
B'	—	16·89	7·18	5·67	11·38 (13)	11·54	3·88	22·94 (9)
G_2	—	—	·43	—	11·11 (8)	·03		1·29 (6)
H'	1·66	·01	12·07	1·62	9·84 (18)	6·26	·15	4·89 (13)
100 G_2/G_1' (or G_2/G_1)	—	—	—	—	8·77 (4)	—	—	1·99 (2)
100 $G'H/GB$	—	—	4·86	3·76	8·23 (9)	7·14	1·24	10·46 (7)
B	·98	1·21	1·03	14·26	7·54 (18)	·07	3·01	5·00 (13)
100 O_2/O_1' (or O_2/O_1)	—	—	4·76	·44	7·46 (9)	1·53	·90	7·48 (7)
100 NB/NH' (or NB/NH)	·35	0·00	11·49	6·14	7·08 (18)	2·32	·06	7·42 (13)
$G'H$	·03	7·15	2·58	1·88	5·75 (16)	·80	·03	2·68 (12)
L	6·32	2·01	1·83	·06	5·19 (18)	9·03	5·13	13·46 (13)
J	0·00	5·53	1·76	1·89	5·07 (18)	12·52	·05	8·76 (13)
U	—	—	·09	3·66	4·95 (11)	1·89	·00	7·93 (9)
100 B/H'	2·63	·31	3·90	4·77	4·78 (18)	3·46	·85	5·09 (13)
NB	2·35	·63	4·58	·54	4·29 (18)	2·15	·18	5·70 (13)
NH' (or NH)	·86	1·22	·95	5·62	4·18 (18)	·11	·77	2·97 (13)
O_3	—	—	1·55	5·11	3·96 (12)	·49	·56	1·98 (9)
$A\angle$	1·04	8·15	4·59	1·24	2·60 (15)	·14	·57	1·38 (11)
$N\angle$	·29	5·01	·71	·01	2·39 (15)	2·38	3·52	8·33 (11)
G_1' (or G_1)	—	—	—	·12	2·18 (8)	—	·26	6·33 (4)
fmb	—	—	·78	3·04	2·16 (11)	·40	·00	1·66 (8)
C	—	—	·93	·61	2·13 (7)	1·94	·10	1·39 (6)
$P\angle$	—	—	1·35	2·18	1·99 (7)	1·98	2·53	·94 (6)
S	—	—	·38	·14	1·63 (11)	·72	3·04	5·07 (9)
fml	—	—	1·80	2·81	1·47 (11)	·55	·36	1·10 (8)
LB	1·30	·60	1·71	·73	·95 (17)	8·44	1·84	2·72 (12)
100 *fmb*/*fml*	—	—	·07	·01	·92 (11)	·02	·17	1·04 (8)

4. *A Comparison of Single Characters.* The α's found in computing the coefficients give a convenient measure of the significance of the differences between single mean measurements. The difference may be supposed to be definitely significant if the α is greater than 10. All values available for the 31 characters are given in Table V for the 4 ♂ and 2 ♀ series which most closely resemble the Sedment type. The mean α's are also given for the 18 series with which comparisons are made in Table III. As is usually found, the average contributions which the characters make to the coefficients vary greatly. Mean α's have been given based on 820 comparisons between 37 European and 4 North African series*. The value for the cephalic index is almost twice as great as that found for any other character, and next in order are B, 100 B/H', L and O_1. The characters 100 H'/L, the

TABLE VI.

Male Means of Egyptian and other Series†.

		L	B	H'	$100\,B/L$	$100\,H'/L$	$Oc.\,I.$	B'
Egyptians: 9th Dynasty (Sedment)		181·9	138·3	137·4	76·1	75·5	64·0	92·6
Modern Cretans		180·2	139·4	137·5	77·5	76·3	—	96·1
Modern Egyptians (Cairo)		176·1	137·6	134·6	78·1	76·4	—	94·3
Egyptians: 4th and 5th Dynasties (Deshasheh and Medum)		184·9	139·3	136·0	75·4	72·6	—	—
Egyptians: 18th—20th Dynasties (Thebes)		183·5	137·3	133·7	74·8	72·8	60·7	94·9
Egyptians: Middle Dynasties (El-Kubanieh N.)		182·2	134·2	136·9	73·7	74·7	61·9	90·4
Other Egyptian series: predynastic to Roman times‡	Range	181·4—185·8	130·8—139·2	130·7—135·1	71·7—76·0	71·3—73·9	59·9—61·9	91·1—96·2
	No. of Series	16	16	16	16	16	5	7

occipital index ($Oc.\,I.$), H' and B' are nearer the middle of the range and hardly any significant differences were found for $N\angle$, $A\angle$, NB, 100 NB/NH' and 100 fmb/fml. In the present comparison the indices 100 H'/L, $Oc.\,I.$ and 100 B/L are the most variable characters, and these are the ones most likely to distinguish the Sedment from all the other series. The height-length index is the only character showing a significant difference in the comparison with the 4th and 5th dynasty Egyptians from Deshasheh and Medum and the minimum frontal diameter is the only one which differentiates the Cretans from the Sedment series. Mean values for some of the characters which differ most significantly are shown in Table VI. The length of the Sedment type is slightly greater than the smallest mean recorded for a

* *Biometrika*, Vol. xx^B. (1928), Table XVI, facing p. 886.

† All the means in this table are based on 30 or more skulls.

‡ The means of these series are given in *Biometrika*, Vol. xvii. (1925), pp. 14—36 (Fouquet's measurements being excluded) and *Ibid.* Vol. xix. (1927), Table II, facing p. 117.

dynastic Egyptian series; the breadth is a millimetre less than the largest of these means and the cephalic index is 0·1 greater than any other recorded for this group of closely allied racial types. Both the basio-bregmatic height (H') and the height-length index ($100\ H'/L$) are more clearly differentiated from the continuous series given by previously measured early Egyptian samples. In the case of all five of these characters the tendency to diverge, or the actual divergence, of the Sedment constant from the inter-racial range is in the direction of the still more divergent mean for modern Cretans. The evidence of these measurements is not independent evidence, of course, since they are known to be quite highly correlated with one another both intra- and inter-racially. The high mean value of α found for the occipital index (*Oc. I.*) is seen from Table VI to have been occasioned by the fact that the Sedment value is appreciably higher than any other found for an Egyptian series. Means of this character have been given for a considerable number of racial types*. They range from 58·0 to 68·8; all the lowest values are for Western European and all the highest for African Negro races. The highest as yet found for European races are for Rumanians (62·7), Serbo-Croats (62·8), Greeks (62·9) and Turks (63·3). The occipital index is unfortunately not available for the modern Cretans. It may be suggested that the high means found for the Sedment series (♂ 64·0, ♀ 63·5) indicate Negroid admixture, but a comparison of characters which are better criteria of the Negro skull, such as angular measurements of prognathism and the nasal index, fails entirely to substantiate that view. Minimum frontal breadths (B') are also compared in Table VI and the Sedment mean falls within the range furnished by other early Egyptian types. The same has been found for every other character measured, including a number which are not used in computing the coefficient of racial likeness. The nasal breadth, index and angle, however, are almost as low as any of the other means. The distinctiveness of the Sedment type is thus seen to depend on very few of the characters which may be compared.

5. *Type Contours.* The ♂ and ♀ type contours for the Sedment series were constructed from the mean measurements of the individual contours in the way usually employed in the Biometric Laboratory†. They are given in Figs. I—VI and the means themselves will be found in Tables VII—IX. A number of chords and angular measurements of the types should agree very approximately with absolute mean readings and a check on both is thus obtained. Comparisons were made in the case of the auricular heights of the transverse section and a number of measurements taken on the sagittal figures. The maximum difference between the readings obtained by the two methods was 0·5 mm. for chords and 0°·4 for angles. This is a satisfactory agreement‡.

* *Biometrika*, Vol. XVI. (1924), pp. 334 and 335. † See *Ibid.* Vol. XIV. (1923), pp. 227—244.

‡ Perfect correspondence between all the absolute and contour measurements which are usually compared is not to be expected, even if the personal equations of both measuring and drawing were zero, owing to the methods employed in tracing individual and constructing type contours. For example, the craniophor auricular height is the *maximum* projective height from the auricular axis, but the highest point on the transverse type is obtained from the mean of the vertical heights bisecting the auricular

The transverse vertical contour is drawn through the auricular points—the "porions" of Martin—perpendicular to the Frankfurt horizontal plane. The ♂ and ♀ types (Figs. I and II) are almost symmetrical, the maximum difference between the right and left sides of the same ordinate being 0·9 mm. in favour of the right

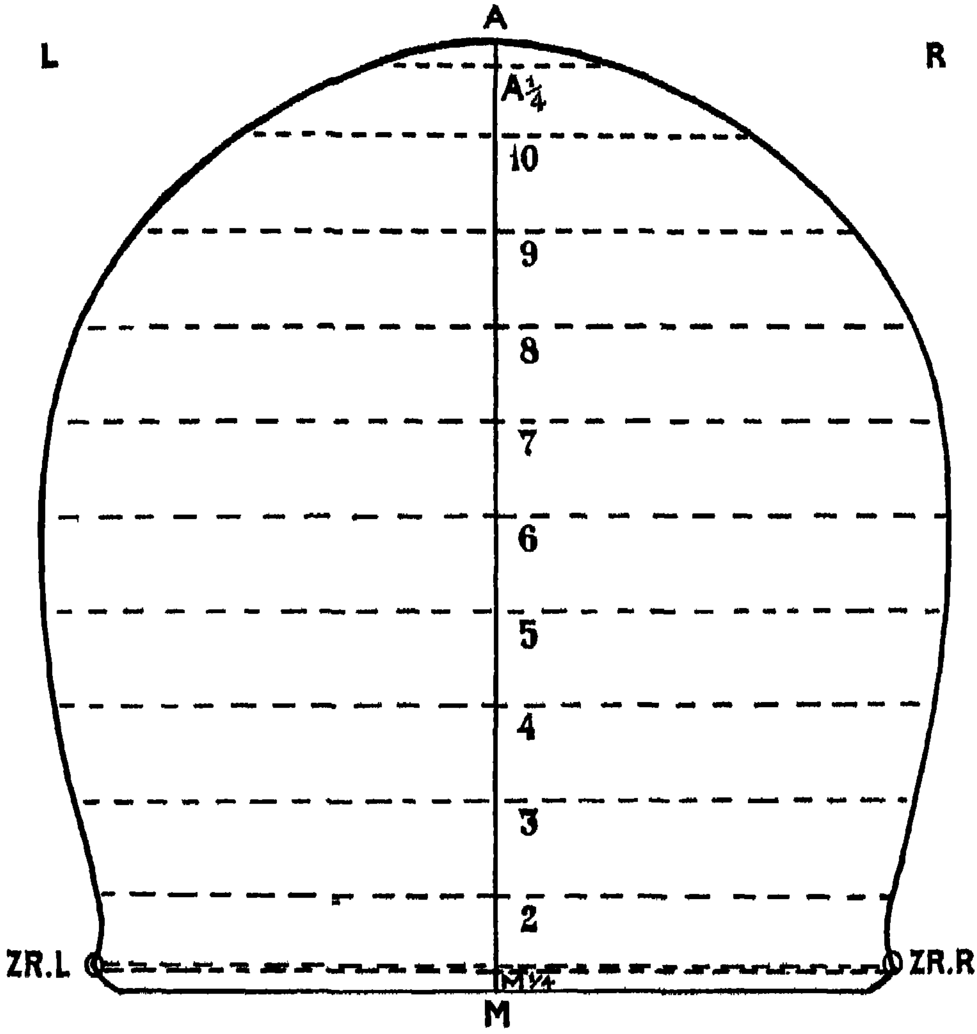


Fig. I. Transverse Type Contour of 39 ♂ Egyptian Skulls from Sedment.

side on the ♂ figure and 1·5 mm. in favour of the left side on the ♀ figure. The maximum breadths of both are close to the 6th parallel. For all racial type contours which have yet been published the maximum breadths lie between the

axis. The two may not correspond on an asymmetrical skull. The fact that the point of the tracer was in a few cases raised or lowered to pass through some standard points shown on the sagittal figures will lead to projected lengths which are slightly different from the direct calliper readings.

4th and 6th parallels, and in several cases they are between the 4th and 5th. The relative position of the major diameter appears to be a character which distinguishes the Sedment Egyptian type from all others available. The point where the line joining the most lateral points right and left meets the axis *MA* may be taken to

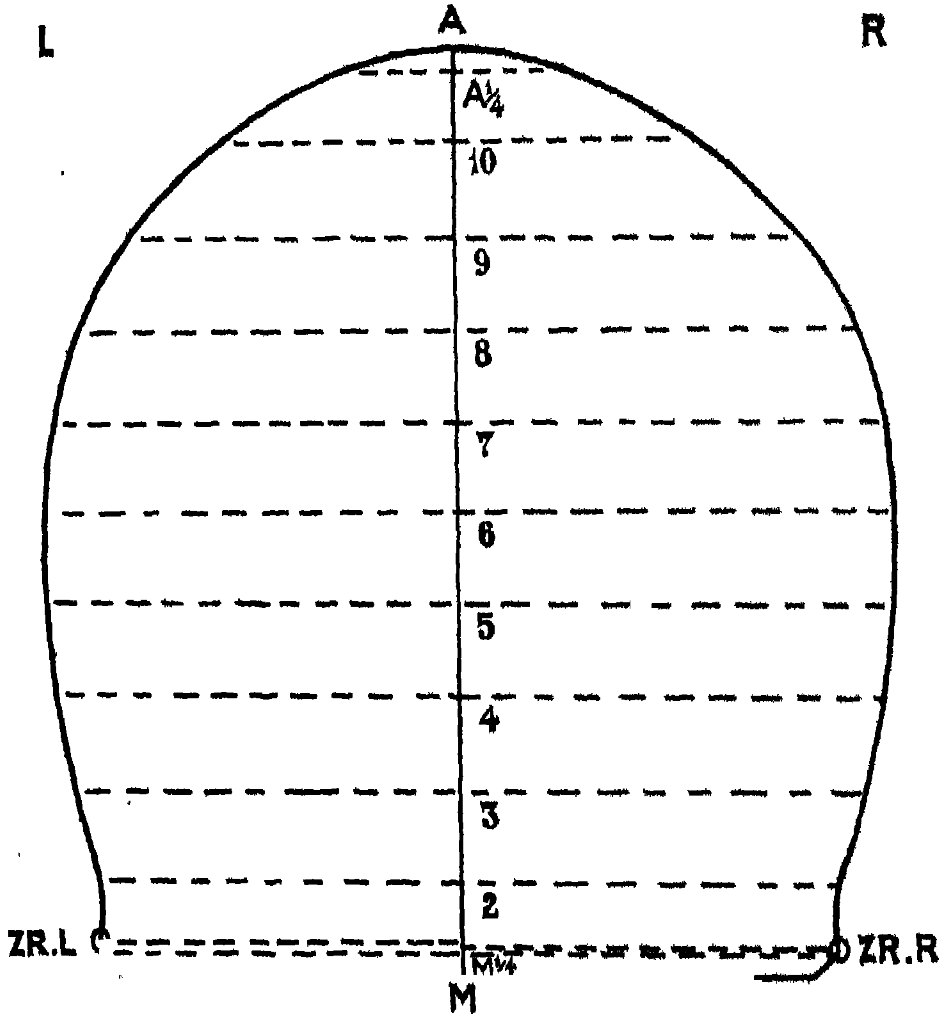


FIG. II Transverse Type Contour of 30 ♀ Egyptian Skulls from Sedment.

indicate the height of the maximum horizontal diameter above the auricular axis. The character in question can be conveniently measured by expressing the distance of this point from *M* as a percentage of *MA*. The indices on the following page are given by type contours each based on 20 or more skulls; the index is for the ♂ type unless otherwise indicated.

45—50. 9th Dynasty Egyptians: Sedment 48·7 (♀ 47·4), Congo Negroes: Fernand Vaz, 1880* 45·6.

40—45. Predynastic Egyptians: Badari 44·4 (♀ 43·2)†, 26th—30th Dynasty Egyptians: Gizeh 43·3*, 1st Dynasty Egyptians: Abydos 43·1‡, Tamils 41·6§, Burmese A 41·4 (♀ 38·0)‖, Congo Negroes: Batetelu* 41·3, Northern Chinese 40·3¶, Hokien Chinese 40·1§.

35—40. Prehistoric Chinese 39·0¶, Congo Negroes: Fernand Vaz, 1864* 38·8, Basques 38·4**, Nepalese 37·5††, Tibetan A 37·3††.

30—35. 17th century English: Whitechapel 34·8*, 17th century English: Farringdon St‡‡ 34·7 (♀ 34·4), Eskimo 34·5*, Anglo-Saxons 31·8 (♀ 30·9)§§.

TABLE VII.

Mean Measurements of Transverse Vertical Contours.

Sex	Cases	*MA*	1*R*=1*L*	¼*R*	¼*L*	2*R*	2*L*	3*R*	3*L*	4*R*	4*L*
♂	39	114·9	54·1	58·2	58·2	57·8	57·7	61·3	61·0	64·5	64·1
♀	30	109·9	51·5	55·1	54·9	55·9	55·5	59·3	58·8	62·5*	61·9

Sex	Cases	5*R*	5*L*	6*R*	6*L*	7*R*	7*L*	8*R*	8*L*	9*R*	9*L*
♂	39	65·9	65·5	66·3	65·8	65·3	64·8	61·4	60·9	52·6	52·0
♀	30	63·9	63·3	63·9	63·2	62·6	61·9	58·9	58·1	50·9	49·9

Sex	Cases	10*R*	10*L*	*A*¼*R*	*A*¼*L*	*ZR, R*		*ZR, L*	
						y	*x*	*y*	*x*
♂	39	37·8	36·9	17·4	17·6	58·5	3·3	58·5	3·4
♀	30	35·4	35·4	16·3	17·8	55·6	3·2	55·6	3·6

* Number of cases = 29.

The four Egyptian series available are all among the types found with the five highest values of this index. The Western European races are at the other extreme of the range and, unfortunately, no data for Eastern European types can be given. It is interesting to observe that the ♂ index is greater than the ♀ in every case. For most races the mean auricular height is less than the mean auricular breadth, so that the index which expresses *MA* as a percentage of the parallel 1 is less than

* Benington: *Biometrika*, Vol. VIII. (1911), pp. 157—198.
† Stoessiger: *Ibid.* Vol. XIX. (1927), pp. 186 and 187.
‡ Motley: *Ibid.* Vol. XVII. (1925), p. 47.
§ Harrower: *Transactions of the Royal Society of Edinburgh*, Vol. LIV. (1926), pp. 592 and 594.
‖ Tildesley: *Biometrika*, Vol. XIII. (1921), pp. 188 and 191.
¶ Black: *Palaeontologia Sinica*, Series D, Vol. VII. (1928), p. 47.
** Morant: *Biometrika*, Vol. XXI. (1929), p. 78.
†† Morant: *Ibid.* Vol. XVI. (1924), pp. 76 and 77.
‡‡ Hooke: *Ibid.* Vol. XVIII. (1926), pp. 48 and 44.
§§ Morant: *Ibid.* Vol. XVIII. (1926), pp. 90 and 91.

100. This is so for the 26th—30th and for the 1st Dynasty Egyptian contours. The indices for the Badari Egyptians (♂ 107·9, ♀ 110·0) are the highest that have yet been found however. The value given by the ♂ Tamil type is 107·7 and the Sedment indices (♂ 106·2, ♀ 106·7) are also close to the extreme. Other measurements of the transverse type contour which have been used for comparative purposes do not distinguish our Sedment from the other series. Figs. I and II are differentiated from all others available by having their maximum breadths peculiarly high up and their heights are also large in proportion to the auricular widths. In these and other respects they bear a closer resemblance to the Badari than to the 1st or the 26th—30th Dynasty sections. Superposing the Sedment and Badari types, with the aid of the tracings provided, a close correspondence is found for both sexes. The former outlines are slightly higher and broader than the others.

TABLE VIII.

Mean Measurements of Horizontal Contours.

Sex	Cases	FO	$F\frac{1}{4}R$	$F\frac{1}{4}L$	$F\frac{1}{2}R$	$F\frac{1}{2}L$	$2R$	$2L$	$2\frac{1}{2}R$	$2\frac{1}{2}L$	$3R$	$3L$
♂	38	180·6	22·2	22·7	33·2	33·4	45·3	45·1	46·9	45·9	47·6	48·0
♀	30	172·4	21·3	21·2	32·9	32·8	44·2	43·8	44·5	44·5	47·0	47·1

Sex	Cases	$4R$	$4L$	$5R$	$5L$	$6R$	$6L$	$7R$	$7L$	$8R$	$8L$
♂	38	54·2	54·4	60·9*	60·8	65·6	66·4	67·4	67·1	65·5	65·2
♀	30	53·1	53·0	59·7†	59·3	64·5†	63·7	65·5	64·5	62·9	62·1

Sex	Cases	$9R$	$9L$	$10R$	$10L$	$O\frac{1}{4}R$	$O\frac{1}{4}L$	T_R		T_L	
								y	x	y	x
♂	38	59·3	58·9	46·6	45·9	28·0	26·6	48·0	20·1	47·4	19·4
♀	30	56·3	55·6	44·0	43·5	26·3	25·9	45·2	17·9	44·7	17·4

* Number of cases = 37. † Number of cases = 29.

The glabella horizontal section (Figs. III and IV) is drawn through the glabella parallel to the Frankfurt horizontal plane. The point F is the glabella, T_R and T_L mark the crossing of the temporal lines and O is the occipital point in the median sagittal plane. The type figures are almost exactly symmetrical. The maximum difference between the right and left sides of the same ordinate (see Table VIII) is 1·4 mm. in favour of the right side for the ♂ figure and 1·0 mm. in favour of the right side for the ♀. As for all other types which have yet been constructed, the maximum breadths are between the 6th and 7th parallels. The outlines have no characteristics which are very distinctive. The temporal fossae are shallow though more marked, as usual, on the ♂ than on the ♀ figure. A number of indices derived from measurements of the horizontal type contour have been used to compare degrees of frontal development and other features. The variation shown

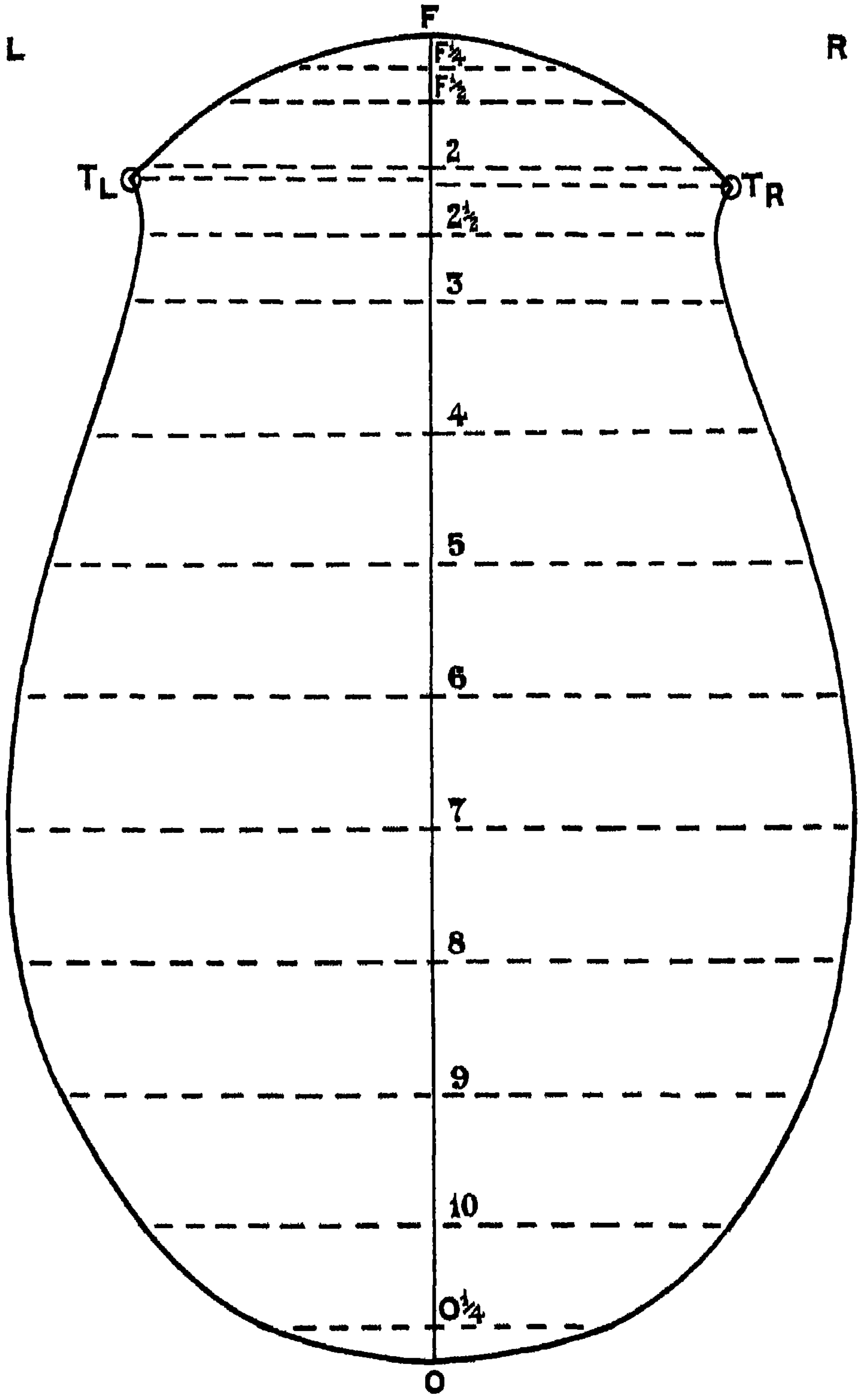


Fig. III. Horizontal Type Contour of 38 ♂ Egyptian Skulls from Sedment.

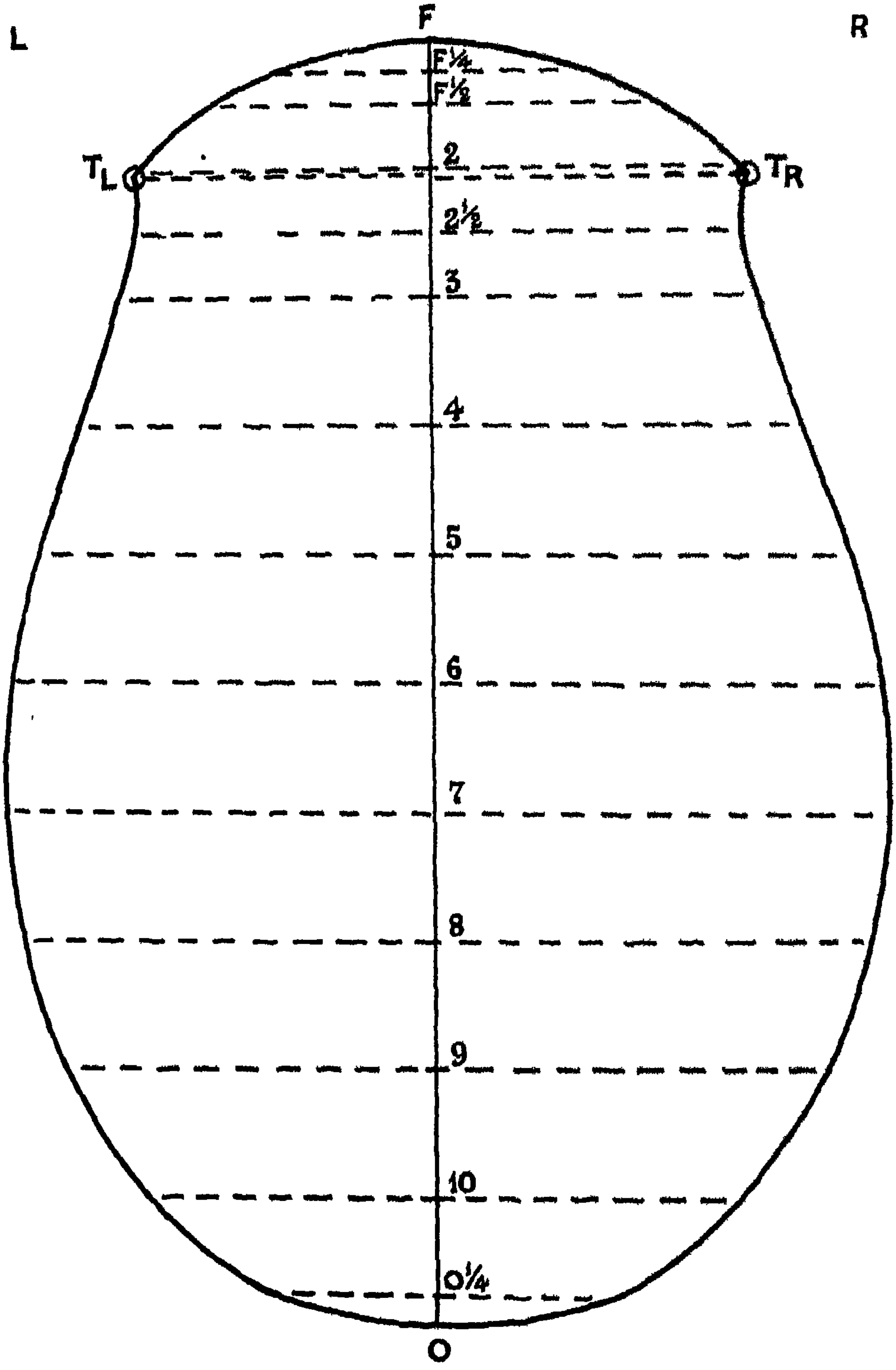


FIG. IV. Horizontal Type Contour of 30 ♀ Egyptian Skulls from Sedment.

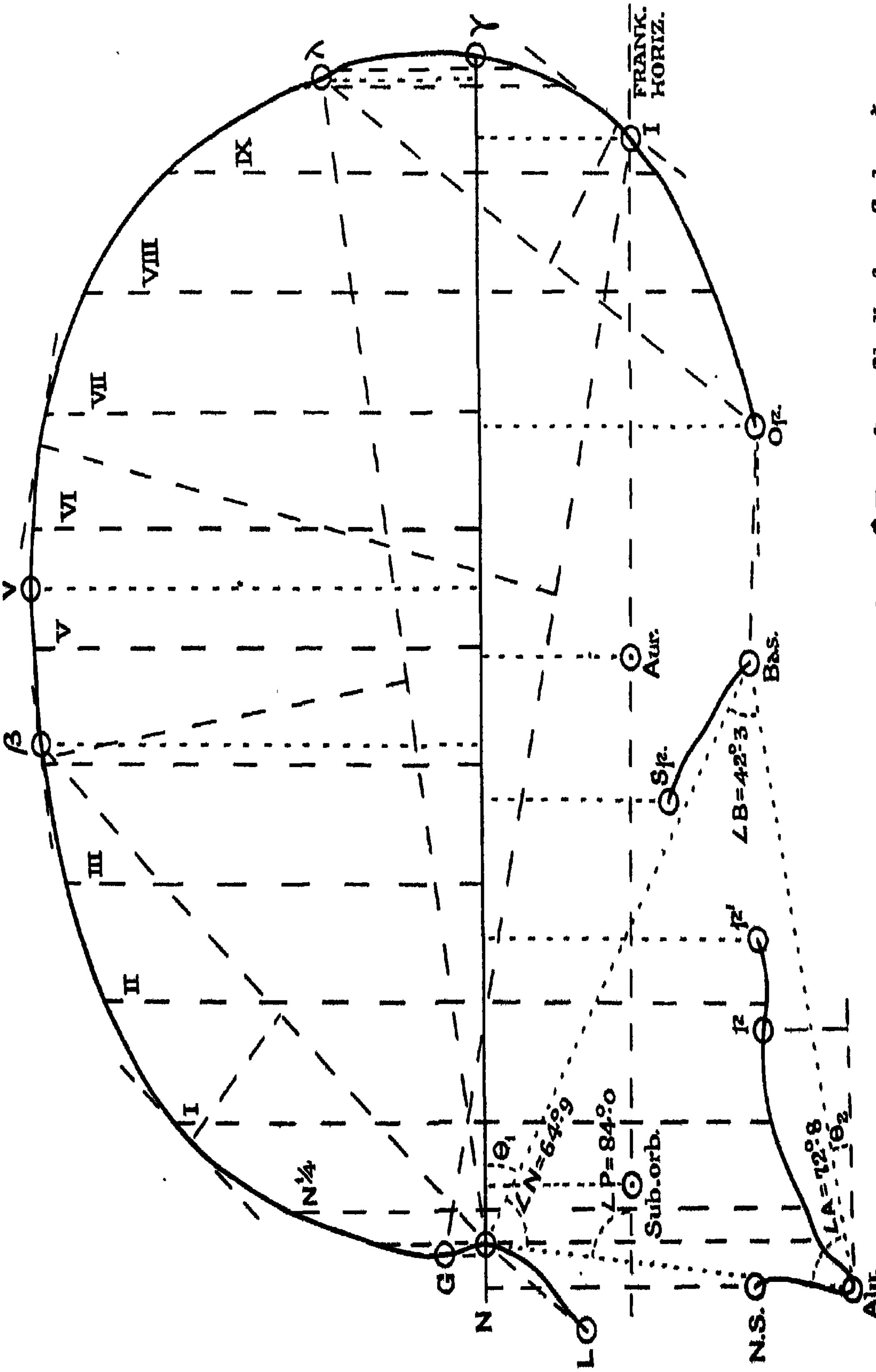


FIG. V. Sagittal Type Contour of 39 ♂ Egyptian Skulls from Sedment.

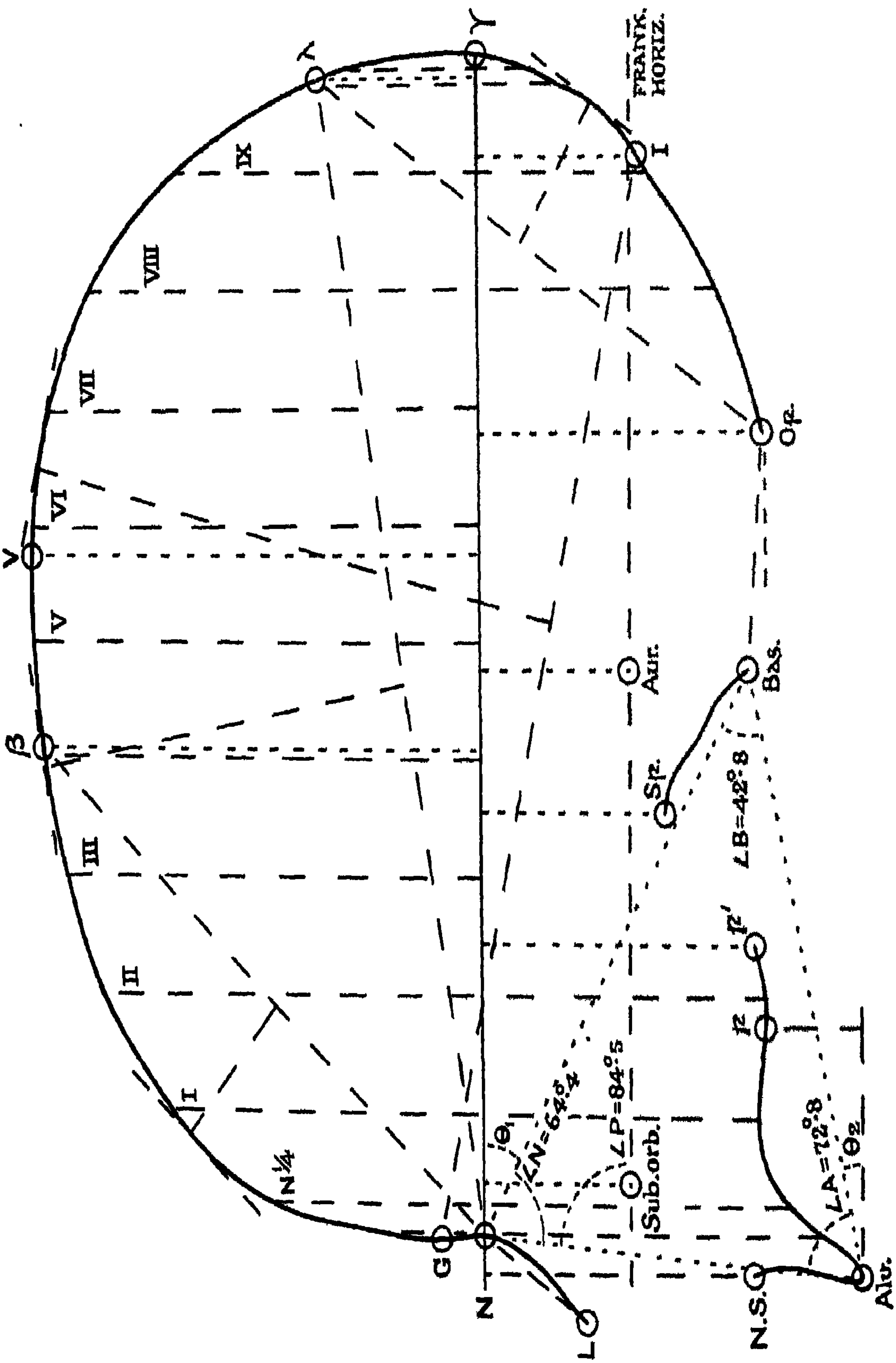

FRANK. HORIZ.
Y
λ
I
IX
VIII
VII
VI
V
V
β
III
II
I
N¼
G
N
L
Op.
Aur.
Bas.
Sp.
∠B=42°·8
pr'
pr
∠N=64°·4
∠P=84°·5
Sub.orb.
θ₁
∠A=72°·8
θ₂
N.S.
Alv.

Fig. VI. Sagittal Type Contour of 30 ○ Egyptian Skulls *from Sedment.*

A

B

A

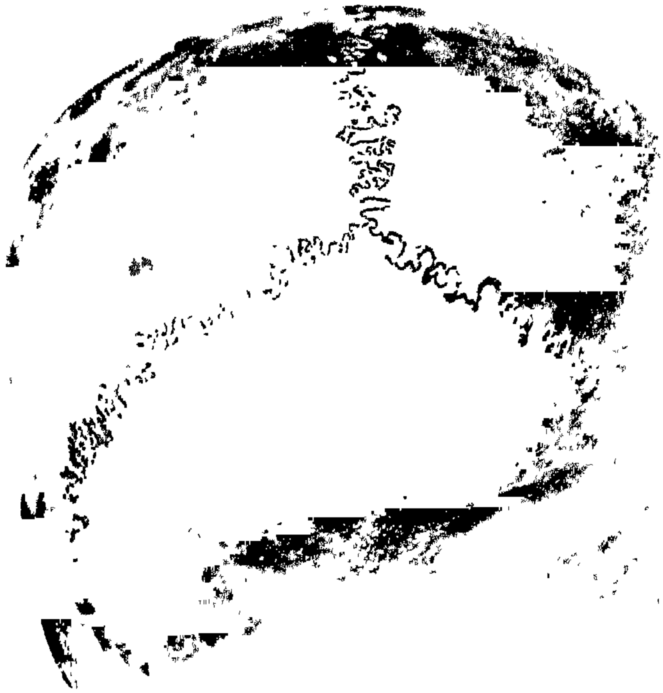
B

A

B

Normal Sedment Egyptian Skulls. *Norma verticalis* (circa 0·7 natural size). A No. 21 ♀. B No. 27 ♂.

A



B

Normal Sedment Egyptian Skulls. *Norma occipitalis* (circ. 0.75 natural size)

A. Basi-occipital fissure on the left side. No. 65 ♂ (circa 1·5 natural size)

B. Large protruding ossicle above parieto-mastoid suture. No. 83 ♂? (circa natural size).

Anomalous Sedment Egyptian Skulls.

by modern races is small and the Sedment values fall within the ranges given by the other types in all except one case. The distance of the parallel $F\frac{1}{2}$ from F—i.e. $\frac{1}{20}$th of FO—expressed as a percentage of the total length of that parallel gives a measure of the curvature of the most anterior part of the frontal section. The highest ♂ index found is 13·6 for the Sedment type contour, but there are several others greater than 13. The lowest index, indicating the frontal bone which is most flattened transversely, is 11·0 (Prehistoric Chinese) and all the lowest values are for Asiatic races. By superposing the contours it is found that the Sedment type corresponds more closely to the horizontal section given for the 26th—30th Dynasty skulls from Gizeh than to the Badari or 1st Dynasty Egyptian figures.

The median sagittal contours (Figs. V and VI) again exhibit few distinctive features. As the series is an Egyptian one, the most characteristic are the unusual height and the flattened occipital section. The vertices lie between the 5th and 6th parallels and the same has been found for all other types as yet constructed. A number of indicial and angular measurements have been devised to aid in the comparison of the most important features of this section*. The ♂ and ♀ Sedment values were determined for each of these, and in all cases they fall within the ranges furnished by the type contours which had been constructed previously. Considerable differences are found from the sagittal figures available for other Egyptian series. The Sedment sections have quite the least protruding and flattest occiputs: they are also the greatest in height. A curious relationship is found on superposing the Sedment and Badari ♂ figures with the nasions and $N\gamma$ lines coincident. The outlines from nasion to lambda almost cover one another and no difference can be detected between the heights of the vertices. Below the lambda the Badari occipital section protrudes beyond the Sedment, but the outlines cross between the inion and opisthion. Quite marked differences are found between the outlines of the basi-occipital and the palate, the Sedment bones being immediately below the Badari but appreciably further removed from the $N\gamma$ line. The difference between the basio-bregmatic, or vertical heights, of the two types is thus due solely to differences associated with the base of the skull. The most significant differences between the Egyptian types are found for this section. The comparison of type contours confirms the conclusion suggested by direct measurements that the 9th Dynasty series from Sedment is not closely related to any one of the other three Egyptian ones which have been described in this way. More adequate comparative material will be needed in order to determine its racial affinities more exactly.

6. *Measurements of Mandibles.* There are 62 of the skulls in the Sedment series with more or less complete mandibles, 36 being ♂, 25 ♀ and 1 juvenile. Notes on the condition of the teeth and any dental anomalies noted are given in Appendix I. Measurements were taken according to the technique described by Morant†. Individual values are given in Appendix II and means in Table X. There is far less comparative material than for the skull, and no standard deviations or correlations

* See *Annals of Eugenics*, Vol. II. (1927), pp. 365—368.

† "A First Study of the Tibetan Skull," *Biometrika*, Vol. XIV. (1923), pp. 193—260.

precisely the same technique as we have used have only been published for one other Egyptian series: that is the Badari studied by Stoessiger*. Mean values of the principal measurements for the Egyptian and for a Tibetan and an Anglo-Saxon series are given in Table XI. Some of the differences between the extremes are

TABLE XI.

Mean Male Mandibular Measurements for Egyptian and other Series.

	9th Dynasty Egyptians (Sedment)	Predynastic Egyptians (Badari)	Tibetan A †	Anglo-Saxons ‡
Maximum breadth at condyles (w_1)	114·3 (26)	109·5 (30)	117·0 (25)	123·7 (25)
" " angles (w_2)	92·0 (32)	88·8 (32)	96·2 (25)	103·2 (45)
Height of symphysis ... (h_1)	33·7 (33)	32·6 (34)	30·6 (25)	33·1 (40)
Minimum breadth of ramus (rb')	33·3 (35)	33·6 (39)	32·1 (25)	33·2 (61)
Condylion to coronion ... ($c_y c_r$)	34·7 (32)	33·8 (37)	34·2 (25)	33·9 (40)
Maximum length of condyle ($c_y l$)	20·8 (30)	20·3 (30)	18·8 (25)	21·7 (38)
Height of coronion ... ($c_r h$)	67·7 (33)	61·8 (33)	60·8 (25)	65·7 (48)
Height of incisura ... (ih')	13·3 (32)	12·2 (33)	15·0 (25)	13·6 (35)
Length of ramus ... ... (rl)	63·0 (35)	57·6 (33)	58·4 (25)	64·0 (45)
Total projective length ... (ml)	102·6 (36)	101·2 (33)	106·2 (25)	107·2 (31)
$100 c_r h/ml$	66·5 (33)	61·0 (32)	57·8 (25)	60·0 (27)
$100 rb'/rl$	53·0 (34)	58·5 (33)	56·3 (25)	51·8 (45)
$M\angle$	121°·0 (36)	120°·0 (34)	125°·3 (25)	120°·3 (47)

surprisingly large. For the condylar breadth (w_1) the greatest mean exceeds the least by 14·2 mm. and for the angular breadth (w_2) the extreme difference is 14·4 mm. These values are almost as great as the *maximum* inter-racial difference found for the larger diameters of the skull such as the length, breadth and height. Some of the other characters—notably rb' and $c_y c_r$—are almost constant for the four series in the table. It is probable that several of the differences between the Sedment and Badari means are significant.

7. *Conclusions.* The series of 9th Dynasty skulls from Sedment, in Upper Egypt, is not more variable than other Egyptian dynastic ones. Judging by a generalised measure of resemblance—Professor Karl Pearson's coefficient of racial likeness—the type is more closely related to that of 4th and 5th Dynasty skulls from neighbouring graveyards at Deshasheh and Medum than to any other which has been adequately described. In spite of this close link, the Sedment series stands still closer to one of modern Cretans than to any Egyptian series at present available. No other close connections have been found with European races, and all those of a rather less intimate order are with dynastic series from Upper Egypt except one with a modern series from Cairo. The last, and the modern Cretans, are only connected

* "A Study of the Badari Crania recently excavated by the British School of Archaeology in Egypt." *Biometrika*, Vol. XIX. (1927), pp. 110—150.

† *Ibid.* Vol. XVI. (1924), pp. 103 and 104.

‡ *Ibid.* Vol. XVIII. (1926), p. 96.

Oi		Angle			
$O_1'R$	$100\frac{fmb}{fml}$	$N\angle$	$A\angle$	$B\angle$	
37·9	—	—	—	—	w before death, teeth considerably worn.
41·8	81·2	63·2	74·6	42·2	
43·4	77·4	63·0	74·6	42·4	massive.
39·0	81·8	68·2	68·9	42·9	ose; ossicles at asteria R and L; teeth complete but considerably worn; JR.
36·8	91·6	68·9	67·8	43·3	greatly worn; bone eroded by disease (?) at roots of both upper 1st molars
41·8	76·2	64·0	72·0	44·0	ars lost from upper jaw, 2 incisors lost from lower jaw, teeth considerably
—	87·0	—	—	—	h complete in lower jaw but exceedingly worn; large epipteric bone R; JR.
40·0	—	68·0	69·8	42·2	t before death, teeth exceedingly worn; 2 small precondyles; JR.
—	80·3	61·4	76·7	41·9	ossicle of lambda and a large ossicle in sagittal suture above it; 8 molars nce.
39·0	81·7	61·0	80·0	39·0	2 1st molars lost from upper jaw, R 1st molar lost from lower jaw; J =.
38·8	84·5	62·6	70·9	46·5	
38·7	90·8	67·0	67·6	45·4	orn; tympanic perforation R; faint trace of metopic suture; protruding
36·2	90·0	61·8	74·1	44·1	
38·2	91·2	63·1	71·2	45·7	
41·6	82·3	61·3	77·2	41·5	ediately above lambda; single wormian in L lambdoid suture; no teeth lost
40·6	83·4	67·1	69·9	43·0	ondyle L; JR.
40·2	94·4	72·1	67·6	40·3	
46·5	85·9	72·1	67·7	40·2	oth jaws, teeth greatly worn; single precondyle; JR.
39·6	88·1	60·0	78·6	41·4	JL.
42·3	79·3	65·1	71·4	43·5	
—	—	—	—	—	h complete (?) and very worn.
41·2	90·6	67·5	69·1	43·4	
43·2	81·3	67·2	69·0	43·8	small precondyles; JR; large and strong skull.
35·2	77·3	61·8	78·6	39·6	worn; tympanic perforations R and L; J =.
38·8	82·7	70·3	66·0	43·7	
38·8	83·4	62·7	70·1	47·2	
38·1	87·9	65·8	75·5	38·7	
41·0	81·4	62·8	74·3	42·9	
38·6	87·6	61·9	73·3	44·8	aw but teeth considerably worn; JR.
36·4	84·1	67·9	73·1	39·0	rom lower jaw, teeth greatly worn; JR; JL divided.
40·6	76·1	70·1	68·9	41·0	
36·0	77·3?	63·0	74·7	42·3	
—	81·1	66·6	73·5	39·9	plete and considerably worn; JR.
39·8	82·2?	—	—	—	mastoid; teeth complete in lower jaw but very worn; tympanic perfora-
37·6	87·8	65·4	72·5	42·1	
41·0	83·6	63·4	72·0	44·6	l wormians in coronal and lambdoid sutures; trace of transverse occipital ly retreating frontal bone and prominent superciliary ridge; basi-occipital
42·1	76·6	63·7	72·5	43·8	
—	90·7	62·8	76·2	41·0	wormians in lambdoid suture R and L; 4 or 5 teeth lost from upper jaw
40·0	86·8	64·7	73·3	42·0	cles in lambdoid suture; several teeth lost from upper jaw, lower teeth
39·6	85·1?	66·4	72·1	41·5	eeth very worn; large epipteric bone L; JL.
39·7	84·1	65°·1	72°·4	42°·5	
35	37	36	36	36	

with the slightly differing early Egyptian types by our present series. These relationships suggest that we are dealing with a sample from a population which was predominantly of Egyptian origin and they may be taken to indicate that at some unknown period there was a direct, or indirect, link between the native Egyptians of Sedment and the Cretan people. The evidence is not sufficient to warrant any more definite statement. The Sedment has a higher cephalic index, a greater height and a higher height-length index than any other early Egyptian type: it is also differentiated by a high occipital index indicating that the arc from lambda to opisthion is less convex than usual. For the first three of these characters the divergences are in the direction of the modern Cretan type, but data for that type are not available in the case of the occipital index. Male and female type contours and mandibular measurements are presented, but owing to the lack of sufficient comparative material no definite conclusions can be deduced from these. The maximum breadth of the transverse section is relatively higher for the Sedment than for any other series for which transverse type contours have been published.

In conclusion I should like to express my great gratitude to Professor Karl Pearson for permitting me to undertake this research in his Laboratory, to Dr Morant for aid in many ways and to Miss McLearn for drawing the type contours.

A STATISTICAL STUDY OF CERTAIN ANTHROPOMETRIC MEASUREMENTS FROM SWEDEN.

BY P. C. MAHALANOBIS, Presidency College, Calcutta.

1. *Introduction.* The present paper consists of a statistical study of certain anthropometric data from Sweden, which have been taken from *The Racial Characters of the Swedish Nation*, edited by H. Lundborg and F. J. Linders, and published by the Swedish State Institute for Race Biology, Uppsala, in 1926*. My aim is to make a first application of the Coefficient of Racial Likeness to the discrimination of racial differences to be ascertained from measurements on the living. Hitherto the method of the C.R.L. has been applied chiefly to craniometric data†. As I have indicated in an earlier memoir (*Biometrika*, Vol. XX^{A}, pp. 1—31) the want of standardisation renders analysis of measurements on the living by this, or indeed by any other, method largely futile.

The material consists of measurements of 46,983 conscripts and regular soldiers belonging to the Swedish Army and Navy. The subjects were all born in Sweden and were over 20 and under 22 years of age. The measurements were taken in 1922 and 1923, each person being measured by two observers. Special precautions were taken to ensure the same standards being maintained by all observers. One of the examiners measured the entire naval force, and another examined nearly half of the persons included in the investigation. The total number of observers was small, and all of them were connected with the Swedish State Institute for Race Biology. It may be assumed therefore that the present series of measurements are standardised and comparable *inter se*. Measurements of 404 persons born in foreign countries are available, but they were excluded from my analysis.

The birthplace of the person examined was chosen as the basis for the regional grouping of the material into five territories:

(A) *North Sweden*, comprising the provinces of Lappland, Västerbotten and Ångermanland.

(B) *West Sweden*, comprising Jämtland, Härjedalen, Dalarne, Värmland, Västmanland, Närke, Dalsland, Bohuslän and Västergötland.

(C) *East Sweden*, comprising Medelpad, Hälsingland, Gästrikland, Uppland, Södermanland, Östergötland, Småland and Öland Island.

(D) *South Sweden*, comprising Halland, Skåne, Blekinge and Gotland Island.

(E) The four biggest *Cities*: Stockholm, Göteborg, Malmö and Norrköping.

* I am much indebted to Professor H. Lundborg for kindly sending me a copy of this book immediately after its publication.

† In *Biometrika* it has been applied to racial characters in silkworms and to those of Macedonian local groups.

The material from each territory was further classified into four groups on an occupational basis:

(α) *Agricultural* communities in which, according to the 1910 Census, more than 60 °/₀ of the inhabitants earned their livelihood through agriculture, forestry and fishing.

(β) *Mixed* communities in which the corresponding percentage was under 60 but over 30.

(γ) *Industrial* communities in which the percentage was less than 30.

(δ) A fourth group, the *Urban* communities, consisted of the inhabitants of the cities, towns and market towns (exclusive of (E)).

We thus get the scheme (shown in Table I) for the whole of Sweden divided into 17 sections.

TABLE I.

Divisions of the Population of Sweden.

Territory	Occupational Group	Section Number	Number of persons examined
(A) North	Agricultural	1	2993
	Mixed ...	2	1059
	Industrial	3	406
	Urban ...	4	337
(B) West	Agricultural	5	7054
	Mixed ...	6	3200
	Industrial	7	1245
	Urban ...	8	1723
(C) East	Agricultural	9	6496
	Mixed ...	10	4642
	Industrial	11	1894
	Urban ...	12	2465
(D) South	Agricultural	13	3687
	Mixed ...	14	2665
	Industrial	15	625
	Urban ...	16	1737
(E) Four Largest Cities		17	4755
		Total	46,983

Mean values of the different characters for each section are shown in Table III on the following page.

Pooling certain of the above sections we obtain the geographical territories and occupational classes shown in Table II.

Several of the mean values for the occupational classes given in Table III were calculated by me; the other figures were taken from the published volume.

I may note here that after a careful comparison with the Census figures for the whole of Sweden the authors came to the conclusion that the geographical as well

as the occupational and social distributions of the persons measured were representative of the whole population. In other words, the present material may be considered to be a fair sample of the male Swedish population for the age-group 20—22 years*.

The values of the general means and standard deviations for the total sample are given in Table IV. The standard deviations for Bi-acromial Index, Supra-

TABLE II.

Divisions of the Population of Sweden (continued).

Territory	Number examined	Occupational Class	Number examined
(A) North	4,795	(α) Agricultural	20,230
(B) West	13,222	(β) Mixed ...	11,566
(C) East	15,497	(γ) Industrial	4,170
(D) South	8,714	(δ) Urban ...	6,262
(E) Cities	4,755	(E) Cities ...	4,755
Total	46,983	Total	46,983

TABLE IV.

General Means, Standard Deviations and Coefficients of Variation, with their Probable Errors†, based on the total Population of 46,983.

(Body measurements are in cms. and head measurements in mms.)

Character	Mean	Standard Deviation	Coefficient of Variation
1. Stature	$172 \cdot 23 \pm \cdot 018$	$5 \cdot 93 \pm \cdot 013$	$3 \cdot 44 \pm \cdot 008$
2. Supra-sternal Height ...	$140 \cdot 89 \pm \cdot 013$	$5 \cdot 29 \pm \cdot 012$	$3 \cdot 75 \pm \cdot 008$
3. Trunk Length	$52 \cdot 37 \pm \cdot 007$	$2 \cdot 41 \pm \cdot 005$	$4 \cdot 60 \pm \cdot 010$
4. Arm Length	$78 \cdot 46 \pm \cdot 010$	$3 \cdot 34 \pm \cdot 007$	$4 \cdot 26 \pm \cdot 009$
5. Leg Length	$92 \cdot 02 \pm \cdot 013$	$4 \cdot 30 \pm \cdot 009$	$4 \cdot 67 \pm \cdot 010$
6. Bi-acromial Diameter ...	$39 \cdot 23 \pm \cdot 005$	$1 \cdot 67 \pm \cdot 004$	$4 \cdot 26 \pm \cdot 009$
7. Inter-iliocristal Breadth ...	$28 \cdot 80 \pm \cdot 005$	$1 \cdot 52 \pm \cdot 003$	$5 \cdot 27 \pm \cdot 012$
8. Trunk Length Index ...	$30 \cdot 49 \pm \cdot 004$	$1 \cdot 18 \pm \cdot 003$	$3 \cdot 88 \pm \cdot 009$
9. Leg Length Index	$53 \cdot 43 \pm \cdot 004$	$1 \cdot 29 \pm \cdot 003$	$2 \cdot 41 \pm \cdot 005$
10. Bi-acromial Index	$22 \cdot 80 \pm \cdot 003$	$0 \cdot 92 \pm \cdot 002$	$4 \cdot 04 \pm \cdot 009$
11. Head Length	$193 \cdot 84 \pm \cdot 019$	$6 \cdot 19 \pm \cdot 014$	$3 \cdot 20 \pm \cdot 007$
12. Head Breadth	$150 \cdot 44 \pm \cdot 016$	$5 \cdot 10 \pm \cdot 011$	$3 \cdot 39 \pm \cdot 007$
13. Face Breadth	$136 \cdot 02 \pm \cdot 015$	$4 \cdot 84 \pm \cdot 011$	$3 \cdot 56 \pm \cdot 008$
14. Morphological Face Height	$126 \cdot 57 \pm \cdot 022$	$6 \cdot 92 \pm \cdot 015$	$5 \cdot 46 \pm \cdot 012$
15. Minimum Frontal Diameter	$104 \cdot 57 \pm \cdot 013$	$4 \cdot 33 \pm \cdot 010$	$4 \cdot 14 \pm \cdot 009$
16. Cephalic Index	$77 \cdot 69 \pm \cdot 010$	$3 \cdot 14 \pm \cdot 007$	$4 \cdot 04 \pm \cdot 009$
17. Morphological Face Index...	$93 \cdot 14 \pm \cdot 017$	$5 \cdot 61 \pm \cdot 012$	$6 \cdot 02 \pm \cdot 013$

* Lundborg and Linders say: "the geographical distribution of the primary material must be regarded as satisfactory" (*op. cit.* p. 18), and again: "the agreement (with Census figures) must be regarded as good and the primary material fully representative from the social standpoint" (p. 20).

† Standard Errors are given throughout *The Racial Characters of the Swedish Nation*, and the probable errors in this table were found as those in Table III.

*

Divisions		Size of Sample (n)	Stature	Breadth	Morphological Face Height	Minimum Frontal Diameter	Cephalic Index	Morphological Face Index
Sections	1	2,993	171·34 ± ·07	3 ± ·06	127·90 ± ·09	105·26 ± ·05	78·86 ± ·04	93·10 ± ·07
	2	1,059	171·30 ± ·12	3 ± ·10	127·67 ± ·16	105·02 ± ·09	78·82 ± ·07	93·26 ± ·13
	3	406	169·86 ± ·19	9 ± ·16	125·86 ± ·24	104·53 ± ·15	79·01 ± ·11	92·81 ± ·20
	4	337	171·67 ± ·23	5 ± ·17	125·91 ± ·27	104·47 ± ·16	78·47 ± ·12	93·28 ± ·21
	5	7,054	172·50 ± ·05	2 ± ·04	127·35 ± ·06	104·53 ± ·03	77·38 ± ·02	93·77 ± ·04
	6	3,200	172·45 ± ·07	4 ± ·06	126·94 ± ·08	104·54 ± ·05	77·25 ± ·04	93·59 ± ·07
	7	1,245	172·01 ± ·11	4 ± ·09	126·78 ± ·12	104·70 ± ·08	77·25 ± ·06	93·75 ± ·10
	8	1,723	172·31 ± ·10	7 ± ·08	126·11 ± ·11	104·20 ± ·07	77·20 ± ·05	93·45 ± ·09
	9	6,496	172·35 ± ·05	2 ± ·04	126·74 ± ·06	104·48 ± ·04	77·49 ± ·03	93·18 ± ·05
	10	4,642	172·19 ± ·06	9 ± ·05	126·30 ± ·07	104·29 ± ·04	77·64 ± ·03	93·00 ± ·06
	11	1,894	172·28 ± ·09	3 ± ·07	125·66 ± ·10	104·00 ± ·07	77·41 ± ·05	92·76 ± ·09
	12	2,465	172·49 ± ·08	7 ± ·07	125·56 ± ·09	104·19 ± ·06	77·50 ± ·04	92·78 ± ·07
	13	3,687	171·92 ± ·06	1 ± ·05	126·61 ± ·08	105·13 ± ·05	78·02 ± ·03	92·69 ± ·06
	14	2,685	171·78 ± ·08	1 ± ·06	126·16 ± ·09	104·94 ± ·06	78·18 ± ·04	92·56 ± ·07
	15	625	171·61 ± ·16	3 ± ·13	126·32 ± ·18	105·16 ± ·12	77·82 ± ·08	92·70 ± ·15
	16	1,737	172·04 ± ·09	0 ± ·08	126·26 ± ·11	104·81 ± ·07	77·84 ± ·05	92·92 ± ·09
(E)	17	4,755	173·03 ± ·06	8 ± ·05	125·61 ± ·07	104·21 ± ·04	77·53 ± ·03	92·87 ± ·05
(North)	A	4,795	171·23 ± ·06	1 ± ·05	127·53 ± ·07	105·09 ± ·04	78·84 ± ·03	93·12 ± ·06
(West)	B	13,222	172·43 ± ·03	1 ± ·03	127·03 ± ·04	104·50 ± ·03	77·21 ± ·02	93·68 ± ·03
(East)	C	15,497	172·32 ± ·03	9 ± ·03	126·29 ± ·04	104·32 ± ·02	77·63 ± ·02	93·01 ± ·03
(South)	D	8,714	171·88 ± ·04	5 ± ·03	126·38 ± ·05	105·01 ± ·03	78·02 ± ·02	92·69 ± ·04
(Agric.)	α	20,230	172·17 ± ·03	6 ± ·02	127·10 ± ·03	104·92 ± ·02	77·75 ± ·01	93·28 ± ·03
(Mixed)	β	11,566	172·10 ± ·04	7 ± ·03	126·67 ± ·04	104·67 ± ·03	77·76 ± ·02	93·08 ± ·04
(Industr.)	γ	4,170	171·86 ± ·06	7 ± ·05	126·10 ± ·07	104·43 ± ·05	77·58 ± ·03	93·05 ± ·06
(Urban)	δ	6,262	172·27 ± ·05	8 ± ·04	125·92 ± ·06	104·37 ± ·04	77·56 ± ·03	93·02 ± ·05

* Standard Errors are given throughout *Then* values calculated from the frequency distributions themselves. Neither were available for the cases of the Sutimate values calculated for the total sample (see p. 97).

(b) Coefficients for Territories and Occupational Classes.

D (South)	E (Cities)	α (Agric.)	β (Mixed)	γ (Industrial)	δ (Urban)	
1·18±·004	3·13±·005	—	—	—	—	A (North)
0·84±·002	1·25±·003	—	—	—	—	B (West)
0·43±·002	0·80±·003	—	—	—	—	C (East)
D	0·99±·004	—	—	—	—	D (South)
	E	1·56±·003	0·97±·003	0·60±·005	0·23±·004	E (Cities)
		α	0·11±·002	0·78±·003	1·03±·002	α (Agricultural)
			β	0·30±·004	0·49±·003	β (Mixed)
				γ	0·08±·005	γ (Industrial)
					δ	

9							
0·21±·004	10						
1·05±·008	0·33±·009	11					
1·01±·006	0·36±·007	0·06±·011	12				
0·50±·005	0·39±·006	1·07±·009	0·96±·008	13			
0·79±·006	0·39±·007	0·79±·010	0·67±·009	0·09±·007	14		
1·21±·020	0·61±·021	0·66±·024	0·51±·023	0·38±·022	0·14±·023	15	
1·72±·008	0·87±·009	0·53±·013	0·34±·011	0·86±·010	0·51±·011	0·11±·025	16
1·53±·004	0·89±·005	0·45±·009	0·18±·007	1·45±·006	1·12±·007	0·92±·021	0·50±·009
9	10	11	12	13	14	15	16
C (East)				D (South)			

To face p. 97

sternal Height, and Leg Length Index were not given by the authors. They were obtained indirectly in the way explained below. The Bi-acromial Index is defined as the ratio of the Bi-acromial Diameter to the Stature, i.e.,

$$\text{Bi-acromial Index } (z) = 100\,\frac{\text{Bi-acromial diameter } (x)}{\text{Stature } (y)}.$$

Therefore writing r_{xy} as the correlation between Bi-acromial Diameter (x) and Stature (y), we have approximately:

$$v_z^2 = v_x^2 + v_y^2 - 2r_{xy}\,v_x v_y,$$

where v_z, v_x, v_y are the coefficients of variation ($100\,\sigma/M$) for the Bi-acromial Index, Bi-acromial Diameter and Stature respectively. Substituting the constants for the total sample (Table IV):

$$M_y = 172{\cdot}23,\quad \sigma_y = 5{\cdot}93,\quad v_y = 3{\cdot}44,\quad r_{xy} = +0{\cdot}47,$$
$$M_x = 39{\cdot}23,\quad \sigma_x = 1{\cdot}67,\quad v_x = 4{\cdot}26,\quad M_z = 22{\cdot}80,$$

we obtain $\sigma_z = 0{\cdot}92$ approximately.

Again the authors define* (p. 73)

$$\text{Supra-sternal Height } (z) = \text{Trunk Length } (x) + \text{Leg Length } (y) - 3{\cdot}5 \text{ cm.}$$

so that $$\sigma_z^2 = \sigma_x^2 + \sigma_y^2 + 2r_{xy}\sigma_x\sigma_y.$$

Since $$\sigma_x = 2{\cdot}41,\quad \sigma_y = 4{\cdot}30,\quad \text{and}\quad r_{xy} = +0{\cdot}18$$

we have $\sigma_z^2 = 28{\cdot}0288$, and $\sigma_z = 5{\cdot}29$ approximately.

Finally, the standard deviation for Leg Length Index was directly calculated from the frequency distribution given in the Swedish work, Table VIII (Supplement, p. 34).

The authors state that the measurements were taken to the nearest millimetre with an "Anthropometer" (compass callipers or Tasterzirkel), and sliding callipers (Gleitzirkel) supplied by P. Hermann of Zürich.

The following notes on measurements are given (pp. 10—11):

"*Bi-acromial Diameter*, defined as the distance between the acromial points, is measured, in departure from the instructions given in Martin†, from the back, the immediate reason for this being to control the posture during measurement."

"*Morphological Face Height* must be regarded as less exactly determined than the other measurements of the head, since the examiner often cannot locate the nasion (sutura naso-frontalis) with certainty."

"*Trunk Length* was calculated as the difference between the supra-sternal height and the height of symphysion."

"*Height of Symphysion* (upper border of symphysis pubis in the middle line). Measurement is rendered difficult in rare cases of excessive corpulence."

"*Arm Length* is the difference between the height of acromion and the height of dactylion, and *Leg Length* is obtained by adding 35 mm. to the height of symphysion (all according to Martin†)."

* [An arbitrary definition, which does not allow for personal or racial variation. ED.]

† It is clear from the remarks under "Bi-acromial Diameter" and "Leg Length" that Martin's directions (presumably those given in his *Lehrbuch der Anthropologie*, 1st edition, 1914) were followed in all cases unless otherwise mentioned.

"*Height of Acromion* could not always be determined with accuracy, since the processus acromialis sometimes showed malformation or at least considerable deviations from the normal form."

"*Height of Dactylion* presented difficulties of measurement in certain cases when the subject could not fully extend the right arm, also when malformations existed in the fingers of the right hand. In such cases this measurement was made from the left."

The *Bi-acromial, Leg Length, Trunk Length* and *Arm Length Indices* all have the stature as the denominator.

2. *Comparisons by the Method of the Coefficient of Racial Likeness.* The main object of the present paper, as I have said, is to present the results of comparisons between the various groups of the Swedish material, described above, made by Professor Karl Pearson's method of the Coefficient of Racial Likeness. In *The Racial Characters of the Swedish Nation* detailed comparisons are made between the means and standard deviations of the characters considered singly, and between the correlations for some pairs of characters calculated for different divisions of the total population. These correlations are shown to be remarkably constant and few significant differences in variability are observed The means are less constant, and it was felt that a far clearer conception of the anthropological significance of these differences would be given by a generalised criterion, which takes into account a number of characters at the same time, than by the more usual method which deals with individual characters. The coefficient of racial likeness has been extensively used in craniometric work, but little has yet been done in applying it to measurements on the living. One of the principal objections against its use in this case has been the fact that the technique of measurement has not been standardised satisfactorily, and thus the data provided by different observers can seldom be compared with safety*. Such an objection does not apply to the material now under consideration; it constitutes the most complete and most valuable description of the population of a single country which has hitherto been provided. Numbers of individuals large enough to form statistically adequate samples are dealt with, which unfortunately can seldom be the case for cranial series. The number of characters determined is less satisfactory as we can only use 17, and intra-racial correlations between some pairs of these are known to be high. The problem of determining a sufficient number of head and body measurements which are all uncorrelated, or at least lowly correlated, with one another is yet unsolved, and the characters which are customarily determined have certainly not been chosen with this object in view.

If m_s is the mean of the sth character in the first group, σ_s its standard deviation and n the size of the sample, while m_s', σ_s' and n' are the corresponding

* See P. C. Mahalanobis: "On the Need for Standardisation in Measurements on the Living," *Biometrika*, Vol. xx^A. (1928), pp. 1—31.

quantities for the second sample, then Professor Pearson's coefficient of racial likeness is defined to be

$$S\left\{\frac{1}{M}\frac{(m_s-m_s')^2}{\frac{\sigma_s^2}{n}+\frac{\sigma_s'^2}{n'}}\right\}-1+\frac{1}{M}\pm\cdot 67449\sqrt{\frac{2}{M}}\ \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where there are M characters compared*. If pairs of samples are drawn from the same population the coefficients between them will vary round zero with the probable error shown. In the present investigation the number of characters used (17) is the same in every comparison and the term $\frac{1}{M}(=\cdot 06)$ has been neglected. The standard deviations are those of the samples and when these are small, as in craniometric work, it has been customary to suppose that they are equal to one another and to the values available for the longest related racial series. The constants have been provided for the Swedish data and they are practically identical for different sections in the case of a particular character and also equal to the general standard deviations calculated for the total sample of 46,983. If the last be denoted by $\bar{\sigma}_s$, then the calculation is greatly simplified by assuming that $\sigma_s=\sigma_s'=\bar{\sigma}_s$. The coefficient becomes

$$S\left(\frac{1}{M}\frac{nn'}{n+n'}\frac{(m_s-m_s')^2}{\bar{\sigma}_s^2}\right)-1\pm\cdot 67449\sqrt{\frac{2}{M}}\ \ldots\ldots\ldots\ldots\ldots(2),$$

when the term $\frac{1}{M}$ is neglected. This is the form which has been used. Values of $\bar{\sigma}$ for the 17 characters are given in Table IV. The characters used should theoretically be uncorrelated with one another, but this condition is far from being satisfied. We are dealing with five indices and the two component lengths from which each is derived are used in addition. The spurious correlations in such cases are probably all greater than 0·5. A number of the absolute measurements also cover one another. Several of the correlations are given in *The Racial Characters of the Swedish Nation*, and in the case of stature and leg length the values for five groups and for the total sample are between 0·86 and 0·88. If the condition were made that no pairs of the measurements used should have correlations greater than 0·5 with one another, then all except three or four of the 17 would have to be rejected. The inclusion of highly correlated measurements is necessitated if the Swedish material is to be dealt with by the method of the coefficient of racial likeness, although these high correlations are far from satisfactory. The procedure is partly justified, perhaps, by the fact that precisely the same group of characters is used in every case. The comparison of these coefficients of racial likeness with others calculated for a different group of measurements would not be justified.

The coefficient provides a measure of the probability that the two samples compared were drawn from the same population. This probability will depend on the sizes of the samples available. It has been suggested that comparable measures of

* Karl Pearson: "On the Coefficient of Racial Likeness," *Biometrika*, Vol. XVIII. (1926), pp. 105—117.

the absolute divergences of the populations represented by the samples may be obtained by reducing each coefficient to the value it would have if each sample were of a standard size*. In the present paper the coefficients have been reduced to values they would have had if each series in the comparison had contained 100 individuals. These values are given by

$$50 \times \frac{\bar{n}+\bar{n}'}{\bar{n}\bar{n}'}\left\{S\left(\frac{1}{M}\frac{nn'}{n+n'}\frac{(m_s-m_s')^2}{\bar{\sigma}_s^2}\right)-1\right\} \pm \cdot 67449 \times 50 \times \frac{\bar{n}+\bar{n}'}{\bar{n}\bar{n}'}\sqrt{\frac{2}{M}} \ldots (3).$$

Crude coefficients of racial likeness, calculated from formula (2), were first found for the 17 sections defined in Table I and for the territories and occupational classes defined in Table II. The occupational samples for the whole country were made up by pooling the relevant sub-groups of the four major territorial divisions and hence some of the larger groups are not independent samples. No comparisons were made in such cases. Every crude coefficient differs significantly from zero. The values for the sections range from $1{\cdot}04 \pm {\cdot}23$ to $142{\cdot}54 \pm {\cdot}23$ and the mean of the 136 coefficients is 24·85. The values between the territories and occupational classes range from $3{\cdot}99 \pm {\cdot}23$ to $169{\cdot}34 \pm {\cdot}23$ and the mean of the 20 coefficients is 68·22. The difference between these means must be attributed to the fact that the samples are larger in the one case than in the other. All the samples are large and hence it is not surprising to find that the majority of the coefficients are of an order which would indicate marked racial divergence if found for short cranial series. The coefficients clearly increase with the sizes of the series compared and no direct comparison can be made between them until correction is made for this varying factor.

Reduced coefficients of racial likeness calculated from formula (3) are given in Table V*a* for the 17 sections and in Table V*b* for the five territories and four occupational classes. The reduction when all the means are supposed to be based on 100 individuals only is very great in all cases, and values as low as many shown have seldom been found for cranial comparisons. All differ significantly from zero. The 136 coefficients between the sections range from $0{\cdot}05 \pm {\cdot}005$ to $5{\cdot}98 \pm {\cdot}030$ and their mean is 1·38: the 20 coefficients between the territories and occupational classes range from $0{\cdot}08 \pm {\cdot}005$ to $3{\cdot}13 \pm {\cdot}005$ and their mean is 0·97. All the lowest reduced coefficients between the sections are indicated in Fig. I. These measures of relationship suggest an arrangement of the territories (Table V*b*) which is almost linear. The north and west divisions occupy extreme positions, with the east very close to the west and the south closest to the east and rather closer to the west than to the north†. A comparison of the sections of the territories representing any one particular occupational class leads to almost precisely the same geographical arrangement. The urban sections Nos. 4, 16, 12 and 8 are of the north, south, east and west territories respectively, and their reduced coefficients of racial likeness (Table V*a*) give the same arrangement as the total samples for the territories, except

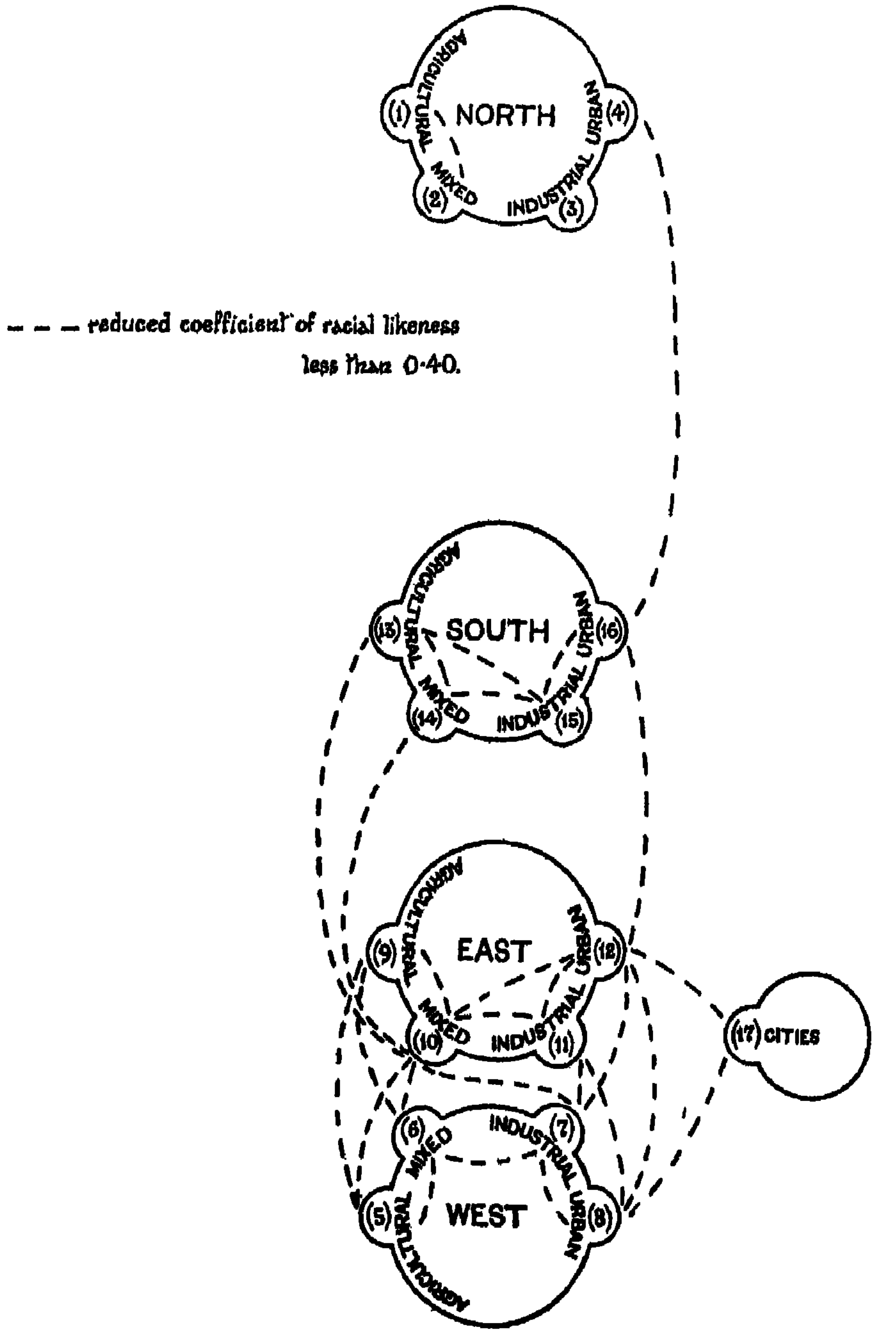
* Karl Pearson: "Note on Standardisation of Method of using the Coefficient of Racial Likeness," *Biometrika*, Vol. xx[B]. (1928), pp. 376—378.

† To correspond more exactly to this arrangement the distance between the circles representing the north and south territories in Fig. I should be increased considerably.

that the south is now rather nearer to the north than to the west. The same is found for the industrial sections except that the south is rather nearer to the west than to the east. For the mixed sections the south is again rather nearer to the north than to the west, and the single inversion in the case of the agricultural sections is the slightly closer approach of the north to the west than to the east. In spite of these

Fig. I

Inter-relationships of various Groups of the Population of Sweden.



small discrepancies, it is true to say that the best linear arrangement of the territories is precisely the same whether we consider their total samples, or the samples for any single one of the four occupational classes. The underlying geographical, or racial, differences can be appreciated nearly as well by considering one particular class only as by considering the total populations irrespective of class. It does not follow from this fact, of course, that the relationships between the territories are precisely the same for one class as for another and, indeed, the contrary can be easily demonstrated. In the comparison of any pair of the four geographical divisions the reduced coefficient between the urban sections is always less than the coefficient between the sections representing any other occupational class. The mixed sections have the next closest degree of relationship in four out of the six possible comparisons, and in five cases the coefficients between the industrial sections are the greatest found. The classes can thus be arranged fairly definitely in the sequence: urban—mixed—agricultural—industrial, with the first on the average showing the minimum and the last the maximum racial differences. All the most intimate connections between the sections are shown in Fig. I, the upper limit being fixed arbitrarily as a reduced coefficient of 0·40. The territories were arranged by considering their total samples and these closest links are now only found between the sections of contiguous territories, and there are far more of them between the east and west sections than between those of the east and south, or south and north territories. A comparison of the total occupational samples for the whole of Sweden (Table V*b*, facing p. 97) gives the definite sequence: agricultural—mixed—industrial—urban, and precisely the same order is given by the sections within any one of the territories west, east or south. The connections between any two classes are approximately the same for all these three. The sections of the north territory have different relationships. The agricultural and mixed sections are still as closely connected as for the other territories, but the resemblances of all other pairs of sections are far less close. The same two also stand nearer to the urban than to the industrial sections and the urban stands nearer to the mixed than to the industrial. It is clear that the occupational arrangement which applies uniformly to the south, east and west territories is different in the case of the north owing to the less homogeneous racial constitution of the last territory. Its agricultural and mixed sections are closely linked to one another and they are distinct from all other samples and must therefore be supposed to contain a peculiarly large proportion of a racial element which is foreign to the bulk of the Swedish population. The industrial community of the north territory also stands apart but the urban is not distinguished in this way (see Fig. I). The racial significance of the observed relationships will be considered later.

3. *Comparisons of Individual Characters.* In making comparisons by the method of the coefficient of racial likeness it has been constantly observed that on the average the differences between the various characters vary greatly in significance. The values of the α's* have been used in examining this point, but one

$$* \quad \alpha = \frac{nn'}{n+n'}\left(\frac{m_s - m_s'}{\sigma_s}\right)^2.$$

objection to their use is that they are influenced, like the coefficients, by the sizes of the samples compared. Since all characters for any one of our samples are based on the same number of individuals it was not necessary to calculate the individual α's in the present investigation. A more direct method of grading the characters can be employed, however. In Table VI are given the inter-group standard deviations (Σ) for the 17 sections of the Swedish material. The co-group standard deviations ($\bar{\sigma}$) in the same table are the general values given for the total sample

TABLE VI.

Inter- and Co-Group Standard Deviations with their Probable Errors.

Character	Inter-group Standard Deviation (Σ)*	Co-group Standard Deviation ($\bar{\sigma}$)†	$\frac{\Sigma}{\bar{\sigma}}$
Cephalic Index	·565 ± ·065	3·14 ± ·007	·180
Head Breadth	·804 ± ·093	5·10 ± ·011	·158
Arm Length	·509 ± ·059	3·34 ± ·007	·153
Bi-acromial Index	·132 ± ·015	0·92 ± ·002	·143
Face Breadth	·655 ± ·076	4·84 ± ·011	·135
Inter-iliocristal Breadth ...	·205 ± ·024	1·52 ± ·003	·135
Bi-acromial Diameter ...	·214 ± ·025	1·67 ± ·004	·128
Stature	·680 ± ·079	5·93 ± ·013	·115
Leg Length	·478 ± ·055	4·30 ± ·009	·111
Head Length	·686 ± ·079	6·19 ± ·014	·111
Supra-sternal Height ...	·569 ± ·066	5·29 ± ·012	·108
Morphological Face Height	·688 ± ·080	6·92 ± ·015	·099
Trunk Length Index ...	·106 ± ·012	1·18 ± ·003	·091
Minimum Frontal Diameter	·372 ± ·043	4·33 ± ·010	·086
Leg Length Index	·106 ± ·012	1·29 ± ·003	·082
Trunk Length	·175 ± ·020	2·41 ± ·005	·072
Morphological Face Index...	·368 ± ·043	5·61 ± ·012	·066

of 46,983 individuals and these are almost precisely the same as the values found for any one of the 17 sections. The $\bar{\sigma}$'s are the ones which were used in computing the coefficients of racial likeness. It is clear that the ratio of Σ to $\bar{\sigma}$ will give a measure of the average significance of the differences found for the various characters. The inter-group variability is small compared with the intra-group variability in every case, but there are still marked differences between the measurements in this respect. The cephalic index tends to show more significant differences than any other character and this has been confirmed in the case of several other comparisons of measurements made either on the living or on the skull. It has been usual to find, too, that the head breadth varies more significantly than the head length and much more significantly than the minimum frontal diameter. The stature is less capable of differentiating the groups than several of the other characters. An index, such as the cephalic or bi-acromial, may vary more significantly than either of its component lengths, or the reverse may hold, as for the leg length and morphological face indices.

* These standard deviations are for the means of the 17 sections given in Table III.

† These standard deviations are for the total sample of 46,983 individuals.

The 17 characters may now be considered individually with regard to both geographical position and occupational class. They can be divided into a number of groups by considering whether the order in which each arranges the 17 sections is controlled more by one of these factors than by the other. The cephalic index is extreme in this respect. The four lowest means are for the sections of the west territory, the sections of the east and the urban section (No. 17) follow next and then the four of the south, while the cephalic indices for the sections of the north territory are greater than any others. There is thus a clear distinction between the territories, and they are arranged in the order shown in Fig. I. The maximum difference between the sections of the same territory with extreme cephalic indices is only 4·0 times its probable error (south and east territories) and no significance whatever can be attached to the orders in which the occupational classes are arranged within the territories. This character is clearly controlled by geographical position and there is no evidence of any significant association with occupational class. The bi-acromial index affords an example of a measurement which is affected by conditions entirely different from these. The order in which the means arrange the 17 sections appears to have no geographical significance whatever, but the three highest indices are for agricultural sections, the lowest is for the sample from the four largest cities (No. 17) and the next four lowest are for the other urban sections. For each territory the highest index is for the agricultural section, the second highest for the mixed, the next for the industrial and the lowest for the urban section. The differences between the agricultural and urban sections of the same territory are very significant in every case, being 8·4, 12·2, 16·9 and 15·1 times their probable errors for the north, south, east and west divisions respectively. The bi-acromial index is thus clearly controlled by the occupational class, and there is no evidence of any significant association with geographical position. These two characters are at opposite extremes in so far as they are controlled by one or other of the factors on the basis of which the groupings were made, but in most other cases there is a definite tendency for a measurement to approach one extreme in this respect rather than the other. The orders in which the sections are arranged may be supposed to have been influenced by both factors in the majority of cases. Whenever there is a clear territorial sequence, or the suggestion of such a sequence, it is always: north, south, east and west. Whenever there is a clear occupational sequence common to all the territories, or the suggestion of such a sequence, it is always: agricultural, mixed, industrial and urban. Paying due regard to the significance of the differences, the following classification of the characters can be made:

(*a*) Characters showing markedly significant territorial differences, but no occupational sequence within the territories—cephalic index, stature, supra-sternal height.

(*b*) Characters showing significant territorial differences and a significant occupational sequence within the territories—head breadth, head length, inter-iliocristal breadth.

(*c*) Characters showing a suggestion of territorial differences, but no occupational sequence within the territories—minimum frontal diameter, leg length.

(*d*) Characters showing a faint suggestion of territorial differences and a markedly significant occupational sequence within the territories—face breadth, arm length, bi-acromial diameter.

(*e*) Characters showing a faint suggestion of territorial differences and a significant occupational sequence within the territories—leg length index, trunk length index.

(*f*) Characters showing no territorial differences and a markedly significant occupational sequence within the territories—morphological face height, bi-acromial index.

(*g*) Characters showing no territorial differences and no occupational sequence within the territories—morphological face index, trunk length.

The comparison of individual characters has confirmed in a very satisfactory way the scheme of relationships suggested by the coefficients of racial likeness. It can now be seen that there are marked differences between the characters not only in their average effect on the coefficients, but also according as they are more or less capable of discriminating between regional or occupational samples. The two in group (*g*) above are the only ones which appear to be quite incapable of serving either purpose and these are the two with the lowest values of $\Sigma/\bar{\sigma}$ (see Table VI). By making a suitable selection from the other 15 it would clearly be possible to obtain coefficients which would emphasise the geographical differences and obscure the occupational, while the reverse effect could be obtained by making a different selection. The characters which show territorial differences will be considered again in the next section. There are seven absolute and three indicial measurements which furnish either a significant, or a markedly significant occupational sequence, and for all except one the agricultural section tends to have the greatest mean and the urban the least. The trunk length index is greater for the urban than for the rural populations, but the reverse is found for the bi-acromial diameter and index, the arm length, the head and face breadths, the head length, the morphological face height, the inter-iliocristal breadth and the leg length index. The fact that the first three of the last nine measurements are greater for rural than for urban samples was to be expected. The relations observed in the case of the others suggest that the agricultural workers have skeletons which are broader in all ways and with relatively longer limbs than town dwellers, though no differences between the statures of the groups can be detected. The differences between the extreme means are all very small, however. The bi-acromial diameter provides a more definite occupational sequence than any other absolute measurement, but the largest mean for a section only exceeds the smallest by 7·4 mm. Whether any of the differences between occupational classes are due to use, or whether they are occasioned by selection, cannot be decided from these Swedish data. With smaller samples, or in the case of a more racially heterogeneous population, it would probably be impossible to prove their existence.

4. *The Racial Constitution of the Swedish Population.* The present study is restricted, on the regional side, to a comparison of the four territorial divisions into

which the whole of Sweden was divided, and some important facts may be overlooked by taking such large areas. All the individuals examined were born in Sweden. The remarkable constancy of the coefficients of variation and correlation, provided in *The Racial Characters of the Swedish Nation*, suggests that the populations are now thoroughly hybridised if they once had diverse racial origins. The coefficients of racial likeness suggested the simple linear arrangement shown in Fig. I and the reasonableness of this order was emphasised by finding that all the characters which are capable of making definite distinctions between the territories show the same sequence from the north at one extreme to the west at the other. Of the 17 measurements there are only six which give significant or markedly significant regional differences when a single occupational class is considered. The total means for the territories are given for these in the table below and this comparison is now not quite so convincing since the relative proportions of the different classes are not the same for all the territories.

Territories	Cephalic index	Stature	Supra-sternal height	Head breadth	Head length	Inter-iliocristal breadth
North	78·84	171·23	140·13	152·27	193·33	28·43
South	78·02	171·88	140·62	150·69	193·33	28·79
East	77·53	172·32	140·96	150·14	193·85	28·86
West	77·31	172·43	141·04	150·23	194·52	28·95

Some pairs of these six measurements are lowly correlated with one another and the fact that they provide the same sequence is all the more significant on that account. The coefficients between the sections are so low that it can only be assumed that all divisions of the total population of Sweden belong predominantly to the same racial type. The observed relationships can be explained on the hypothesis that this basic type has been modified slightly, but in different degrees in different territories, by admixture with another race. The north territory was more modified than any other by this means, the south considerably less, the east still less and the west territory may have been unaffected, or modified to a less extent than any other. The racial crossing seems to have resulted in a perfect blending of all the characters for which data are available and those which show no territorial differences may be assumed to have been the same for the two racial types. These are the conclusions suggested by a purely statistical analysis of the material and we may attempt to reconcile them with what is known of the ethnic history of the country. The following particulars are taken from the section written by Rolf Nordenstreng in *The Racial Characters of the Swedish Nation* *.

"The Swedes have inhabited their country since later neolithic times. The main body of the prehistoric population seems to have been of rather distinctly Nordic

* "Origin, Growth and Racial Components of the Swedish Nation," *op. cit.*, pp. 41—49 and summary on p. 174.

race, though other types also occur ...The finds from the Bronze Age and the Iron Age do not present any new types, but agree with those from the Stone Age....The early Swedish kingdom did not consist of more than the present central territories about Lake Mälaren; but gradually other parts of the present kingdom were conquered, the people of the Gauts south of the Swedish settlements between the Baltic Sea and the North Sea being the most important of those incorporated into the nation. All these peoples on the Scandinavian Peninsula were Teutons like the Swedes, of much the same race, and using similar languages. Only in the northernmost part of the country lived Lapps, roving since prehistoric times. The Swedish dominion was early extended to territories east of the Baltic, whence in the course of time came an influx of the East Baltic race, especially in a Finnish immigration in the last years of the 16th, and the first half of the 17th century....That the Nordic race has been the chief stock of Sweden's ancient population, as of the present, is beyond all doubt. But as to what extent it was mixed and with which races, we can venture nothing more than a guess....It is not impossible that the East Baltic race is very ancient in this country, more ancient even than the Nordic, but this cannot be proved and is hardly very likely; the possibility should not, however, be wholly dismissed. The most noteworthy support is given by the type demonstrated by Arbo in South Norway and often called 'the blond brachycephal,' a type which reminds one not a little of the Finnish....According to Lönborg's calculations Sweden (except Norrbotten) and parts of East Norway had at the close of the 17th century a Finnish population of between twelve and thirteen thousand persons. This figure is very likely too low, but nevertheless is highly appreciable, considering that Sweden's entire population then amounted to hardly 1½ millions, and that the parts of the country in which the Finns were living were certainly very sparsely populated. As these immigrants were unusually prolific, their offspring undoubtedly increased at a proportionately higher rate than those of the real Swedes....There is also a Lappic race-admixture in the Finnish population of North Sweden....The number of Finnish-speaking persons in Northern Sweden probably amounts at present to about 30,000....How strong the race-mixture with East Baltic blood has been in Sweden is at present impossible to state. But it would hardly be an exaggeration to assert that at least some hundred thousand present-day Swedes and perhaps many more evince more or less East Baltic characters."

The only foreign races which are known to have influenced the population of Sweden to any marked extent since neolithic times are thus the Finns and, to a lesser extent, the inhabitants of the East Baltic states; the Lapps, as far as is known, have lived in the north as long as the country was inhabited at all. All these alien races are closely allied to one another, and, where they differ from the Swedish type, they apparently do so in the same direction as, for example, in possessing higher cephalic indices and shorter statures. The miscegenation with the so-called nordic population must have been extremely thorough, since the variabilities for all sections are almost identically the same. Even in the north, the bulk of the population must be of "nordic" origin, and it is not surprising to find that the effects of slight differences between the types with which admixture was made in

different regions cannot be detected at all by considering large groups of the existing population. Comparisons made in that way only suggest that there was a crossing with a single racial type resulting in a perfect blending of all the characters considered. The alien element is far more evident in the north than in any other territory; it produced a greater effect in the south than in the central regions of Sweden, and the east was slightly more affected than the west.

Note added in proof.

This paper was originally written as an integral part of an empirical study of certain alternative formulae for the measurement of racial divergence. Very extensive and substantial editing of the text was therefore necessary in publishing it in the present form. I am deeply indebted to Dr G. M. Morant for having carried out the editing work much more satisfactorily than I could have done myself. I also wish to acknowledge the help I received from my assistant Mr Sudhir Kumar Banerjee in reducing the statistical material for the paper. P. C. Mahalanobis, Calcutta, 22nd July, 1930.

SKEW BIVARIATE FREQUENCY SURFACES, EXAMINED IN THE LIGHT OF NUMERICAL ILLUSTRATIONS.

By S. J. PRETORIUS, M.Sc.

CONTENTS.

I.

A. *Introductory.*

Since the development of the theory of normal correlation by Galton and Dickson (1886) several attempts have been made to describe analytically a distribution of two correlated variables when both variables follow a law of skew variation. These

attempts may be considered as of two types: those that are founded on one or another hypothesis, and those that are purely empirical. This discrimination, however, is not of great practical importance. The superiority of one frequency function over another depends rather on the success with which that function can be applied to graduate data than on the manner in which it originated. The univariate functions that are in common use can, as a rule, be fitted fairly easily. But in the case of bivariate functions, the process of fitting is extremely laborious. Comparative results are rare, and such illustrations as do exist are of little final value. Either the test has not been stringent enough, or the paucity of observations has made it impossible for a dispassionate judgment to be passed on the goodness of fit. In short, the descriptive power of the various surfaces has as yet not been extensively investigated.

The purposes of this study are: (i) to present an account of the surfaces that have been evolved; (ii) to analyse geometrically a few observed distributions, each containing a large number of observations; hence (iii) to put to a practical test some of the hypotheses from which these surfaces have been developed; and (iv) finally, to compare the adequacy of the surfaces by fitting their marginal and partial moment curves to the observations.

B. *Notation and Terminology.*

To avoid unnecessary repetition, a description is given below of the notation and terminology that will be adopted.

The two variables will be denoted by x and y; the total number of observations by N; the number of observations in an x-array of y's and in a y-array of x's by n_x and n_y respectively; any cell frequency by n_{xy}; the usual correlation coefficient between x and y by r; the correlation ratio of y on x and of x on y by η_{yx} and η_{xy} respectively. The ss'th product-moment coefficient calculated from the observations about any origin will be denoted by $\mu'_{ss'}$ and $\nu'_{ss'}$, according as corrections for grouping have or have not been applied; the dashes will be dropped when the origin is the arithmetic mean $(\bar{x}, \bar{y})$. Thus

$$\nu'_{ss'} = \frac{1}{N} \Sigma\Sigma (n_{xy} \cdot x^s y^{s'}),$$

and
$$\mu_{ss'} = \frac{\iint F(x, y)(x-\bar{x})^s (y-\bar{y})^{s'}\, dx\, dy}{\iint F(x, y)\, dx\, dy},$$

where $F(x, y)\, dx dy$ expresses the probability that x lies between x and $x+dx$, y between y and $y+dy$, and the integrations are taken over the entire surface.

Still following the usual notation, I shall write:

$$q_{ss'} = \frac{\mu_{ss'}}{\sigma_1^{s} \cdot \sigma_2^{s'}}, \qquad \begin{array}{ll} \beta_{10} = \dfrac{\mu_{30}^2}{\mu_{20}^3}, & \beta_{20} = \dfrac{\mu_{40}}{\mu_{20}^2}, \\[2ex] \beta_{01} = \dfrac{\mu_{03}^2}{\mu_{02}^3}, & \beta_{02} = \dfrac{\mu_{04}}{\mu_{02}^2}, \end{array}$$

where $\sigma_1 \equiv \sqrt{\mu_{20}}$ and $\sigma_2 \equiv \sqrt{\mu_{02}}$ are the standard deviations of the x- and y-marginal totals respectively.

The moments of an array distribution will be termed *array moments*; those of a section of a theoretical surface parallel to the zx or zy plane, will be termed *partial moments*. The corrected sth moment coefficient of an x-array of y's about the mean of the array will be denoted by $\mu_s(y)$; that of a y-array of x's by $\mu_s(x)$; the x- and y-array means by $\mu_1'(y)$ and $\mu_1'(x)$ respectively. The same notation will be used for the partial moments, but it must be remembered that in this case the variable assumes "singular" and not "plural" values; this distinction is brought out by the terms introduced above. The curves in which $\mu_1'(y)$, $\sigma_y \equiv \sqrt{\mu_2(y)}$, $\sqrt{\beta_1(y)} \equiv \dfrac{\mu_3(y)}{\mu_2^{\frac{3}{2}}(y)}$ and $\beta_2(y) - 3 \equiv \dfrac{\mu_4(y)}{\mu_2^2(y)} - 3$ are plotted to x are the *regression, scedastic, clitic and kurtic curves of y on x**. A system is either *homoscedastic* or *heteroscedastic* according as the arrays "are equally scattered about their means," or not.

C. *Historical.*

1. *Writers before Galton.* The normal probability surface discussed by Lagrange, Laplace, Plana, Gauss, Bravais, has little bearing, if any at all, on the theory of correlation. With admirable clarity Pearson† pointed out a few years ago that the quantities Gauss and Bravais were observing, were absolutely independent of one another. Only by the introduction of quantities linearly related to those observed, did the product terms in their expressions arise. In a more recent paper, dealing with Plana's work, Pearson‡ again indicated that the writers on the theory of observations up to the time of Galton were concerned merely with finding a mathematical expression for the probability of the simultaneous occurrence of two or more errors and not with finding a measure of relationship between two variables organically associated.

Galton, on the other hand, started with the conception that the observed quantities are dependent. In studying the inheritance of traits, he developed in

* Pearson originally defined the scedastic curve as the curve in which the ratio of the standard deviation of the array to the standard deviation of the character in the population at large is plotted to position, and the clitic curve as the curve in which the skewness of the array is plotted to position. For the Pearson Type III curve: Skewness $= \dfrac{\text{Mean} - \text{Mode}}{\text{Standard Deviation}} = \frac{1}{2}\sqrt{\beta_1}$.

Wicksell calls the curve $y = -\frac{1}{2}\sqrt{\beta_1(y)}$ ($\equiv$ "skewness") the clitic curve, and the curve $y = \frac{1}{2}(\beta_2(y) - 3)$ ($\equiv$ "excess") the synagic curve. I prefer, however, to retain Pearson's original term "kurtosis" as expressing that deviation of frequency curves from the normal type which corresponds to forms more or less flat-topped. Further, being concerned merely with how the skewness varies from array to array and not with the degree of skewness of any particular array distribution, I shall omit all constant multiplying factors for $\sqrt{\beta_1(y)}$ and $\beta_2(y) - 3$.

† "Notes on the History of Correlation," *Biometrika*, Vol. XIII. 1920—21, pp. 25—45.

‡ "The Contribution of Giovanni Plana to the Normal Bivariate Frequency Surface," *Biometrika*, Vol. XXA. 1928, pp. 295—298. See also Walker, Helen M.: "The Relation of Plana and Bravais to the Theory of Correlation," *Isis*, Vol. X. No. 34, 1928, pp. 466—484.

a series of papers, from 1877, the ideas of regression and correlation. Dickson*, in 1886, investigated mathematically the system of concentric ellipses that would correspond to the ellipses deduced by Galton in his study of "Regression towards Mediocrity in Hereditary Stature" (1885).

2. *Skew Univariate Distributions.* At the time when Galton developed his theory of correlation, writers on mathematical statistics realised that the univariate normal law of De Moivre and Laplace could not be regarded as a universal law of frequency distribution; the presence of skewness in homogeneous material was certainly as common as that of normality. Attempts to describe analytically this skew variation, led up to the work of Gram, Thiele, Pearson, Edgeworth, Bruns, Charlier, and Kapteyn, to mention only the most prominent contributors.

It would be superfluous to give here more than a short summary of these curves such as will be required for further reference. We may conveniently treat them under the following three divisions:

(*a*) Pearson's system of skew frequency curves derived from the generalised probability equation:

$$\frac{1}{y}\cdot\frac{dy}{dx}=-\frac{x+a}{b_0+b_1x+b_2x^2};$$

(*b*) (i) Edgeworth's generalised law of error,

(ii) The Gram-Charlier Type A and Type B series;

(*c*) The translated, or transformed, curves of Edgeworth, and of Galton and MacAlister, as treated by Pearson and Wicksell.

Only (*b*) and (*c*) will be shortly discussed.

(*b*) (i) Edgeworth† deduces his generalised law of error from a consideration of a large number, n, of elemental frequency groups which satisfy certain conditions. The most important of these conditions are: that selections from different groups are independent of one another; that the chance of obtaining a particular magnitude from one group is independent of previous selections; that $\frac{\mu_p}{\sigma^p}$ is finite for all values of p in the elemental groups. On these assumptions the frequency locus of the aggregate formed, is found to be:

$$F(x)=e^{-\frac{1}{3!}k_3\frac{d^3}{dx^3}+\frac{1}{4!}k_4\frac{d^4}{dx^4}-\cdots}\cdot\phi(x),$$

where

$$\phi(x)=\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{1}{2}x^2},$$

$$k_3=\sqrt{\beta_1},\quad k_4=\beta_2-3,\ \ldots\ldots$$

* Galton, Francis: "Family Likeness in Stature." With an appendix by J. D. Hamilton Dickson, *Proc. Roy. Soc.* Vol. XL. 1886, pp. 42—73.

† "The Asymmetrical Probability Curve," *Phil. Mag.* Vol. XLI. 1896, pp. 90—99; "The Law of Error," *Camb. Phil. Trans.* Vol. XX. 1905, pp. 36—65, 113—141.

It is further shown that k_{p+2} is of the order $n^{-\frac{1}{2}p}$, so that to an approximation of the order $\frac{1}{n}$:

$$F(x)=\left[1-\frac{k_3}{3!}\frac{d^3}{dx^3}+\frac{k_4}{4!}\frac{d^4}{dx^4}+\frac{1}{2}\left(\frac{k_3}{3!}\right)^2\cdot\frac{d^6}{dx^6}\right]\phi(x) \quad \text{............(1)}.$$

Introducing Pearson's definition of the tetrachoric functions, viz.,

$$\tau_s(x)=\frac{1}{\sqrt{s!}}\left(-\frac{d}{dx}\right)^{s-1}\cdot\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{1}{2}x^2},$$

we have from (1):

$$F(x)=\tau_1+\sqrt{\tfrac{2}{3}}\cdot\sqrt{\beta_1}\cdot\tau_4+\sqrt{\tfrac{5}{24}}\cdot(\beta_2-3)\cdot\tau_5+\sqrt{\tfrac{35}{36}}\cdot\beta_1\cdot\tau_7 \quad \text{......}(1)^{bis}.$$

(*b*) (ii) Several other writers—Gram, Bruns, Charlier—have discussed a series almost identical with (1). Starting from the hypothesis of elementary errors, Charlier* deduces two forms of the frequency function, called by him Type A and Type B. Type A is an extension of the De Moivre-Laplace approximation to the binomial; Type B is an extension of the Poisson limit to the binomial. The transition from Type A to Type B cannot be expressed mathematically. Usually Type B is employed when there is a marked asymmetry, while for slightly asymmetrical curves the type can be determined only by trial.

Type A. $$F(x)=\phi(x)+\sum_{p\geq 3}(-1)^p\frac{A_p}{p!}\frac{d^p\phi(x)}{dx^p},$$

where $$A_3=\sqrt{\beta_1}=\lambda_3,\quad A_4=(\beta_2-3)=\lambda_4,$$

$$A_5=\beta_3'-10\sqrt{\beta_1}=\lambda_5,\quad A_6=\beta_4-15\beta_2+30=\lambda_6+10\lambda_3^2,$$

the λ's being the third, fourth, ... semi-invariants and $\beta_3'=\mu_5/\sigma^5$. Expressed in terms of tetrachoric functions:

$$F(x)=\tau_1+\sqrt{\tfrac{2}{3}}\cdot\sqrt{\beta_1}\cdot\tau_4+\sqrt{\tfrac{5}{24}}\cdot(\beta_2-3)\cdot\tau_5 \quad \text{..................(2)},$$

up to moments of the fourth order only.

The two approximations $(1)^{bis}$ and (2) are not quite identical. It is partly on this ground that Edgeworth† criticised the Gram-Charlier series as not being the true generalisation of Laplace's law of error. In a later paper Charlier‡ has shown the order of magnitude of the coefficients $A_p'\equiv\frac{A_p}{p!}$ to be:

* "Über das Fehlergesetz," *Arkiv för Mat., Astr. och Fysik*, Bd. 2, No. 8, 1905, pp. 1—9; "Über die Darstellung willkürlicher Funktionen," *Arkiv för Mat., Astr. och Fysik*, Bd. 2, No. 20, 1905, pp. 1—22; "Die strenge Form des Bernoulli'schen Theorems," *Arkiv för Mat., Astr. och Fysik*, Bd. 5, No. 15, 1909, pp. 1—22; "Contributions to the Mathematical Theory of Statistics. 5. Frequency Curves of Type A in Heterograde Statistics," *Arkiv för Mat., Astr. och Fysik*, Bd. 9, No. 25, 1914, pp. 1—17.

† "On the Representation of Statistical Frequency by a Series," *Journ. Roy. Stat. Soc.* Vol. LXX. 1907, pp. 102—106.

‡ "Die strenge Form des Bernoulli'schen Theorems," *Arkiv för Mat., Astr. och Fysik*, Bd. 5, No. 15, 1909, pp. 1—22.

$$\left.\begin{array}{llll} A_3' \text{ is of the order of magnitude of} & \dfrac{1}{\sqrt{n}} \\ A_4' \quad " \quad " \quad " & \dfrac{1}{n} \\ A_6' \quad " \quad " \quad " & \dfrac{1}{n} \\ A_5' \quad " \quad " \quad " & \dfrac{1}{n\sqrt{n}} \end{array}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where n is the number of "error-sources." It is further shown* that λ_6 is of the order $\frac{1}{n^2}$, so that

$$\frac{A_6}{6!} = \frac{1}{2}\left(\frac{A_3}{3!}\right)^2 + \text{term of order } \frac{1}{n^2},$$

and hence to an approximation of the order $\frac{1}{n}$:

$$F(x) = \tau_1 + \sqrt{\tfrac{2}{3}} \,.\, \sqrt{\beta_1} \,.\, \tau_4 + \sqrt{\tfrac{5}{24}} \,.\, (\beta_2 - 3) \,.\, \tau_5 + \sqrt{\tfrac{35}{36}} \,.\, \beta_1 \,.\, \tau_7,$$

which is identical with $(1)^{bis}$. In the paper just cited, Wicksell shows the orders of magnitude (3) not to be a necessary consequence of the hypothesis of elementary errors; they can be deduced only when the skewness of the error distributions is regarded as independent of n. The convergency of tetrachoric expansions has been discussed from a more practical point of view by Pearson†. He assigns definite values to β_1 and β_2 and demonstrates that, unless the skewness be chiefly in the one direction or the other, any tetrachoric term in the series is not negligible as compared with those preceding it. A good illustration of the oscillatory nature of expansions in terms of tetrachoric functions and of their practical non-convergency, is provided in a paper by James Henderson‡. Closely associated with the problem of convergency, is the appearance of negative frequencies in the tails of the curve and the impossibility of making the curve start at a fixed point. Although a fairly good description of the central part of the observations is likely to be obtained, the curve fails us almost entirely in the determination of the significance of outlying observations.

For convenience we shall refer to equation (2) as Type Aa; to $(1)^{bis}$ as Type Ab; to both or either of the two as Type A.

Type B. $\quad F(x) = k_0 \,.\, \psi_\lambda(x) - \dfrac{k_1}{1!}\Delta\psi_\lambda(x) + \dfrac{k_2}{2!}\Delta^2\psi_\lambda(x) - \ldots,$

where

$$\Delta\psi_\lambda(x) = \psi_\lambda(x) - \psi_\lambda(x-1),$$
$$\Delta^2\psi_\lambda(x) = \psi_\lambda(x) - 2\psi_\lambda(x-1) + \psi_\lambda(x-2),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

* Wicksell, S. D.: "The Correlation Function of Type A and the Regression of its Characteristics," *Kungl. Sv. Vet. Akad. Handl.* Bd. 58, No. 3, 1917, pp. 1—48.

† "The Fifteen Constant Bivariate Frequency Surface," *Biometrika*, Vol. XVII. 1925, pp. 277—280.

‡ "On Expansions in Tetrachoric Functions," *Biometrika*, Vol. XIV. 1922—23, pp. 157—185.

and
$$\psi_\lambda(x) = \frac{e^{-\lambda}}{\pi}\int_0^{+\pi} e^{\lambda\cos w} . \cos(\lambda \sin w - xw)\, dw$$
$$= \frac{e^{-\lambda} . \lambda^m}{m!}, \text{ when } x \text{ is a positive integer, } m.$$

The function $\psi_\lambda(x)$ and the more general function
$$\vartheta_{\lambda,\eta}(x) = \frac{e^{-\lambda}}{\pi}\int_0^{+\pi} e^{\lambda\cos w} . \cos(\eta \sin w - xw)\, dw$$

have been introduced by Charlier* as continuous functions representing the Poisson exponential. Charlier confined his treatment to the function $\psi_\lambda(x)$ and determined the coefficients k by an approximate method. The more general function $\vartheta_{\lambda,\eta}(x)$ has been discussed by Jørgensen†. He finds the exact values of the coefficients and considers special cases of a linear transformation of the argument. The order of magnitude of the coefficients necessitates the use of an even number of terms in successive approximations to the series.

Because of the theoretical and practical objections that can be adduced against the use of these continuous generating functions, I shall not give a detailed account of the Type B distribution. In a critical note on Jørgensen's proof of these functions Steffensen‡ has shown that the moment integrals $\int_0^\infty \vartheta(x) . x^p dx$ are divergent. Apart also from the negative frequencies that arise in applying the series, the curve assumes a sinusoidal form for fractional values of the variate: "for brudne Værdier af Abscisserne svinger de i Virkeligheden sinusoideformet og giver for store Abscisser negative Ordinater, hvad der navnlig for $\lambda = \eta$ for negative Abscisser træder stærkt frem§."

(c) *The Method of Translation.* Let $\eta = \phi(\xi)$ be the frequency curve of a hypothetical variate ξ. Replace ξ by a function $f(x)$ of x, x being the quantity observed. If the areas between corresponding ordinates of the generating curve $\phi(\xi)$ and the generated ("translated") curve $F(x)$ are to remain unchanged, then
$$y = F(x) = \phi[f(x)] . f'(x).$$

By a suitable choice of ϕ and f, the form of $F(x)$ might be such as is commonly observed in practical statistics.

* "Die Zweite Form des Fehlergesetzes," *Arkiv för Mat., Astr. och Fysik*, Bd. 2, No. 15, 1905, pp. 1—8; "Weiteres über das Fehlergesetz," *Arkiv för Mat., Astr. och Fysik*, Bd. 4, No. 13, 1907, pp. 1—9.

† "Note sur la fonction de répartition de Type B de M. Charlier," *Arkiv för Mat., Astr. och Fysik*, Bd. 10, No. 15, 1914, pp. 1—18; *Undersøgelser over Frequensflader og Korrelation.* København: Arnold Busck, 1916.

‡ *Svenska Aktuarieföreningens Tidskrift*, Nos. 4—5, 1916. See also *Matematisk Iagttagelselære*, København, 1923, p. 71.

§ Jørgensen; *loc. cit.* p. 28.

(i) *Edgeworth's Translated Curves**. Take $\phi(\xi)$ to be the normal curve:

$$\eta = \frac{1}{\sqrt{\pi}} \cdot e^{-\xi^2},$$

and consider the particular equation of translation:

$$x = a(\xi + k\xi^2 + \lambda\xi^3) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

The ordinate of the translated curve is:

$$y = \frac{1}{\sqrt{\pi}} \cdot e^{-\xi^2} \cdot \frac{1}{a(1 + 2k\xi + 3\lambda\xi^2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

and the rth moment about the origin—the median—is:

$$M_r' = \int_{-\infty}^{+\infty} y \cdot x^r dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} a^r (\xi + k\xi^2 + \lambda\xi^3)^r \cdot e^{-\xi^2} d\xi.$$

From the moment coefficients about the mean, the constants a, k, and λ can be determined. The equations are:

$$\left.\begin{array}{c} \chi(\frac{3}{2} + \chi + 9\lambda + \frac{135}{8}\lambda^2)^2 - j^2(1 + \chi + 3\lambda + \frac{15}{4}\lambda^2)^3 = 0 \\ 6\chi + 3\lambda + 3\chi^2 + 54\chi\lambda + 27\lambda^2 + 135\chi\lambda^2 + \frac{405}{4}\lambda^3 + \frac{1215}{8}\lambda^4 \\ - i(1 + \chi + 3\lambda + \frac{15}{4}\lambda^2)^2 = 0 \end{array}\right\} \ldots\ldots\ldots(6),$$

where $\qquad \chi = k^2, \quad j^2 = \beta_1/8, \quad i = \frac{1}{4}(\beta_2 - 3).$

The area subtended by the translated curve between any two values of x can be obtained, after solving the cubic (4), from tables of the normal probability integral. An ambiguity arises when the values of k and λ are such that for a certain range of x, the cubic has three real roots. The translated curve then loses its typical shape of rising continuously from a practically zero value to a maximum and falling at the same or at a different rate down to zero again. The singularities that occur are of two types. In Edgeworth's terminology: there is a "*break*" if $\frac{dx}{d\xi}$, the quadratic expression in the denominator of (5), becomes negative; there is a "*stop*" if the ordinate of the curve has a relative minimum value, that is to say, if $\frac{dy}{d\xi}$ has real roots other than the mode. After passing through the minimum value the curve ascends and ultimately changes abruptly from $+\infty$ to $-\infty$ at that value of x which corresponds to a root of $\frac{dx}{d\xi} = 0$. Edgeworth claimed that the method of translation is applicable especially to slightly and moderately abnormal curves; and he considered the construction as sufficiently accurate if no peculiarity occurs within a distance from the median of the translated curve corresponding to a distance of $|\xi| = 2$ from the mean of the generating curve. The tail areas cut off outside this range of about 2·83 times the standard deviation of the normal curve from its mean amount to only about 5 per-mille of the total frequency,

* Edgeworth's papers on the mathematical representation of statistical data appeared chiefly in the *Journ. Roy. Stat. Soc.* For a complete bibliography see A. L. Bowley: "F. Y. Edgeworth's Contributions to Mathematical Statistics," London, 1928, *Roy. Stat. Soc.*

and are therefore practically insignificant as compared to the central portion of the curve. They are folded over or swung round, so to speak, in the process of translation, the central portion being extended or contracted according to the nature of the data.

Now $\frac{dx}{d\xi}$ will be positive for all values of $|\xi|$ from 0 to 2, provided $k^2 < 9(\lambda + \frac{1}{12})^2$. Also, the derived function

$$\frac{1}{y} \cdot \frac{dy}{d\xi} = \frac{-2[3\lambda\xi^3 + 2k\xi^2 + (3\lambda + 1)\xi + k]}{1 + 2k\xi + 3\lambda\xi^2},$$

equated to zero, will have no real root within the region $|\xi| = 2$ other than the mode, provided $k^2 < \frac{100}{9}(\lambda + \frac{1}{15})^2$. These conditions, together with $k^2 < 3\lambda$, form a lower boundary to the χ, λ field within which the method can be applied. By assuming $\beta_2 = 15$ to be a fairly extreme case, Edgeworth obtained from the second of equations (6) an upper boundary to the χ, λ area which is to be searched for values of χ and λ, satisfying equations (6). Professor Bowley* utilised these conditions in constructing a table which shows the values of χ and λ to three decimal places for given $\beta_1/8$ and $\epsilon = \frac{1}{12}(\beta_2 - 3)$ by intervals of ·01.

The portion of the β_1, β_2 plane within which Edgeworth's hypothesis holds good, subject to the conditions laid down above, extends upwards—towards higher β's—from the broken line shown in Diagram (1). I obtained this locus by computing the values of β_1 and β_2 from equations (6) corresponding to the values of χ and λ which satisfy the lower boundary of the restricted χ, λ area. When the β's of an observed distribution lie in Pearson's Type I area below the broken line, the translated curve will present singularities within a region of $|\xi| = 2$.

To illustrate, not so much the application of the method as the nature of the singularities, I take the distribution of single births arranged according to the age of mother at birth of child (Table II, p. 153). The observed constants are:

$$\beta_1 = \cdot 100{,}603, \quad \sigma = 3\cdot 083{,}148 \text{ (2-year unit)},$$
$$\beta_2 = 2\cdot 430{,}327, \quad N = 631{,}682.$$

Using Diagram (1) we note that the β's fall outside the limited area; hence at least one singularity within the range $|\zeta| = |\sqrt{2}\xi| = 2\cdot 83$ is to be expected.

The constants of the translated curve are:

$$\lambda = -\cdot 07659, \quad k = \cdot 08746, \quad a = 4\cdot 87522,$$

and Mean-Median = ·21320.

Hence: $$y = \frac{N}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\zeta^2} \cdot \frac{1}{3\cdot 44730(1 + \cdot 12369\zeta - \cdot 11488\zeta^2)},$$

where $\zeta = \sqrt{2}\xi$, the origin of the curve being at the median. The curve fitted to the observed frequencies is shown in Diagram (2).

The denominator, $\frac{dx}{d\zeta}$, becomes zero for $\zeta_1 = -2\cdot 46076$ and $\zeta_2 = +3\cdot 53749$,

* *Loc. cit.* pp. 123—128. The table must be entered with $\beta/8$ and not with β.

corresponding to ages of mother 18·7 and 47·1; the ordinate of the curve becomes infinite at these two points. For values of $|\zeta|$ greater than ζ_1 and ζ_2, $\frac{dx}{d\zeta}$ is negative (or, x decreases with increasing ζ) and we get the two lower (negative) branches shown in the figure. They both asymptote to the x-axis as $x \to \pm\infty$, or as $\zeta \to \mp\infty$.

The relative advantages and disadvantages of the method will be discussed more fully in a later section.

EDGEWORTH'S TRANSLATED CURVES IN RELATION TO THE PEARSON TYPES.

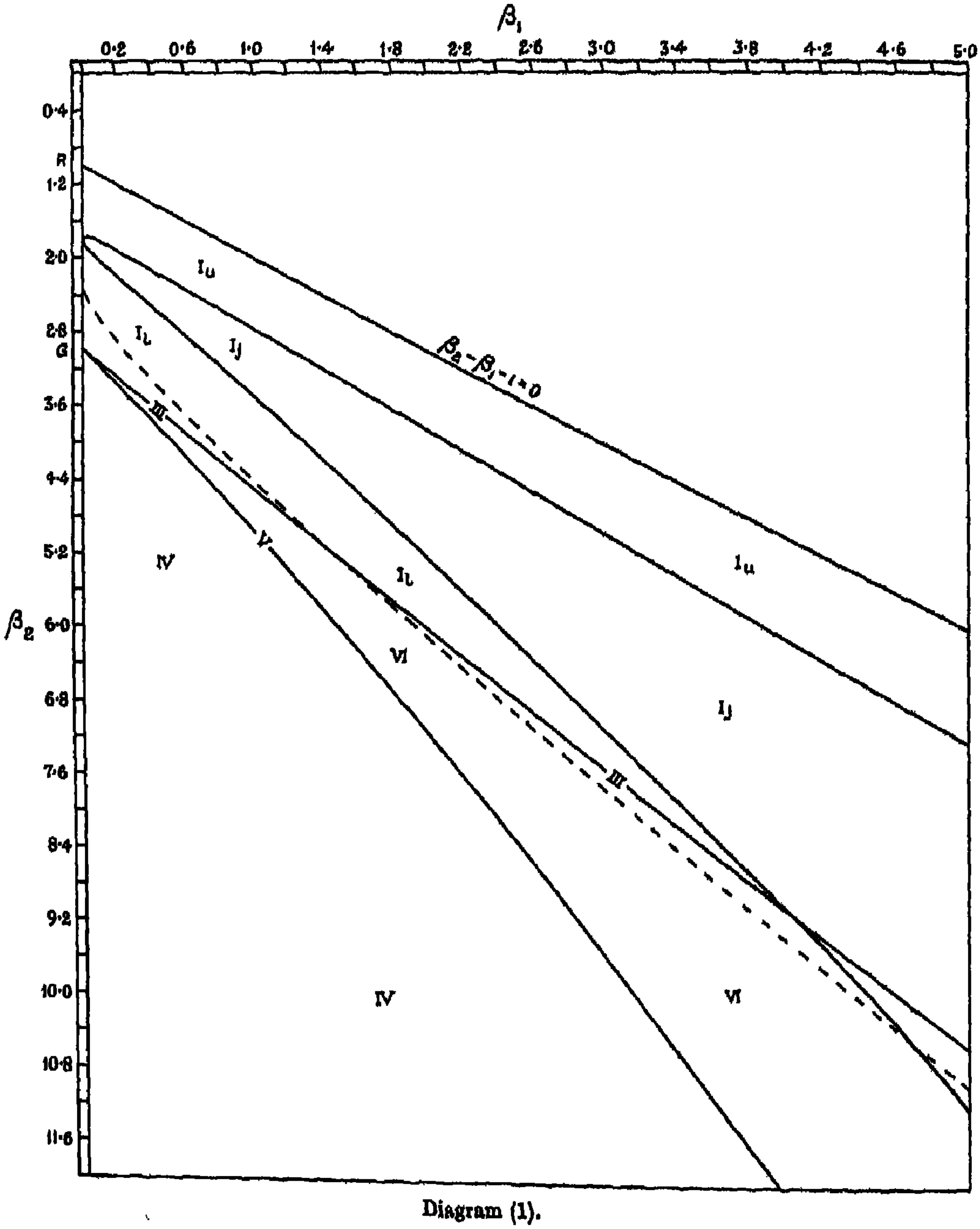


Diagram (1).

DISTRIBUTION OF SINGLE BIRTHS ACCORDING TO AGE OF MOTHER AT BIRTH OF CHILD

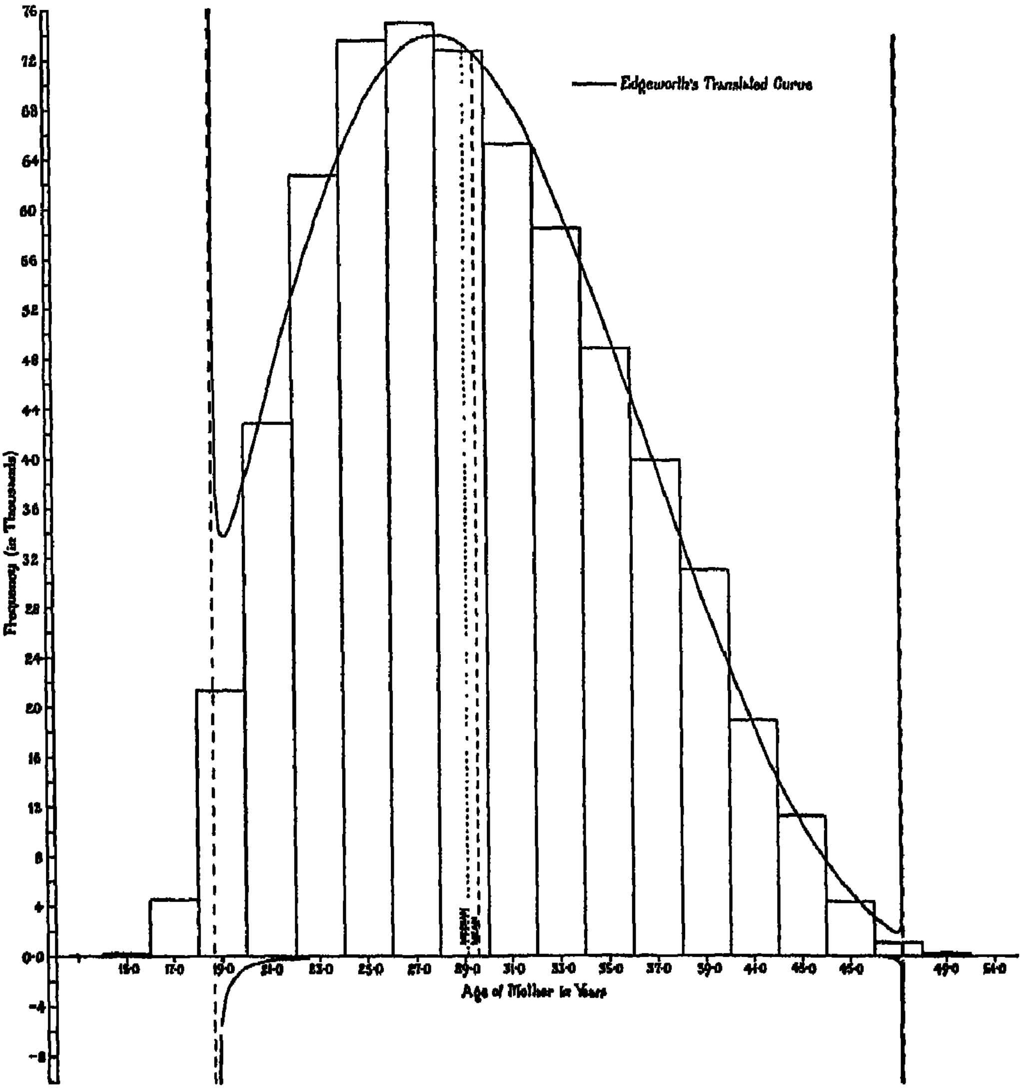


Diagram (2).

(ii) *Logarithmic Transformation.* Galton* suggested in 1879 that in many vital phenomena the geometrical mean and not the arithmetical mean is likely to be the most probable value of the quantity measured. The corresponding law of frequency was deduced by MacAlister† in the same year. The fitting of the curve by the method of moments was discussed by Pearson‡ (1905) and more recently by Jørgensen§, Wicksell|| and several other writers.

* "The Geometric Mean, in Vital and Social Statistics," *Proc. Roy. Soc.* Vol. XXIX. 1879, pp. 365—367.

† "The Law of the Geometric Mean," *Proc. Roy. Soc.* Vol. XXIX. 1879, pp. 367—376.

‡ "'Das Fehlergesetz und seine Verallgemeinerungen durch Fechner und Pearson.' A Rejoinder." *Biometrika*, Vol. IV. 1905—1906, pp. 193—196.

§ *Loc. cit.* pp. 47—49.

|| "On the Genetic Theory of Frequency," *Arkiv för Mat., Astr. och Fysik*, Bd. 12, No. 20, 1917, pp. 1—56.

Consider the normal curve:

$$\eta = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\xi^2},$$

and transform it by writing:

$$\xi = \frac{\log_{10} x - l}{s} \quad \text{.....................................(7).}$$

The resulting frequency curve is of the form:

$$y = y_0 \cdot \frac{1}{x} \cdot e^{-\frac{1}{2}\left(\frac{\log x - l}{s}\right)^2} \quad \text{.............................(8).}$$

The pth moment about the start of the curve is given by:

$$M_p' = \int_0^\infty y \cdot x^p \cdot dx$$
$$= y_0 \cdot \sqrt{2\pi} \cdot b \cdot s \cdot e^{blp + \frac{1}{2} b^2 s^2 p^2},$$

where $\log_{10} e = 1/b$.

For the areas under the two curves to be equal, we must have:

$$y_0 = \frac{N}{\sqrt{2\pi} \cdot b \cdot s}.$$

Hence: $\mu_r' = e^{blp + \frac{1}{2} b^2 s^2 p^2}$.

Or, taking moments about the mean:

$$\left.\begin{aligned} \mu_2 &= e^{2bl + b^2 s^2} \cdot [e^{b^2 s^2} - 1] \\ \mu_3 &= e^{3bl + \frac{3}{2} b^2 s^2} \cdot [e^{3b^2 s^2} - 3 \cdot e^{b^2 s^2} + 2] \\ \mu_4 &= e^{4bl + 2b^2 s^2} \cdot [e^{6b^2 s^2} - 4 \cdot e^{3b^2 s^2} + 6 \cdot e^{b^2 s^2} - 3] \end{aligned}\right\} \text{................(9).}$$

The following relations* hold between the fourth and lower order moments about the start:

$$\left.\begin{aligned} \mu_4' \cdot (\mu_1')^2 &= (\mu_3')^2 \\ \mu_4' \cdot (\mu_1')^5 &= (\mu_2')^3 \cdot \mu_3' \\ (\mu_4') \cdot (\mu_1')^8 &= (\mu_2')^6 \end{aligned}\right\} \text{............................(10).}$$

If the observed distribution fixes its own start, then l and s can be determined from μ_1' and μ_2':

$$\left.\begin{aligned} bs^2 &= \log\left(\frac{\mu_2'}{\mu_1'^2}\right) \\ l &= \log\left(\frac{\mu_1'^2}{\sqrt{\mu_2'}}\right) \end{aligned}\right\} \text{...............................(11).}$$

In the majority of cases, however, it will be better to find the start of the curve from the moment coefficients about the mean. Let ξ_1 be the distance between the start and mean of the curve, then from (9):

$$\mu_3 \xi_1^3 - 3\mu_2^2 \cdot \xi_1^2 - \mu_2^3 = 0 \text{ ....(12),}$$

* For a more complete analysis of the range of applicability of this curve, see pp. 146—150.

from which ξ_1 is to be determined. Further:

$$\left.\begin{aligned} l &= 2\log\xi_1 - \tfrac{1}{2}\log(\mu_2+\xi_1^2) \\ bs^2 &= 2\log\xi_1 - 2l \\ &= \log(\mu_2+\xi_1^2) - 2\log\xi_1 \end{aligned}\right\} \quad \text{(13)}.$$

The possibility of extending this method to the Type A series, has been pointed out by Jørgensen and Wicksell in the papers cited.

3. *Schols and Perozzo.* To preserve completeness in the historical survey, as far as possible, it seems desirable before passing on to the extension of the univariate formulae to the problem of correlation, to give a brief account of Schols'* treatment (1875) of errors in space, and of Perozzo's† analysis (1882) of Italian marriage statistics.

A full theory of errors of observations in space was for the first time worked out by Schols. He dealt generally with the principal axes of inertia, and showed that for the normal surface they were axes of independent probability. Generalising this it would signify that if $z = F(x, y)$ be the expression for the frequency surface, then by a rotation of axes it could be put into the form:

$$z = f_1(x') \cdot f_2(y').$$

Sections parallel to the principal axes are thus not only similar but also similarly situated. Any justification for applying this idea to frequency surfaces in general necessarily rests on the geometrical analysis of observed data; Schols does not seem to have attempted this.

Perozzo's investigation is, as far as I am aware, the first attempt to analyse graphically a skew bivariate distribution and to give general formulae for its representation. From the table exhibiting the number of marriages contracted in Italy during the years 1878–79 Perozzo obtains the contours of equal probability; he points out that they are not concentric and are tending to symmetry with respect to one axis only. In other words, the normal surface—which at that time was of interest only in ballistics—no longer applies. As an approximation to the binomial Perozzo gives the asymmetrical curves:

$$z = z_0 \cdot e^{\pm a_1 x - a_2 x^2 \pm a_3 x^3 - \dots}$$

and

$$z = a\left(x - \sqrt{\frac{n}{2a_2}}\right)^n \cdot e^{-a_2\left(x - \sqrt{\frac{n}{2a_2}}\right)^2}.$$

Similarly for the asymmetrical surface

$$z = z_0 \cdot z_0' \cdot e^{-a_2 x^2 \pm a_3 x^3 - \dots - a_2' y^2 \pm a_3' y^3 - \dots}$$

and

$$z = a \cdot a' \cdot \left(x - \sqrt{\frac{n}{2a_2}}\right)^n \left(y - \sqrt{\frac{n'}{2a_2'}}\right)^{n'} \cdot e^{-a_2\left(x - \sqrt{\frac{n}{2a_2}}\right)^2 - a_2'\left(y - \sqrt{\frac{n'}{2a_2'}}\right)^2}.$$

* "Théorie des erreurs dans le plan et dans l'espace," *Ann. de l'École Polytechn. de Delft*, 1886, pp. 123—175. Published in Dutch in the *Verhandelingen van de Koninklijke Akademie van Wetenschappen*, Deel 15, 1875, Amsterdam.

† "Nuove Applicazioni del Calcolo delle Probabilità," *Acta, Reale Accademia dei Lincei*, 1881—82, pp. 1—88.

Perozzo gives no underlying theoretical basis for these formulae, nor does he fit them to his observations.

4. *Double Hypergeometrical Series.* After having discussed the development of his system of curves (1895), Pearson remarked that if material obeyed a law of skew distribution, the Galton-Dickson theory of correlation would have to be considerably modified. The curves of equal probability derived from the correlation of cards of the same suit in two players' hands at whist, and from the correlation of ages of husband and wife at marriage, indicated a distinct deviation from the ellipses of normal correlation. The analytical description of skew bivariate distributions thus claimed immediate attention.

The idea of axes of independent probability marked the starting point of Pearson's researches on this problem. By an analysis such as that mentioned in the preceding paragraph, he was however able to convince himself that if principal inertial axes of the contour system existed, they were not axes of independent probability. The next step taken was an endeavour to extend the idea underlying his system of skew curves, i.e. to determine a family of surfaces from the two general differential equations to a certain double hypergeometrical series. These equations, as given by Rhodes*, were of the form:

$$\frac{1}{z}\cdot\frac{dz}{dx}=\frac{\text{Cubic in } x, y}{\text{Quartic in } x, y},$$

$$\frac{1}{z}\cdot\frac{dz}{dy}=\frac{\text{Another cubic in } x, y}{\text{Same quartic in } x, y}.$$

Without limitation on the constants, however, integration was found to be impossible. Special forms were thereafter considered by Pearson, Filon (1901) and Isserlis (1913), but these again led to surfaces of little value. In each case there existed a relation between the β's of the two marginal distributions, while also the correlation could be expressed as a function of them. That these and similar restrictions upon the characteristics of the distribution could not lead to satisfactory bivariate frequency surfaces, has over and over again been emphasised by Pearson. Freedom can be given to the variation of the characteristics only by having enough independent constants in the equation of the surface. The following surface is given by Pearson† as one of those obtained by Filon and Isserlis:

$$z=z_0\left(\frac{x}{b_1}\right)^{p_1}\left(\frac{y}{b_2}\right)^{p_2}\left(1-\frac{x}{b_1}-\frac{y}{b_2}\right)^{q} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14).$$

Here
$$\frac{\beta_{20}+3}{\beta_{10}+4}=\frac{\beta_{02}+3}{\beta_{01}+4},$$

$$r=\pm\sqrt{\frac{(p_1+1)(p_2+1)}{(p_1+q+2)(p_2+q+2)}}.$$

The marginal and array distributions are Pearson Type I curves. Regression and scedasticity are linear.

* "On a Certain Skew Correlation Surface," *Biometrika*, Vol. XIV. 1922—23, p. 355.

† "Notes on Skew Frequency Surfaces," *Biometrika*, Vol. XV. 1923, pp. 224—230.

The fitting of data with a discontinuous double hypergeometrical series was accomplished by Isserlis* in 1914. The corresponding problem in probability may be stated as follows:

Suppose a limited population of size N to contain m marked and $N-m$ unmarked characters; a sample of n is drawn and not replaced; a second sample of n' is drawn. The chance of s marked characters in the first sample and s' in the second, is

$$z(s, s') = \frac{n!\,n'!}{s!\,s'!\,(n-s)!\,(n'-s')!} \cdot \frac{(N-n-n')!}{N!} \cdot \frac{m!}{(m-s-s')!} \cdot \frac{(N-m)!}{(N-n-n'-m+s+s')!} \quad \text{......(15).}$$

Let $-n=\alpha$, $-n'=\alpha'$, $-m=\beta$, $N-m-n-n'+1=\gamma$, then it can be shown that

$$\Sigma\Sigma z(s, s') = \frac{(N-n-n')!}{N!} \cdot \frac{(N-m)!}{(N-m-n-n')!} \cdot F(\alpha, \alpha', \beta, \gamma, 1, 1),$$

where $F(\alpha, \alpha', \beta, \gamma, x, y)$ denotes the double hypergeometrical series

$$\Sigma\Sigma \frac{\alpha_s \alpha'_{s'} \beta_{s+s'}}{s!\,s'!\,\gamma_{s+s'}} \cdot x^s y^{s'},$$

in which

$$a_s = a(a+1)(a+2)\dots(a+s-1).$$

Isserlis expresses the parameters n, n', m and N in terms of moment and product-moment coefficients. He evaluates them for three numerical examples but to only one of the examples the equivalent series is fitted, namely, the distribution in 25,000 deals of trumps in the first two hands in whist with ordinary shuffling. The annexed photographs of the theoretical and observational surfaces superposed do not give us a clear idea of the goodness of fit. It is however not likely to be very good: the experimental returns show too marked discrepancies from the theoretical frequencies computed from the double hypergeometrical series†.

The range of applicability of the hypergeometrical was to some extent defined by Wicksell‡ (1917) when he showed that its regression curves are linear. In this connection it may be of interest to point out that while the discontinuous has linear regression, the two general differential equations (p. 122) lead to a surface with cubic regression§.

Wicksell‖ has further shown (1923) that the Type A and Type B series are analytical expressions for the representation of the hypergeometrical as well

* "The Application of Solid Hypergeometrical Series to Frequency Distributions in Space," *Phil. Mag.* Vol. XXVIII, 1914, pp. 379—403.

† Pearson, Karl: "On a Certain Double Hypergeometrical Series and its Representation by Continuous Frequency Surfaces," *Biometrika*, Vol. XVI. 1924, p. 186.

‡ "The Application of Solid Hypergeometrical Series to Frequency Distributions in Space," *Phil. Mag.* Vol. XXXIV, 1917, pp. 389—394.

§ Pearson, Karl: "Notes on Skew Frequency Surfaces," *Biometrika*, Vol. XV, 1923, p. 222.

‖ "Contributions to the Analytical Theory of Sampling," *Arkiv för Mat., Astr. och Fysik*, Bd. 17, No. 19, 1923.

as of the binomial. The double hypergeometrical leads to correlation functions of these two types.

In 1924 Pearson returned to the representation of a double hypergeometrical series by continuous frequency surfaces. The regression and scedasticity* are shown to be linear and parabolic respectively; a symmetrical surface† with similar forms for the regression and scedasticity is fitted to the special case of whist correlation: $N=52$, $n=n'=13$; also the Filon-Isserlis surface is fitted. From a comparison of the marginal distributions and of the contours, neither of the two surfaces seems to be really adequate.

5. *Skew Correlation and Non-linear Regression.* The preceding section clearly indicates that the earliest attempts at describing skew correlation, as based on the "correlation surface method," were not very profitable. Recourse had therefore to be had to a more general method which would not involve any assumptions as to the form of the frequency distribution. In a paper on multiple correlation (1897) Yule‡ showed that if the regression be linear, irrespective of the type of frequency surface, the multiple regression "plane" as reached by the method of least squares was identical in form with that flowing from a multiple normal surface. This method of approaching the problem of correlation, i.e. from the form of the regression curves, was extended by Pearson§ (1905) to non-linear regression.

Now while it is of great advantage that no assumptions as to the frequency distribution are made, this generality is, as has been pointed out by Pearson, also the chief defect of the method. Without some knowledge of the array distributions the probability of an individual observation falling within certain limits as measured from the regression curves cannot be determined.

The types of regression dealt with are: linear, parabolic, cubic and quartic. The parameters of these polynomials are expressed in terms of moments and product moments. Theoretically there is no limit to the order of the curve; in practice it depends largely on the rapidly increasing probable errors of the moments. The correlation ratio, η, is introduced as a measure of relationship when the regression is not linear; the conceptions of scedasticity and clisy are formulated, and measures of their heterogeneity are given. Finally, the regression formulae are illustrated on four examples.

A general method of determining the successive terms in a skew regression line was published by Pearson‖ in 1921. The form of the regression curve is assumed to be

$$y=f(x)=a_0\psi_0+a_1\psi_1+\ldots+a_n\psi_n,$$

* See pp. 141—142 for the third and fourth array moments.

† See p. 137.

‡ "On the Significance of Bravais' Formulae for Regression, etc., in the case of Skew Correlation," *Proc. Roy. Soc.* Vol. LX. 1896—97, pp. 477—489.

§ "On the General Theory of Skew Correlation and Non-Linear Regression," *Drapers' Company Research Memoirs*, Biometric Series, II. 1905, pp. 1—54.

‖ "On a General Method of determining the Successive Terms in a Skew Regression Line," *Biometrika*, Vol. XIII. 1920—21, pp. 296—300.

where $a_0, a_1, \dots a_n$ are constants to be determined and the ψ's form an orthogonal system of functions of x. The regression orthogonal functions up to the fourth order are obtained.

The results of Pearson have been put into a still more general form by Neyman*. Certain results of the theory of continued fractions are used, but no appeal is made to their theory, nor to the theory of orthogonal functions. The nth order regression parabola is expressed in determinantal form.

6. *The Correlation Function of Type A.* The surface whose sections parallel to the coordinate planes zx and zy are curves of Type A, has been discussed by Van der Stok† (1907–1908), Charlier‡ (1914), Jørgensen§ (1916), Wicksell|| (1917), and others on the Continent; in England by Edgeworth¶ (1896, 1905, 1917), Pearson** (1925) and Rhodes†† (1925). For brevity we shall adopt Jørgensen's notation for this surface, viz. Type AA. Its general equation can be written in the form:

$$\left.\begin{aligned} F(x, y) &= \phi(x, y) + \sum_{(p+q)\geqslant 3}\sum (-1)^{p+q} \cdot \frac{A_{pq}}{p!\,q!} \cdot \frac{\partial^{p+q}\phi(x, y)}{\partial x^p . \partial y^q} \\ &\text{where the generating function is the normal surface:} \\ \phi(x, y) &= \frac{1}{2\pi\sqrt{1-r^2}} \cdot e^{-\frac{1}{2(1-r^2)}(x^2 - 2rxy + y^2)} \end{aligned}\right\} \quad \dots\dots\dots(16).$$

The various contributions may be dealt with as follows: (*a*) special forms of $F(x, y)$, (*b*) determination of the coefficients of the differential terms, (*c*) the partial moment curves, (*d*) the curves of equal probability, (*e*) applications. The marginal distributions are identical with the Type A curves treated in Section 2.

(*a*) *Special forms of* $F(x, y)$. With the exception of Edgeworth and Van der Stok, all the authors mentioned above start with equation (16). The ensuing discussions of Charlier, Jørgensen and Pearson are confined to the approximation $3 \leqslant (p+q) \leqslant 4$ (Type AaAa); Wicksell discusses both this approximation and that given in Section 2 where all terms of the order $1/n$ are included (Type AbAb); Edgeworth extends his generalised law of error to two dimensions but considers thereafter terms involving moments up to the third order only, $(p+q) = 3$; this approximation is discussed more fully by Rhodes who applies it to the problem of ranks and grades.

* "Further Notes on Non-Linear Regression," *Biometrika*, Vol. XVIII. 1926, pp. 257—262.

† "On the Analysis of Frequency Curves according to a General Method," *Proc. Kon. Ak. v. Wet.* (Amsterdam), 1907—1908, pp. 799—817.

‡ "Contributions to the Mathematical Theory of Statistics. 6. The Correlation Function of Type A," *Arkiv för Mat., Astr. och Fysik*, Bd. 9, No. 26, 1914, pp. 1—18.

§ *Undersøgelser over Frequensflader og Korrelation.* København, 1916: Arnold Busck.

|| "The Correlation Function of Type A, and the Regression of its Characteristics," *Kungl. Sv. Vet. Akad. Handl.* Bd. 58, No. 3, 1917, pp. 1—48.

¶ "The Compound Law of Error," *Phil. Mag.* Vol. XLI. 1896, pp. 207—215; "The Law of Error," *Camb. Phil. Trans.* Vol. XX. 1905, pp. 116—119; "On the Mathematical Representation of Statistical Data," *Journ. Roy. Stat. Soc.* Vol. LXXX. 1917, pp. 266—288.

** "The Fifteen Constant Bivariate Frequency Surface," *Biometrika*, Vol. XVII. 1925, pp. 268—313.

†† "On a Skew Correlation Surface," *Biometrika*, Vol. XVII. 1925, pp. 314—326.

Van der Stok takes as generating function:

$$\phi_1(x, y) = \frac{1}{2\pi} \cdot e^{-\frac{1}{2}(x^2+y^2)} = f_1(x) \times f_2(y) \quad \ldots\ldots\ldots\ldots\ldots\ldots(17),$$

and deduces the surface:

$$F_1(x, y) = \phi_1(x, y) + B_{11}\frac{df_1}{dx} \cdot \frac{df_2}{dy} + \sum_{(p+q)\geq 3}\sum (-1)^{p+q}\frac{B_{pq}}{p!\,q!} \cdot \frac{d^p f_1}{dx^p} \cdot \frac{d^q f_2}{dy^q} \ldots(18).$$

The only comment he makes on this surface is that by a rotation of axes the B_{11}-term can be made to vanish.

Jørgensen observes that the general form (16) is not well adapted to numerical application; he thereupon turns to (17) and (18) where the variables and differential coefficients are separable. With tables of the normal curve and of its derivatives at hand, the arithmetical work can be greatly diminished.

The fitting of equation (16) can be best performed after the differential functions have been expanded as a polynomial in x and y. In this form the surface is discussed by Pearson:

$$\begin{aligned}F(x, y) = \phi(x, y) \cdot [1 - a_0 + a_1 x + a_2 y + b_1 x^2 + 2b_2 xy + b_3 y^2 \\ + c_1 x^3 + c_2 x^2 y + c_3 x y^2 + c_4 y^3 + d_1 x^4 + d_2 x^3 y + 3d_3 x^2 y^2 + d_4 x y^3 + d_5 y^4] \ldots(19).\end{aligned}$$

(*b*) *The coefficients* in the different equations, as given by the respective writers, are as follows:

Type AaAa.

$$\left.\begin{array}{ll} A_{30} = q_{30}, & A_{40} = q_{40} - 3 \\ A_{21} = q_{21}, & A_{31} = q_{31} - 3r \\ A_{12} = q_{12}, & A_{13} = q_{13} - 3r \\ A_{03} = q_{03}, & A_{04} = q_{04} - 3 \\ \multicolumn{2}{c}{A_{22} = q_{22} - 1 - 2r^2} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(20).$$

Type AbAb. The coefficients of the additional terms ($p + q = 6$) are:

$$\left.\begin{array}{ll} A'_{60} = \frac{1}{2}(A'_{30})^2, & A'_{06} = \frac{1}{2}(A'_{03})^2 \\ A'_{51} = A'_{30} \cdot A'_{21}, & A'_{15} = A'_{03} \cdot A'_{12} \\ A'_{42} = A'_{30} \cdot A'_{12} + \frac{1}{2}(A'_{21})^2, & A'_{24} = A'_{03} \cdot A'_{21} + \frac{1}{2}(A'_{12})^2 \\ \multicolumn{2}{c}{A'_{33} = A'_{30} \cdot A'_{03} + A'_{21} \cdot A'_{12}} \end{array}\right\} \ldots\ldots(21),$$

where

$$A'_{pq} = (-1)^{p+q} \cdot \frac{A_{pq}}{p!\,q!}.$$

Equation (18). Except for B_{22}, the B-coefficients are identical with the A's:

$$B_{11} = q_{11} = r, \quad B_{22} = q_{22} - 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(22).$$

Equation (19). Pearson gives the expressions for the coefficients a, b, c and d in the paper already cited. Because of their complexity I shall not re-write them.

(*c*) *The Partial Moment Curves.* The regression and scedastic curves of Type AaAa are derived by Wicksell and Pearson. The third and fourth partial moments are found only by Wicksell; he hereupon treats the special forms all these moment curves will assume when the correlation is moderately skew.

Pearson's forms are:

Regression Curve of y on x:

$$\mu_1'(y) = rx + \frac{\sqrt{\frac{3}{2}}\,(q_{21} - r\sqrt{\beta_{10}})\,\tau_3 + \sqrt{\frac{2}{3}}\,(q_{31} - r\beta_{20})\,\tau_4}{\tau_1 + \sqrt{\frac{2}{3}}\,.\,\sqrt{\beta_{10}}\,.\,\tau_4 + \sqrt{\frac{5}{24}}\,.\,\overline{\beta_{20} - 3}\,.\,\tau_5} \quad \text{............(23)}$$

$$\equiv rx + \frac{A_x}{z_x}.$$

Scedastic Curve of y on x:

$$\sigma^2(y) = 1 - r^2 - \frac{\sqrt{2}\,[r(q_{21} - r\sqrt{\beta_{10}}) - (q_{12} - rq_{21})]\,\tau_2 + \sqrt{6}\,[r(q_{31} - r\beta_{20}) - \frac{1}{2}(q_{22} - 1 - r^2\,.\,\overline{\beta_{20} - 1})]\,\tau_3}{z_x}$$

$$- \left(\frac{A_x}{z_x}\right)^2 \quad \text{.................................(24)}$$

$$\equiv 1 - r^2 - \left(\frac{B_x}{z_x}\right) - \left(\frac{A_x}{z_x}\right)^2.$$

The following form in which Wicksell writes these curves is certainly not as elegant as (23) and (24); the deviation from normality is obscured by not taking out the factors rx and $\overline{1 - r^2}$:

$$\mu_1'(y) = \frac{rx - rA'_{30}\,.\,R_4(x) - A'_{21}\,.\,R_2(x) - rA'_{40}R_5(x) - A'_{31}R_3(x)}{1 + A'_{30}R_3(x) + A'_{40}\,.\,R_4(x)} \quad \text{...(25)},$$

where $R_s(x)$ is the Hermite Polynomial of the sth order. Wicksell develops the denominator of (25) as a power series in $A'_{30}R_3(x) + A'_{40}R_4(x)$, and neglects all terms whose coefficients are of an order less than $\frac{1}{n}$, n being the number of elementary "error-sources." Arranging the resulting expression in powers of x, he finds the regression to be cubic. To the same degree of approximation the scedasticity is parabolic, the clisy linear and the kurtosis constant. The foregoing development is justifiable only for distributions of moderate skewness and within certain ranges from the mean.

For Type AbAb Wicksell derives only the regression and scedastic curves.

Jørgensen derives the regression and scedastic curves for the simplified form (18).

(*d*) *Curves of Equal Probability.* In any correlation distribution, $F(x, y)$, the curves of equal probability are given by*

$$z = F(x, y) = \text{constant} \quad \text{........................(26)}.$$

But more than often the form of $F(x, y)$ is too complicated for these curves to be directly constructed.

For the Type AaAa surface an approximate solution to (26) has been found by Wicksell†. Assuming the correlation to be moderately skew, he shows that in the

* The curves so defined are, strictly speaking, curves of equal ordinates, i.e. the contours of the surface.

† "The Construction of the Curves of Equal Frequency in case of Type A Correlation," *Sv. Akt. Tidskr.* Häft. 2—3, 1917, pp. 1—19.

vicinity of the mode the curves of equal probability are ellipses, while further out they are disturbed ellipses. These outer "ellipses" can be constructed by making use of auxiliary circles; certain quantities expressed in terms of products of the Hermite Polynomials are to be added to the radii of the circles for the radii vectores of the required curves to be obtained. Tables are given for $R_i(\xi_1) \cdot R_j(\eta_1)$, where ξ_1 and η_1 are the points of intersection of 24 radii vectores with circles whose radii correspond to definite values of $z = F(x, y) =$ constant.

A more detailed treatment or a restatement of the derived formulae seems to me not warranted.

(*e*) *Applications.* The fitting of surface (18) is illustrated by Jørgensen on one example. He first considers the possibility of making some of the higher coefficients in the expression negligibly small by a rotation of axes; the new axes are to coincide with the principal axes of inertia. However, in his particular illustration nothing is gained by such a transformation. The mid-ordinates of the frequency cells are calculated and compared with the observed frequencies. Even if allowance be made for the paucity of the observations we are bound to conclude, from an examination of the table, that the graduation is not at all satisfactory.

Wicksell illustrates his method of approximating to the partial moment curves of the Type AaAa surface on four examples representative of moderately and of considerably skew correlation. Both regression curves are fitted for all four examples; the scedastic curves for two of the examples only. The range of applicability of the approximate formulae can to some extent be appreciated from the following values of β_1 and β_2 which I have evaluated for the marginal distributions corresponding to the instances where Wicksell replaces his cubic by the general regression curve (25):

$$\left.\begin{array}{l}\beta_1 = \cdot 109\\ \beta_2 = 2\cdot 952\end{array}\right\}; \quad \left.\begin{array}{l}\beta_1 = \cdot 233\\ \beta_2 = 2\cdot 889\end{array}\right\}; \quad \left.\begin{array}{l}\beta_1 = \cdot 829\\ \beta_2 = 4\cdot 868\end{array}\right\}.$$

Hereafter, Wicksell fits his formulae to three of the examples given by Pearson in his memoir on skew correlation and non-linear regression. The diagrams given by Wicksell seem to indicate that his formulae, with moments up to the fourth order, give virtually as good a description of the observation points as Pearson's formulae involving moments up to the sixth; also the arithmetic is far less. However, not until we have more comparative results before us, will it be possible to vindicate the general use of these formulae.

Pearson tests the value of "The Fifteen Constant Surface" (19) on two examples: the whist double hypergeometrical series, and the distribution of contemporaneous barometric heights at Southampton and Laudale. In both illustrations the theoretical ordinates and frequencies are computed, and the contour lines are constructed. A very close agreement is obtained between the mathematical surface and the double hypergeometrical. For the barometric data, due regard being paid to the sparseness of the observations, the agreement is less satisfactory; the Goodness of Fit Test shows, however, that the graduation is better than that

obtained by Rhodes with his surface*; the regression curves fit the observation points very well.

7. *The Correlation Functions of Type B, and of Type A and Type B.* In his systematic treatise on frequency surfaces and correlation, 1916, Jørgensen discusses the following three types of surfaces: (i) Type AA; (ii) Type BB, where sections parallel to the coordinate planes zx and zy are curves of Type B; (iii) Type AB where sections parallel to the plane zx are curves of Type A and sections parallel to the plane zy are curves of Type B.

Jørgensen takes the generating function of Type BB and of Type AB to be

$$\vartheta(x, y) = \vartheta(x) \times \vartheta(y)$$

and

$$\psi(x, y) = \phi(x) \times \vartheta(y)$$

respectively, where $\phi(x)$ and $\vartheta(x)$ are as defined in Section 2, p. 115. The constants, regression and scedastic curves are determined for these simplified forms. No numerical illustrations are given.

8. *Translation applied to Correlation. Edgeworth*†.

(*a*) *Simple Translation.* Let the generating surface be

$$\zeta = \frac{1}{\pi\sqrt{1-R^2}} \cdot e^{-\frac{1}{(1-R^2)}[\xi^2 - 2R\xi\eta + \eta^2]} \quad \text{.................(27)},$$

and the equations of translation:

$$\left.\begin{aligned} x &= a_1(\xi + k_1\xi^2 + \lambda_1\xi^3) \\ y &= a_2(\eta + k_2\eta^2 + \lambda_2\eta^3) \end{aligned}\right\} \quad \text{..........................(28)}.$$

The constants are to be determined separately for the two equations.

Taking r to be the correlation coefficient between x and y, and R to be that between ξ and η, Edgeworth finds

$$r\sqrt{\mu_{20} \cdot \mu_{02}} = a_1 \cdot a_2 \left[\frac{R}{2} + \tfrac{3}{4}(\lambda_1 + \lambda_2) R + \tfrac{1}{2} k_1 \cdot k_2 R^2 + \tfrac{1}{3}\lambda_1\lambda_2(9R + 6R^3)\right].$$

If cubic terms in k and λ are neglected, then:

$$R = r\left[1 + \tfrac{3}{4}(\lambda_1^2 + \lambda_2^2) + \tfrac{1}{2}(k_1^2 + k_2^2)\right] - k_1 k_2 r^2 - \tfrac{3}{2}\lambda_1\lambda_2 \cdot r^3.$$

After ξ and η have been evaluated from (28) the cell frequencies can be found from (27) with the use of tables for the normal curve.

(*b*) *Composite Translation.* Professor Bowley‡ considers the case

$$x = a_1(\xi + k_1\xi^2 + \lambda_1\xi^3 + \gamma_1\eta^2),$$
$$y = a_2(\eta + k_2\eta^2 + \lambda_2\eta^3 + \gamma_2\xi^2),$$

while Edgeworth omits λ_1 and λ_2.

* See pp. 134—136.

† "On the Use of Analytical Geometry to represent Certain Kinds of Statistics," *Journ. Roy. Stat. Soc.* Vol. LXXVII. 1914, pp. 838—852; Vol. LXXX. 1917, pp. 266—288.

‡ *F. Y. Edgeworth's Contributions to Mathematical Statistics*, Roy. Stat. Soc., London, 1928, pp. 79—81.

Fairly simple expressions can be found for k_1, k_2, γ_1 and γ_2 if squared terms in the k, λ and γ are neglected, i.e. if the correlation be regarded as *moderately* skew. To solve the general moment equations involving these constants would be a severe task.

Composite translation is necessary if the relations

$$\left.\begin{aligned} q_{21} &= \tfrac{1}{3}(2q_{30}\,.\,r + q_{03}\,.\,r^2) \\ q_{12} &= \tfrac{1}{3}(2q_{03}\,.\,r + q_{30}\,.\,r^2) \end{aligned}\right\} \quad \text{..........................(29)}$$

do not hold.

Edgeworth states his views on the relative merits of simple and composite translation and of the generalised law of error in the concluding paragraph of his paper in *J. S. S.*, Vol. LXXX, 1917: "The inadequacy of simple translation, the impracticability of composite translation, constitutes an important point in the comparison between the use of the generalised law of error and the method of translation in two dimensions. The balance between the two methods is altered in one respect. Whereas in one dimension the generalised law is theoretically at least preferable for subnormal curves, while translation has the advantage of being applicable to abnormal cases, this advantage is greatly reduced in two dimensions, while that preference still subsists."

The results of both methods are illustrated on a few frequency groups. The agreement between theory and observation seems to be quite satisfactory; but whether the same degree of agreement holds throughout the surface, Edgeworth did not establish.

9. *Logarithmic Correlation. Wicksell.* The method of logarithmic transformation has been extended to correlation problems by Wicksell* (1917) in two successive papers. If $\log x$ and $\log y$ are assumed to be normally distributed, their correlation function will be

$$F(x, y) = \frac{Z_0}{x\,.\,y}\,.\,e^{-\frac{1}{2}\frac{1}{1-\rho^2}\left[\left(\frac{\log x - l_1}{s_1}\right)^2 - 2\rho\left(\frac{\log x - l_1}{s_1}\right)\left(\frac{\log y - l_2}{s_2}\right) + \left(\frac{\log y - l_2}{s_2}\right)^2\right]}.$$

The regression curves of this surface are however of a form one would not expect to observe in practice; they have (i) no inflexions, (ii) two points of intersection.

In the second paper Wicksell assumes the distribution to be of the form

$$F(\xi, \eta) = \phi(\xi, \eta) + \sum_{(p+q)=s}\sum B_{pq}\frac{\partial^{p+q}\phi(\xi, \eta)}{\partial\xi^p\,.\,\partial\eta^q} \quad \text{...............(30)},$$

where $\xi = \log x$, $\eta = \log y$, and

$$\phi(\xi, \eta) = \frac{N(\log e)^2}{s_1\,.\,s_2\,.\,2\pi\sqrt{1-\rho^2}}\,.\,e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{\xi - l_1}{s_1}\right)^2 - 2\rho\left(\frac{\xi - l_1}{s_1}\right)\left(\frac{\eta - l_2}{s_2}\right) + \left(\frac{\eta - l_2}{s_2}\right)^2\right]} \quad (31).$$

* "On the Genetic Theory of Frequency," *Arkiv för Mat., Astr. och Fysik*, Bd. 12, No. 20, 1917: "On Logarithmic Correlation, with an Application to the Distribution of Ages at First Marriage," *Sv. Akt. Tidskr.* Häft. 4, 1917, pp. 1—21.

(*a*) *Determination of the constants in equation* (30). The moments, M'_{pq}, of x and y about the origin are given by:

$$M'_{pq}=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{pb\xi}\,.\,e^{qb\eta}\,.\,F(\xi,\eta)\,d\xi\,.\,d\eta,$$

where $b=\dfrac{1}{\log_{10} e}$ as before.

Let the origin (ξ_1, η_1) be so chosen that B_{30} and B_{03} vanish; the expressions for the moments of the marginal distributions are then identical with those given on p. 120. The expressions for the product moments about the origin are:

$$\left.\begin{aligned}
\mu'_{11}&=[1-b^3(B_{21}+B_{12})]\,.\,e^{bl_1+bl_2+\frac{b^2}{2}(s_1^2+2\rho s_1\,.\,s_2+s_2^2)}\\
\mu'_{21}&=[1-2b^3(2B_{21}+B_{12})]\,.\,e^{2bl_1+bl_2+\frac{b^2}{2}(4s_1^2+4\rho s_1\,.\,s_2+s_2^2)}\\
\mu'_{12}&=[1-2b^3(B_{21}+2B_{12})]\,.\,e^{bl_1+2bl_2+\frac{b^2}{2}(s_1^2+4\rho s_1\,.\,s_2+4s_2^2)}\\
\mu'_{31}&=[1-3b^3(3B_{21}+B_{12})]\,.\,e^{3bl_1+bl_2+\frac{b^2}{2}(9s_1^2+6\rho s_1\,.\,s_2+s_2^2)}\\
\mu'_{22}&=[1-8b^3(B_{21}+B_{12})]\,.\,e^{2bl_1+2bl_2+\frac{b^2}{2}(4s_1^2+8\rho s_1\,.\,s_2+4s_2^2)}\\
\mu'_{13}&=[1-3b^3(B_{21}+3B_{12})]\,.\,e^{bl_1+3bl_2+\frac{b^2}{2}(s_1^2+6\rho s_1\,.\,s_2+9s_2^2)}
\end{aligned}\right\}\quad\ldots(32).$$

The origin (ξ_1, η_1) is to be determined from

$$\left.\begin{aligned}
\mu_{30}\xi_1^3-3\mu_{20}\xi_1^2-\mu_{20}^3&=0\\
\mu_{03}\eta_1^3-3\mu_{02}\eta_1^2-\mu_{02}^3&=0
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(33).$$

Further

$$\left.\begin{aligned}
l_1&=2\log\xi_1-\tfrac{1}{2}\log(\mu_{20}+\xi_1^2)\\
l_2&=2\log\eta_1-\tfrac{1}{2}\log(\mu_{02}+\eta_1^2)\\
bs_1^2&=\log(\mu_{20}+\xi_1^2)-2\log\xi_1\\
bs_2^2&=\log(\mu_{02}+\eta_1^2)-2\log\eta_1
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(34).$$

Write

$$\left.\begin{aligned}
k'_{11}&=\frac{\mu'_{11}}{\mu'_{10}\,.\,\mu'_{01}}=[1-b^3(B_{21}+B_{12})]\,.\,e^{\rho s_1 s_2 b^2}\\
k'_{21}&=\frac{\mu'_{21}}{\mu'_{20}\,.\,\mu'_{01}}=[1-2b^3(2B_{21}+B_{12})]\,.\,e^{2\rho s_1 s_2 b^2}\\
k'_{12}&=\frac{\mu'_{12}}{\mu'_{02}\,.\,\mu'_{10}}=[1-2b^3(B_{21}+2B_{12})]\,.\,e^{2\rho s_1 s_2 b^2}
\end{aligned}\right\}\quad\ldots\ldots\ldots(35),$$

and assume $(b^3B_{21})^2$ and $(b^3B_{12})^2$ to be negligibly small as compared with 1. B_{12}, B_{21} and ρ are then to be found from

$$\left.\begin{aligned}
k'_{21}-k'^3_{11}&=u(2k'^2_{11}-k'_{21})+v(k'^2_{11}-k'_{21})\\
k'_{12}-k'^3_{11}&=u(k'^2_{11}-k'_{12})+v(2k'^2_{11}-k'_{12})\\
\rho&=\frac{\log k'_{11}-\log(1+u/2+v/2)}{b\,.\,s_1\,.\,s_2}
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(36),$$

where $u=-2b^3B_{21}$ and $v=-2b^3B_{12}$.

The successful application of these formulae depends on the following conditions: (i) that terms of an order higher than the third in (30) may always be neglected when the origin is so chosen that B_{03} and B_{30} vanish; (ii) that $(b^3 B_{21})^2$ and $(b^3 B_{12})^2$ are negligibly small. If this condition be not fulfilled, equations (35) must be solved for B_{21}, B_{12} and ρ. Thus

$$e^{\rho s_1 s_2 b^2} = \tfrac{3}{4} k'_{11} + \sqrt{\tfrac{1}{16}\left(9k'^2_{11} - 4k'_{21} - 4k'_{12}\right)} \quad \text{..............(37)}$$

and two linear equations involving B_{12} and B_{21}.

The following identical relations between moments of the fourth order, about the origin, must be approximately fulfilled:

$$\left.\begin{aligned}
\mu'_{40} \cdot (\mu'_{10})^8 &= (\mu'_{20})^6 \\
\mu'_{31} \cdot (\mu'_{10})^6 \cdot (\mu'_{01})^2 &= (\mu'_{11})^3 \cdot (\mu'_{20})^3 \, \frac{1 - 3b^3 (3B_{21} + B_{12})}{[1 - b^3 (B_{21} + B_{12})]^3} \\
\mu'_{22} \cdot (\mu'_{10})^4 \cdot (\mu'_{01})^4 &= (\mu'_{11})^4 \cdot \mu'_{20} \cdot \mu'_{02} \, \frac{1 - 8b^3 (B_{21} + B_{12})}{[1 - b^3 (B_{21} + B_{12})]^4} \\
\mu'_{13} \cdot (\mu'_{10})^2 \cdot (\mu'_{01})^6 &= (\mu'_{11})^3 \cdot (\mu'_{02})^3 \, \frac{1 - 3b^3 (B_{21} + 3B_{12})}{[1 - b^3 (B_{21} + B_{12})]^3} \\
\mu'_{04} \cdot (\mu'_{01})^8 &= (\mu'_{02})^6
\end{aligned}\right\} \quad \text{......(38).}$$

(*b*) *The Marginal Distributions of the Surface.* These are identical with the curves considered in Section C, 2, p. 120.

(*c*) *The Partial Moment Curves.* Wicksell finds the expression for the sth moment curve of y on x about the origin (ξ_1, η_1) to be

$$\mu'_s(y) = \mu'_{0s} \cdot e^{\lambda^{(s)} \cdot \log x - \gamma^{(s)}} \cdot \left[D_0^{(s)} + D_1^{(s)} \{\log x - d^{(s)}\} + D_2^{(s)} \{\log x - d^{(s)}\}^2\right] \quad \text{......(39),}$$

where

$$\left.\begin{aligned}
\lambda^{(s)} &= s\left(b\rho \cdot \frac{s_2}{s_1}\right), & D_2^{(s)} &= -s\left(b \cdot \frac{B_{21}}{s_1^4}\right) \\
D_0^{(s)} &= 1 + s\left(b \cdot \frac{B_{21}}{s_1^3}\right), & \gamma^{(s)} &= s\left(b\rho \cdot \frac{s_2}{s_1} \cdot l_1\right) + s^2 \left(\tfrac{1}{2} b^2 s_2^2 \rho^2\right) \\
D_1^{(s)} &= -s^2\left(b^2 \cdot \frac{B_{12}}{s_1^2}\right), & d^{(s)} &= l_1 + s\,(b s_1 s_2 \rho)
\end{aligned}\right\} \text{......(40).}$$

(*d*) *Illustration.* The derived formulae are fitted to the marginal and regression curves for the age distribution of bachelors and spinsters married in Sweden, 1901—10. The relative marginal frequencies and a diagram of the regression curves seem to indicate a fair agreement between theory and observation.

10. *Steffensen's Correlation Formulae**. To represent a slight degree of correlation, Steffensen (1922) writes the frequency function in the form:

$$F(x, y) = k f_1(x, y) \times f_2(x, y),$$

which, for special values of the parameters, can be reduced to

$$F(x, y) = k_1 f_1'(x) \times f_2'(y).$$

* 'A Correlation Formula,' *Sk. Akt. Tidskr.* 1922; *Matematisk Iagttagelselære*, København, 1923, pp. 106—132.

Suppose x and y to be linearly related, then

$$F(x, y) = kf_1(x + cy) \times f_2(y + \gamma x) \quad \text{......(41)}$$
$$= kf_1(\xi) \times f_2(\eta),$$

where

$$\left.\begin{aligned} \xi &= x + cy \\ \eta &= y + \gamma x \end{aligned}\right\} \quad \text{......(42)}.$$

Determination of the Constants in (41). Let c_p' and γ_p' denote the moment coefficients of the functions $f_1(\xi)$ and $f_2(\eta)$ respectively; μ'_{pq} the pqth moment coefficient of $F(x, y)$; the dashes are to be dropped when the origin is at the mean. Thus

$$c_p' = \int \xi^p f_1(\xi)\, d\xi, \quad \gamma_p' = \int \eta^p f_2(\eta)\, d\eta,$$

$$\mu'_{pq} = \iint x^p \cdot y^q \cdot F(x, y)\, dx\, dy.$$

The constants on which f_1 and f_2 depend are to be found in the usual way from the moments c_p and γ_p.

From transformation (42):

$$\iint f_1(\xi) f_2(\eta)\, d\xi \cdot d\eta = \iint f_1(x + cy) \cdot f_2(y + \gamma x) \begin{vmatrix} \dfrac{\partial \xi}{\partial x}, & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \xi}{\partial y}, & \dfrac{\partial \eta}{\partial y} \end{vmatrix} dx\, dy$$

$$= \frac{1}{k} \iint F(x, y) \cdot |1 - c\gamma| \cdot dx\, dy,$$

i.e.

$$k = |1 - c\gamma| \quad \text{......(43)}.$$

Also

$$\iint (x + cy)^p \cdot (y + \gamma x)^q \cdot F(x, y)\, dx\, dy = c_p' \cdot \gamma_q' \quad \text{......(44)}.$$

If the origin is assumed to be at the mean, it follows from (44) that

$$c_1 = \mu_{10} + c\mu_{01} = 0,$$
$$\gamma_1 = \gamma\mu_{10} + \mu_{01} = 0,$$
$$\gamma\mu_{20} + (1 + c\gamma)\mu_{11} + c\mu_{02} = 0 \quad \text{......(45)},$$

$$\left.\begin{aligned} \gamma^2\mu_{30} + \gamma(2 + c\gamma)\mu_{21} + (1 + 2c\gamma)\mu_{12} + c\mu_{03} &= 0 \\ \gamma\mu_{30} + (1 + 2c\gamma)\mu_{21} + c(2 + c\gamma)\mu_{12} + c^2\mu_{03} &= 0 \end{aligned}\right\} \quad \text{......(46)}.$$

Equations (45) and (46) are to be solved for c and γ.

Write

$$u = \frac{\gamma}{1 + c\gamma}, \quad v = \frac{c}{1 + c\gamma},$$

whence

$$c = \frac{1 - \sqrt{1 - 4uv}}{2u}, \quad \gamma = \frac{1 - \sqrt{1 - 4uv}}{2v}.$$

Substitute these expressions for c and γ in (45) and (46):

$$u\mu_{20} + v\mu_{02} + \mu_{11} = 0 \quad \text{......(47)},$$

$$\left.\begin{aligned} u\mu_{21} + v\mu_{03} + \mu_{12} &= 0 \\ u\mu_{30} + v\mu_{12} + \mu_{21} &= 0 \end{aligned}\right\} \quad \text{......(48)}.$$

Combine equations (48) by making

$$(u\mu_{21}+v\mu_{03}+\mu_{12})^2+(u\mu_{30}+v\mu_{12}+\mu_{21})^2$$

a minimum. This gives

$$\frac{\mu_{02}}{\mu_{20}}=\frac{u(\mu_{03}\,.\,\mu_{21}+\mu_{30}\,.\,\mu_{12})+v(\mu_{03}^2+\mu_{12}^2)+\mu_{12}(\mu_{03}+\mu_{21})}{u(\mu_{30}^2+\mu_{21}^2)+v(\mu_{03}\,.\,\mu_{21}+\mu_{30}\,.\,\mu_{12})+\mu_{21}(\mu_{30}+\mu_{12})}\quad\ldots\ldots(49).$$

From (47) and (49) c and γ can now be easily determined.

The moments c_p and γ_p are obtained by expanding the binomial in (44). Thus

$$\left.\begin{aligned}c_p&=\mu_{p0}+\frac{p}{1!}\,.\,c\,.\,\mu_{p-1,1}+\frac{p(p-1)}{2!}\,.\,c^2\,.\,\mu_{p-2,2}+\ldots+c^p\mu_{0p}\\ \gamma_p&=\gamma^p\mu_{p0}+\frac{p}{1!}\,.\,\gamma^{p-1}\,.\,\mu_{p-1,1}+\frac{p(p-1)}{2!}\,.\,\gamma^{p-2}\,.\,\mu_{p-2,2}+\ldots+\mu_{0p}\end{aligned}\right\}\ldots\ldots(50).$$

Application. The method is illustrated on the example treated by Jørgensen for the simplified form of Type AaAa, equation (18). The moments c_p and γ_p are evaluated; β_1 and β_2 for each of the functions f_1 and f_2 correspond approximately to a Pearson Type III curve. The resulting equation of the surface is of the form:

$$z=z_0\,.\,e^{-d_1x-d_2y}(1-a_1x+b_1y)^{p_1}(1-a_2x+b_2y)^{p_2}\ldots\ldots\ldots\ldots(51).$$

The cell mid-ordinates are computed and exhibited together with Jørgensen's result. From an inspection of the table it is fairly obvious that Steffensen's method gives the better graduation. Moreover, it does not give rise to the objectionable negative frequencies.

11. *Rhodes' Surface** (1922). The equation† of the surface is

$$z=z_0\,.\,e^{-lx-my}\left(1-\frac{x}{a}+\frac{y}{b}\right)^p\left(1+\frac{x}{a'}-\frac{y}{b'}\right)^{p'}\ldots\ldots\ldots\ldots\ldots\ldots(52),$$

the mode being given by:

$$1-\frac{x}{a}+\frac{y}{b}=\frac{pp'\left(\frac{1}{ab'}-\frac{1}{a'b}\right)}{-\left(\frac{mp'}{a'}+\frac{lp'}{b'}\right)},$$

$$1+\frac{x}{a'}-\frac{y}{b'}=\frac{pp'\left(\frac{1}{ab'}-\frac{1}{a'b}\right)}{-\left(\frac{lp}{b}+\frac{mp}{a}\right)}.$$

(*a*) *Determination of the Constants in* (52). By considering integrals of the form $\iint\frac{dz}{dx}\,.\,x^t\,.\,y^{t'}\,.\,dxdy$ and $\iint\frac{dz}{dy}\,.\,x^t\,.\,y^{t'}\,.\,dxdy$, Rhodes finds the following equations for the determination of θ, ϕ, λ, s and z_0:

$$\beta_{10}=\frac{4(\phi^2+\lambda)^2}{s(\phi^2+\lambda)^3},\qquad \beta_{01}=\frac{4(\theta^2+\lambda)^2}{s(\theta^2+\lambda)^3},$$

* "On a Certain Skew Correlation Surface," *Biometrika*, Vol. xiv. 1922—23, pp. 355—377.

† Equations (51) and (52) are of essentially the same form. Rhodes' Surface can be obtained from the product of two Pearson Type III curves by linear transformations of the arguments.

$$r=\frac{\theta\phi+\lambda}{\sqrt{(\theta^2+\lambda)(\phi^2+\lambda)}},\qquad z_0=\frac{N.X.p^s.p'^{s'}}{e^{R-2}.\Gamma(s).\Gamma(s')},$$

$$\left.\begin{aligned}\frac{q_{21}-r\sqrt{\beta_{10}}}{\sqrt{1-r^2}}&=\frac{2\sqrt{\lambda}}{\sqrt{s}}.\frac{\phi(1-\phi)}{(\phi^2+\lambda)^{\frac{3}{2}}}\\ \frac{q_{12}-r\sqrt{\beta_{01}}}{\sqrt{1-r^2}}&=\frac{2\sqrt{\lambda}}{\sqrt{s}}.\frac{\theta(1-\theta)}{(\theta^2+\lambda)^{\frac{3}{2}}}\end{aligned}\right\}\text{.........................(53),}$$

where
$$\theta=\frac{ap'}{a'p},\quad \phi=\frac{bp'}{b'p},\quad \lambda=\frac{s'}{s},$$

$$X=\frac{1}{a'b}-\frac{1}{ab'},\quad R=p+p'+2,\quad s'=p'+1,\quad s=p+1.$$

In the illustration one equation is formed from the two equations (53) so as not to give greater weight to one part of the table; but Rhodes does not tell us how he combines them.

The following relations hold amongst the moments of the surface:

$$\left.\begin{aligned}\frac{q_{21}-r\sqrt{\beta_{10}}}{\sqrt{1-r^2}}.\sqrt{3}&=\sqrt{2\beta_{20}-3\beta_{10}-6}\\ \frac{q_{12}-r\sqrt{\beta_{01}}}{\sqrt{1-r^2}}.\sqrt{3}&=\sqrt{2\beta_{02}-3\beta_{01}-6}\end{aligned}\right\}\text{......................(54).}$$

The distance of the mean from the mode is

$$\mu'_{10}=\frac{1}{X}\left(\frac{1}{pb'}+\frac{1}{p'b}\right),$$

$$\mu'_{01}=\frac{1}{X}\left(\frac{1}{pa'}+\frac{1}{p'a}\right).$$

(*b*) *The Arrays of the Surface.* The marginal and regression curves are expressed as infinite series:

y-marginal curve:

$$z_y=c.e^{-pa'u}.\left[u^{R-1}-\frac{aa'ls'}{R}.u^R+\frac{(aa'l)^2}{2!}.\frac{s'(s'+1)}{R.R+1}.u^{R+1}-\ldots\right].$$

Regression curve of x on y:*

$$\mu_1'(x)=a'\left(\frac{y}{b'}-1\right)+\frac{as'}{R}.\frac{S_{R+1,s'+1}}{S_{R,s'}},$$

where
$$u=\frac{1}{a}+\frac{1}{a'}+Xy,$$

$$S_{R,s'}=u^{R-1}-\frac{aa'ls'}{R}.u^R+\ldots,$$

$$S_{R+1,s'+1}=u^R-aa'l.\frac{s'+1}{R+1}.u^{R+1}+\ldots.$$

* It can be easily shown that also the scedastic, clitic and kurtic curves are in the form of infinite series.

(c) *Application.* The results of the theory are illustrated on the distribution of barometric heights at Laudale and Southampton. The cell mid-ordinates and frequencies are computed, and the Goodness of Fit Test is applied to the whole surface as well as to the marginal totals*.

12. *Narumi's*† *System of Frequency Surfaces* (1923). Narumi starts his investigation on bivariate frequency surfaces from a consideration of the regression and scedastic curves. The regression curve need not be restricted to the curve of means; it can be any series of points defined in the same manner for each array. Let $x=f_1(y)$ and $y=f_2(x)$ be the two regression curves; and let $\frac{1}{F_1(y)}$ and $\frac{1}{F_2(x)}$ be the scales of measurement which will reduce the system to complete homoscedasticity. Then the most general functional equation to the frequency surface will be

$$z=\phi_1(y)\,\psi_1[\{x-f_1(y)\}\,F_1(y)]=\phi_2(x)\,\psi_2[\{y-f_2(x)\}\,F_2(x)].$$

The corresponding surfaces are determined for definite forms of $f_1(y)$, $f_2(x)$, $F_1(y)$ and $F_2(x)$.

The array distributions reduced to the regression curve as origin and reduced in scale owing to the heteroscedasticity, are similar and similarly situated curves. According to Narumi there is physically much to uphold this conception of the similarity of parallel arrays.

The most interesting cases considered are:

(i) Homoscedasticity and linear regression both ways → normal surface;

(ii) Scedasticity and regression linear both ways → Filon-Isserlis surface;

(iii) Scedastic and regression curves equilateral hyperbolae both ways

$$\rightarrow z=z_0(x+f_1)^{\gamma_1}(y+g_2)^{\gamma_2}\,.\,e^{\gamma(x+f_2)(y+g_1)};$$

the arrays are Pearson Type III curves;

(iv) Parabolic variance and linear regression → Pearson non-skew surface (see next section).

13. *Pearson's*‡ *Non-Skew Frequency Surfaces* (1923). An investigation by

* I would like to draw attention to the fact that by reducing the number of frequency groups in a bivariate distribution to about 25 broad groups and then applying the P, χ^2 Goodness of Fit Test, we are likely to get almost any value for P. Where a single surface has been fitted to an observed distribution too much significance should not be attached to the corresponding value of P in judging the descriptive power of that surface. We really want more comparative results: different equations fitted to the same observed distribution, the grouping not being altered throughout the investigation. The P's will then enable us to arrange the equations in order of merit as to their successful representation of the data. A comparison of this nature has been made by Pearson between "The Fifteen Constant Surface" and Rhodes' surface (see p. 128), and also between the Filon-Isserlis surface and the symmetrical surface described on p. 137 (see p. 124).

† "On the General Forms of Bivariate Frequency Distributions which are Mathematically Possible when Regression and Variation are subjected to Limiting Conditions," *Biometrika*, Vol. xv. 1923, pp. 77—88, 209—221.

‡ "Non-Skew Frequency Surfaces," *Biometrika*, Vol. xv. 1923, pp. 231—244.

Pearson on the range of frequency surfaces which would have symmetrical marginal distributions, led to the surface

$$z = \frac{N}{2\pi . \sigma_1 . \sigma_2 . \sqrt{1-r^2}} . \frac{n-1}{n-2} . \frac{1}{\left[1 + \frac{1}{2(n-2)} . \frac{1}{1-r^2}\left(\frac{x^2}{\sigma_1^2} - \frac{2rxy}{\sigma_1 . \sigma_2} + \frac{y^2}{\sigma_2^2}\right)\right]^n} \quad \text{......(55)},$$

where

$$\frac{3(\beta_{20}-2)}{\beta_{20}-3} = n = \frac{3(\beta_{02}-2)}{\beta_{02}-3},$$

i.e. $\beta_{20} = \beta_{02}$.

The x-marginal distribution is

$$\phi_1(x) = \frac{N}{\sqrt{2\pi} . \sigma_1} . \frac{1}{\sqrt{n-2}} . \frac{\Gamma(n-\frac{1}{2})}{\Gamma(n-1)} . \frac{1}{\left[1 + \frac{1}{2(n-2)} . \frac{x^2}{\sigma_1^2}\right]^{n-\frac{1}{2}}}$$

The surface has double linear regression and double parabolic variance. It has been shown later by Narumi that no other surface than Pearson's has these forms of regression and scedasticity combined (see previous section).

Regression curve of x on y:

$$\mu_1'(x) = -r . \frac{\sigma_1}{\sigma_2} . y.$$

Scedastic curve of x on y:

$$\sigma^2(x) = \sigma_1^2(1-r^2)\left\{1 + \frac{1}{2n+3}\left(1 - \frac{y^2}{\sigma_2^2}\right)\right\}.$$

When $\beta_2 = 3$, (55) reduces to the normal surface.

A number of special cases are considered when $\beta_2 < 3$:

(i) $n = 1$, $\beta_2 = 2{\cdot}25$: (55) $\rightarrow$ upper portion of a paraboloid;

(ii) $n = \frac{1}{2}$, $\beta_2 = 2{\cdot}1429$: (55) $\rightarrow$ upper half of an ellipsoid;

(iii) $n = 0$, $\beta_2 = 2{\cdot}000$: (55) $\rightarrow$ elliptic cylinder;

(iv) $0 > n \geqslant -\frac{1}{2}$, $2 > \beta_2 \geqslant 1{\cdot}8$: (55) $\rightarrow$ surface is cup-shaped; marginal totals are rectangles when $\beta_2 = 1{\cdot}8$;

(v) $-\frac{1}{2} > n \geqslant -1$, $1{\cdot}8 > \beta_2 \geqslant 1{\cdot}5$: (55) $\rightarrow$ also the marginal totals are now U-shaped.

14. *The Dissection of Frequency Surfaces.* In *Medd. från Lunds Astr. Obs.*, Ser. 2, No. 9, Charlier has dealt with the dissection of a bivariate distribution into two normal components with zero correlation:

$$\phi_1(x, y) = \frac{Z_0}{2\pi . \sigma_1^2} . e^{-\frac{1}{2}\left[\frac{(x-m_1)^2 + (y-n_1)^2}{\sigma_1^2}\right]},$$

and

$$\phi_2(x, y) = \frac{Z_0'}{2\pi . \sigma_2^2} . e^{-\frac{1}{2}\left[\frac{(x-m_2)^2 + (y-n_2)^2}{\sigma_2^2}\right]}$$

The dissection into normal components with elliptical contours having their

principal axes parallel to the coordinate axes, has been treated by Åkesson*; the quite general case of normal components with any direction of principal axes has been discussed by Charlier and Wicksell†. Here

$$\phi_1(x, y) = \frac{Z_0}{2\pi \cdot \sigma_1 \cdot \sigma_1' \sqrt{1-r^2}} \cdot e^{-\frac{1}{2(1-r^2)} \cdot \left[\left(\frac{x-m_1}{\sigma_1}\right)^2 - 2r\left(\frac{x-m_1}{\sigma_1}\right)\left(\frac{y-n_1}{\sigma_1'}\right) + \left(\frac{y-n_1}{\sigma_1'}\right)^2\right]},$$

$$\phi_2(x, y) = \frac{Z_0'}{2\pi \cdot \sigma_2 \cdot \sigma_2' \sqrt{1-r'^2}} \cdot e^{-\frac{1}{2(1-r'^2)} \cdot \left[\left(\frac{x-m_2}{\sigma_2}\right)^2 - 2r'\left(\frac{x-m_2}{\sigma_2}\right)\left(\frac{y-n_2}{\sigma_2'}\right) + \left(\frac{y-n_2}{\sigma_2'}\right)^2\right]}.$$

In this last paper general equations are given for the moments of a bivariate distribution in terms of the moments of any two components. The case of normal components is then worked out fully. For the moment coefficients up to the fourth order fifteen equations are obtained involving the twelve unknowns. There are thus certain identical relations between the moments; and these may be regarded as criteria for dissecting the distribution into two normal components.

The determination of the unknowns is shown to depend on the solution of equations of an order not higher than the third. Some special cases, for which the general analysis is not valid, are treated separately.

15. '*Mutually Consistent Multiple Regression Surfaces.*' *Camp*‡ (1925). The object of Professor Camp's study is to determine what forms of the regression surfaces and of the total regressions are mathematically consistent with one another; arbitrary forms for the regression surfaces may not be combined with arbitrary forms for the total regressions.

He confines his study to trivariate distributions and assumes the regression surfaces to be polynomials of the second or higher order. These simple assumptions generally lead to total regressions of the form:

$$y = \frac{\text{polynomial in } x}{\text{polynomial in } x}.$$

E.g. Let the regression of z on x, y be:

$$g(x, y) = a + by + cx + dxy,$$

i.e. all the partial regressions linear, then the regression of z on x is of the form:

$$\alpha(x) = \frac{\text{parabola in } x}{\text{parabola in } x}.$$

If $$g(x, y) = a + by + cx + dxy + ex^2 + fyx^2,$$

then $$\alpha(x) = \frac{\text{cubic in } x}{\text{cubic in } x}.$$

The first of these expressions for $g(x, y)$ is subjected to a detailed treatment, as it is of the form assumed by Isserlis in his paper on the partial correlation ratio.

* "On the Dissection of Correlation Surfaces," *Arkiv för Mat., Astr. och Fysik*, Bd. 11, No. 16, 1916, pp. 1—16.

† "On the Dissection of Frequency Functions," *Arkiv för Mat., Astr. och Fysik*, Bd. 18, No. 6, 1923, pp. 1—64.

‡ *Biometrika*, Vol. xvii, 1925, pp. 443—458.

16. *"On Treating Skew Correlation." Van Uven** (1925—29). The method followed by Van Uven in analysing skew correlation is equivalent to the principle of translation underlying the frequency curves of Edgeworth and of Kapteyn, viz. if x be the directly observed quantity, to find that function $f(x)$ of x which will be normally distributed. Whereas Edgeworth and Kapteyn assumed definite forms for $f(x)$, Van Uven† has developed a scheme for determining $f(x)$ graphically. This method is now extended to the treatment of correlation.

If the observed variates, x and y, are not normally correlated, the problem is to construct two functions t and t' of x and y which will follow the normal law and which will give as high a measure as possible of the correlation between x and y. Each of the new variables may involve both x and y, but it is generally possible, with the use of certain transformations, to express t (or t') as a function only of x (or only of y).

Let $x_{p-h/2}$ and $y_{q-k/2}$ be the mid-points of the pth x-array of y's and of the qth y-array of x's respectively; h and k the grouping units of x and y; and suppose p to vary from 1 to n, q from 1 to n'. Then in our usual notation:

$$n_{x_p} = \sum_{q=1}^{n'} n_{x_p y_q}, \quad n_{y_q} = \sum_{p=1}^{n} n_{x_p y_q}.$$

The relative frequencies of the x- and y-marginal totals will be $\frac{n_{x_p}}{N}$ and $\frac{n_{y_q}}{N}$ respectively; those of the x_p- and y_q-arrays will be $\frac{n_{x_p y}}{n_{x_p}}$ and $\frac{n_{x y_q}}{n_{y_q}}$ respectively.

Assume x and y, measured in terms of their standard deviations, to be normally distributed, i.e.

$$d\phi = \frac{\sqrt{1-r^2}}{2\pi} \, . \, e^{-\frac{1}{2}(x^2 - 2rxy + y^2)} \, . \, dx \, . \, dy.$$

Write $\sqrt{1-r^2} \, . \, x = z$ and $y - rx = \zeta$, then

$$d\phi = \frac{1}{\sqrt{2\pi}} \, . \, e^{-\frac{1}{2}z^2} \, . \, dz \times \frac{1}{\sqrt{2\pi}} \, . \, e^{-\frac{1}{2}\zeta^2} \, . \, d\zeta = du \, . \, dv,$$

where $u = \theta(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}w^2} \, . \, dw$, and $v = \theta(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\zeta} e^{-\frac{1}{2}w^2} \, . \, dw$.

Similarly, by writing $\sqrt{1-r^2} \, . \, y = z'$ and $x - ry = \zeta'$, we get

$$d\phi = \frac{1}{\sqrt{2\pi}} \, . \, e^{-\frac{1}{2}z'^2} \, . \, dz' \times \frac{1}{\sqrt{2\pi}} \, . \, e^{-\frac{1}{2}\zeta'^2} \, . \, d\zeta' = du' \, . \, dv',$$

where

$$u' = \theta(z') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z'} e^{-\frac{1}{2}w^2} \, . \, dw, \text{ and } v' = \theta(\zeta') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\zeta'} e^{-\frac{1}{2}w^2} \, . \, dw.$$

* *Proc. Kon. Ak. v. Wet.* (Amsterdam), Vol. XXVIII. Nos. 8—9, 1925, pp. 797—811; No. 10, pp. 919—935; Vol. XXIX. No. 4, 1926, pp. 580—590; Vol. XXXII. No. 4, 1929, pp. 408—413; see also "Skew Correlation between three and more Variables," *Proc. Kon. Ak. v. Wet.* Vol. XXXII. No. 6, 1929, pp. 793—807; No. 7, pp. 995—1007; No. 8, pp. 1085—1103.

† Kapteyn, J. C. and Van Uven, M. J.: *Skew Frequency Curves in Biology and Statistics.* 2nd Paper. Groningen, 1916, pp. 30—53.

It is easily seen that:

$$\theta(z_p)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z(x_p)} e^{-\frac{1}{2}w^2}\,.\,dw \qquad =\frac{\sum\limits_{i=1}^{p} n_{x_i}}{N},$$

$$\theta(z_q')=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z'(y_q)} e^{-\frac{1}{2}w^2}\,.\,dw \qquad =\frac{\sum\limits_{i=1}^{q} n_{y_i}}{N},$$

$$\theta(\zeta_{p,\,q-k/2})=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\zeta(x_p,\,y_{q-k/2})} e^{-\frac{1}{2}w^2}\,.\,dw=\frac{\sum\limits_{i=1}^{p} n_{x_i y_q}}{n_{y_q}} \text{ (approximately)},$$

and
$$\theta(\zeta'_{p-h/2,\,q})=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\zeta'(x_{p-h/2},\,y_q)} e^{-\frac{1}{2}w^2}\,.\,dw=\frac{\sum\limits_{i=1}^{q} n_{x_p y_i}}{n_{x_p}} \text{ (")}.$$

The conditions for normal correlation are: that the sets of values of the functions $z(x)$, $z'(y)$, $\zeta(x, y)$ and $\zeta'(x, y)$ obtained from these equations, have to satisfy linear relations:

$$z=ax+b, \quad \zeta=a'x+b'y+c', \text{ etc.}$$

If x and y follow a law of skew variation, two new variables t and t' are introduced as functions of z, z', ζ and ζ'—and thus as functions of x and y—to reduce the correlation to normality.

The method is illustrated on the correlation of height and volume of djati trees—a distribution that tends to be J-shaped in the one direction and thus representative of extremely skew correlation. The total number of observations is 916. The correlation is found to be ·948 as against ·831 calculated by the product-moment method. I find the values of the correlation ratios to be in fair agreement with the value of the relationship found by Van Uven; they are* $\eta_{yx}=·916$ and $\eta_{xy}=·988$, where x is the height and y the volume of the trees. No general conclusions can be drawn from this one example, but, on grounds of the agreement obtained, I think that the relatively small amount of labour demanded by the correlation ratio method in comparison with that involved in the application of Van Uven's method, is sufficient to establish, in the absence of further illustrations, the superiority of that method.

II.

D. *Further Analysis of Some of the Proposed Constructions.*

1. *The Array Moments of a Certain Double Hypergeometrical Series.* Let us consider the case of composite sampling specified on p. 123. It was stated there that the chance of s marked characters being drawn in the first sample and s' in the second, is

$$z(s, s')=\frac{n!\,n'!}{s!\,s'!\,(n-s)!\,(n'-s')!}\cdot\frac{(N-n-n')!}{N!}\cdot\frac{m!}{(m-s-s')!}\cdot\frac{(N-m)!}{(N-n-n'-m+s+s')!} \quad \text{.........(56).}$$

* No corrections for grouping or abruptness have been applied.

We have further, that the actual distribution of the terms of an array of second samples corresponding to a definite number s of marked characters in the first sample, is given by*

$$\frac{N-n-(m-s)}{N-n}\;\frac{N-n-1-(m-s)}{N-n-1}\cdots\frac{N-n-n'+1-(m-s)}{N-n-n'+1}$$
$$\times\left[1+n'\cdot\frac{m-s}{N-n-n'+1-(m-s)}\right.$$
$$+\frac{n'(n'-1)}{2!}\cdot\frac{(m-s)(m-s-1)}{(N-n-n'+1-(m-s))(N-n-n'+2-(m-s))}+\cdots$$
$$\left.+\frac{n'!}{s'!\,(n'-s')!}\cdot\frac{(m-s)(m-s-1)(m-s-2)\ldots(m-s-s'+1)}{(N-n-n'+1-(m-s))\ldots(N-n-n'+s'-(m-s))}+\cdots\right] \qquad \text{.........(57).}$$

Write $\alpha=-n'$, $\beta=-(m-s)$, $\gamma=N-n-n'+1-(m-s)$; (57) is then seen to be the hypergeometrical series $F(\alpha,\beta,\gamma,1)$.

From the usual formulae for the moments of a hypergeometrical series, the moments of (57) can be readily written down and these will furnish us with the required expressions for the array moments.

Pearson finds the regression and scedasticity to be

$$\nu_1'(s')=n'\cdot\frac{m-s}{N-n}$$

and $$\nu_2(s')=n'\cdot\frac{N-n-n'}{N-n-1}\cdot\left[\frac{1}{4}-\left(\frac{m-s}{N-n}-\frac{1}{2}\right)^2\right] \text{ respectively.}$$

The third array moment is given by

$$\nu_3(s')=n'\cdot\frac{N-n-n'}{N-n-1}\cdot\frac{N-n-2n'}{N-n-2}\cdot\frac{m-s}{N-n}\cdot\left(1-\frac{m-s}{N-n}\right)\cdot\left(1-\frac{2(m-s)}{N-n}\right)$$
$$=\frac{N-n-2n'}{N-n-2}\cdot\left(1-\frac{2(m-s)}{N-n}\right)\cdot\nu_2(s').$$

Hence

$$\beta_1(s')\equiv\frac{\nu_3^2(s')}{\nu_2^3(s')}=\left(\frac{N-n-2n'}{N-n-2}\right)^2\cdot\frac{N-n-1}{n'(N-n-n')}\cdot\left[\frac{(N-n)^2}{(m-s)(N-n-(m-s))}-4\right] \qquad \text{.........(58).}$$

The fourth array moment is

$$\nu_4(s')=n'\cdot\frac{N-n-n'}{N-n-1}\cdot\frac{m-s}{N-n}\cdot\left(1-\frac{m-s}{N-n}\right)\cdot\left[1-\frac{6(n'-1)(N-n-n'-1)}{(N-n-2)(N-n-3)}\right.$$
$$\left.+3(n'-2)\cdot\frac{m-s}{N-n}\cdot\left(1-\frac{m-s}{N-n}\right)\left\{1-\frac{n'-1}{N-n-2}\cdot\left(\frac{n'-10}{n'-2}+\frac{9}{N-n-3}\right)\right\}\right]$$
$$=\nu_2(s')\left[1-\frac{6(n'-1)(N-n-n'-1)}{(N-n-2)(N-n-3)}+3(n'-2)\frac{N-n-1}{n'(N-n-n')}\cdot\nu_2(s')\right.$$
$$\left.\times\left\{1-\frac{n'-1}{N-n-2}\cdot\left(\frac{n'-10}{n'-2}+\frac{9}{N-n-3}\right)\right\}\right].$$

* Pearson, Karl: "On a Certain Double Hypergeometrical Series and its Representation by Continuous Frequency Surfaces," *Biometrika*, Vol. XVI. 1924, p. 175.

Thus

$$\beta_2(s') \doteq \frac{\nu_4(s')}{\nu_2^2(s')} = \frac{N-n-1}{n'(N-n-2)(N-n-3)(N-n-n')}$$
$$\times\left[3\{n'(N-n-n')(N-n+6)-2(N-n)^2\}\right.$$
$$\left.+\frac{(N-n)^2\{(N-n)(N-n-6n'+1)+6n'^2\}}{(m-s)(N-n-(m-s))}\right] \quad \ldots\ldots\ldots(59).$$

The relation between $\beta_1(s')$ and $\beta_2(s')$ can be obtained from equations (58) and (59). After some reductions I find it to be

$$\beta_1(s')+\frac{(N-n-3)(N-n-2n')^2}{(N-n-2)\{(N-n)(6n'-N+n-1)-6n'^2\}}\beta_2(s')$$
$$=\frac{(N-n-1)(N-n-2n')^2\{3n'^2-(N-n)(3n'-2)\}}{n'(N-n-2)(N-n-n')\{(N-n)(N-n-6n'+1)+6n'^2\}} \ldots(60).$$

Accordingly, β_1 and β_2 of the arrays lie on a straight line in the β-diagram; the curves of both $\beta_1(s')$ and $\beta_2(s')$ are U-shaped. Of course only a portion of these curves may correspond to appropriate values of s.

As an example we shall consider the case of whist correlation. Putting $N=52$, $n=n'=m=13$, we get:

$$\nu_2(s')=\tfrac{1}{171}(13-s)(26+s),$$

$$\beta_1(s')=\frac{(13+2s)^2}{(111)^2 \cdot \nu_2(s')} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(58)^{bis},$$

$$\beta_2(s')=3{\cdot}081{,}081-\frac{{\cdot}351{,}351}{\nu_2(s')} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(59)^{bis},$$

$$\beta_1(s')+{\cdot}351{,}351\,\beta_2(s')-1{\cdot}027{,}027=0 \quad \ldots\ldots\ldots\ldots\ldots(60)^{bis}.$$

The curves $(58)^{bis}$, $(59)^{bis}$, $(60)^{bis}$ are shown in the accompanying diagram (p. 143). As s increases from 0 to 12, $\beta_1(s')$ increases from ·007 to ·500 and $\beta_2(s')$ decreases from 2·903 to 1·500; the rate of increase or decrease being small for low values of s. The diagram besides being instructive as to the form of the β-curves, shows why the Filon-Isserlis surface and the Pearson non-skew surface were found inadequate to represent the double hypergeometrical series*. Both these surfaces have similar parallel sections.

2. *The Third and Fourth Partial Moments of the Type AaAa Surface.* The equation of the surface can be written in the form

$$z=Z-\tfrac{1}{6}[q_{30}Z_{30}+3q_{21}Z_{21}+3q_{12}Z_{12}+q_{03}Z_{03}]$$
$$+B_{40}Z_{40}+Q_{31}Z_{31}+3Q_{22}Z_{22}+Q_{13}Z_{13}+B_{04}Z_{04} \quad \ldots\ldots\ldots\ldots(61),$$

where
$$Z=\frac{1}{2\pi\sqrt{1-r^2}}\cdot e^{-\frac{1}{2(1-r^2)}(x^2-2rxy+y^2)},$$

$$Z_{m,n}=\frac{d^{m+n}Z}{dx^m\,.\,dy^n}, \qquad B_{40}=\tfrac{1}{24}(\beta_{40}-3),$$

$$Q_{31}=\tfrac{1}{6}(q_{31}-3r), \qquad Q_{22}=\tfrac{1}{12}(q_{22}-1-2r^2), \text{ etc.}$$

* See p. 124. See remarks, *Biometrika*, Vol. XVI. p. 185.

The regression and scedastic curves of (61) have been dealt with in Section C, 6. The third and fourth partial moments will now be considered, firstly about the mean of the surface and then about the regression curve as origin. These moments have, in fact, been found before*, but not about the line of means. Moreover, the expressions can be put into forms much simpler than those given by Wicksell. It will thus be possible to perform the numerical applications more readily.

WHIST DOUBLE HYPERGEOMETRICAL SERIES. β_1 AND β_2 OF PARALLEL ARRAYS.

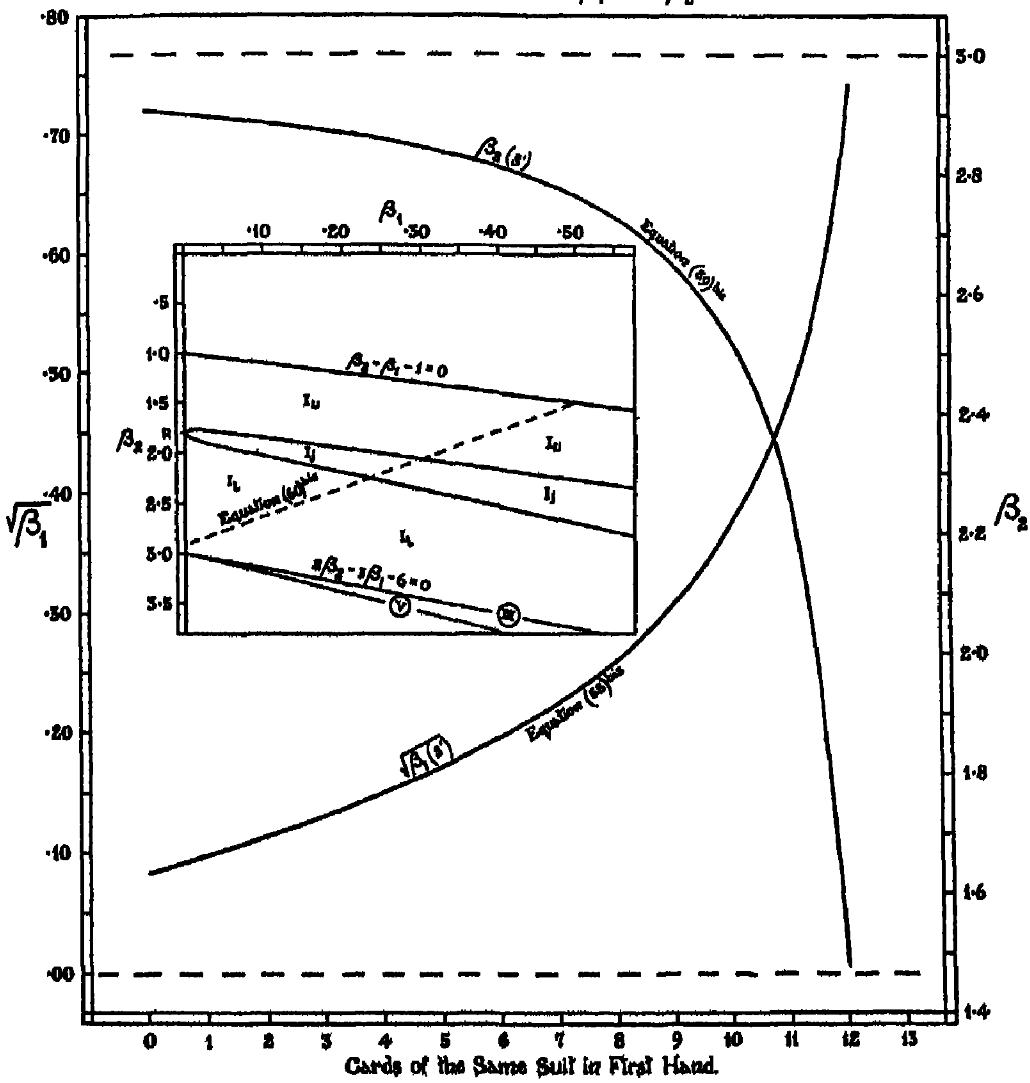


Diagram (8).

The following results are restated as we shall have to refer to them later on.

y-marginal Curve:

$$z_y = \tau_1 + \sqrt{\tfrac{2}{3}}\,.\sqrt{\beta_{01}}\,.\,\tau_4 + \sqrt{\tfrac{5}{24}}\,(\beta_{02}-3)\,\tau_5$$
$$\equiv \tau_1 + a_4\tau_4 + a_5\tau_5 \quad\text{.......................................}(62).$$

* See p. 126.

Regression Curve of x on y:

$$\mu_1'(x) = ry + \frac{\sqrt{\frac{2}{3}}\,(q_{12} - r\sqrt{\beta_{01}})\,\tau_3 + \sqrt{\frac{2}{3}}\,(q_{13} - r\beta_{02})\,\tau_4}{z_y}$$

$$\equiv ry + \frac{b_3\tau_3 + b_4\tau_4}{z_y} \quad \text{...(63)}$$

$$\equiv ry + \frac{A_y}{z_y}.$$

Scedastic Curve of x on y:

$$\mu_2(x) = (1 - r^2)$$
$$-\frac{\sqrt{2}\,[r\,(q_{12} - r\sqrt{\beta_{01}}) - (q_{21} - rq_{12})]\,\tau_2 + \sqrt{6}\,[r\,(q_{13} - r\beta_{02}) - \frac{1}{2}\,(q_{22} - 1 - r^2(\beta_{02} - 1))]\,\tau_3}{z_y}$$
$$- \left(\frac{A_y}{z_y}\right)^2$$

$$\equiv (1 - r^2) - \frac{c_2\tau_2 + c_3\tau_3}{z_y} - \left(\frac{A_y}{z_y}\right)^2 \quad \text{...(64)}$$

$$\equiv (1 - r^2) - \left(\frac{B_y}{z_y}\right) - \left(\frac{A_y}{z_y}\right)^2.$$

(i) *The third partial Moment.*

Write $\int_{-\infty}^{+\infty} Z\,dx = \frac{1}{\sqrt{2\pi}}\,.\,e^{-\frac{1}{2}y^2} \equiv v$ (say), $\frac{d^s}{dy^s}(v) \equiv v_s$.

Consider the integral $\int_{-\infty}^{+\infty} z\,.\,x^3\,dx = z_y\,.\,\mu_3'(x)$.

We have $\int_{-\infty}^{+\infty} Z\,.\,x^3 dx = v\,[r^3(y^3 - 3y) + 3ry]$,

$$\int_{-\infty}^{+\infty} Z_{30}\,.\,x^3 dx = -6v, \quad \int_{-\infty}^{+\infty} Z_{21}\,.\,x^3 dx = -6vr\,(y^2 - 1),$$

$$\int_{-\infty}^{+\infty} Z_{12}\,.\,x^3 dx = -3v\,[r^2(y^4 - 6y^2 + 3) + (y^2 - 1)],$$

$$\int_{-\infty}^{+\infty} Z_{03}\,.\,x^3 dx = v_3\,[r^3(y^3 - 3y) + 3ry] + 9rv_2\,[r^2(y^2 - 1) + 1] + 6r^3v\,[1 - 3y^2],$$

$$\int_{-\infty}^{+\infty} Z_{40}\,.\,x^3 dx = 0, \quad \int_{-\infty}^{+\infty} Z_{31}\,.\,x^3 dx = 6vy,$$

$$\int_{-\infty}^{+\infty} Z_{22}\,.\,x^3 dx = 6rv\,(y^3 - 3y),$$

$$\int_{-\infty}^{+\infty} Z_{13}\,.\,x^3 dx = 3v\,[(y^3 - 3y)(1 - r^2 + r^2y^2) - 6r^2y\,(y^2 - 1) + 6yr^2],$$

$$\int_{-\infty}^{+\infty} Z_{04}\,.\,x^3 dx = v_4\,[r^3(y^3 - 3y) + 3ry] + 12v_3r\,[r^2(y^2 - 1) + 1] + 36v_2r^3y - 24vr^3y.$$

Accordingly:

$$\mu_3'(x) = 3\,(1 - r^2 + r^2y^2)\,A_y - 3ryB_y + [r^3(y^3 - 3y) + 3ry]\,z_y$$
$$+ v\,[6y\,(Q_{31} - 6rQ_{22} + 3r^2Q_{13} - 4r^3B_{04}) + \sqrt{\beta_{10}} - r^3\sqrt{\beta_{01}} - 3r\,(q_{21} - rq_{12})].$$

Using the expressions for $\mu_1'(x)$ and $\mu_2'(x)$, we get

$$\mu_3(x)=\frac{[\sqrt{\beta_{10}}-r^3\sqrt{\beta_{01}}-3r(q_{21}-rq_{12})]\,\tau_1+6\sqrt{2}\,[Q_{31}-6rQ_{22}+3r^2Q_{13}-4r^3B_{02})\,\tau_2}{z_y}$$
$$+3\left(\frac{A_y}{z_y}\right)\left(\frac{B_y}{z_y}\right)+2\left(\frac{A_y}{z_y}\right)^3$$
$$\equiv\frac{d_1\tau_1+d_2\tau_2}{z_y}+3\left(\frac{A_y}{z_y}\right)\left(\frac{B_y}{z_y}\right)+2\left(\frac{A_y}{z_y}\right)^3 \quad\text{.....................................(65)}$$
$$\equiv\left(\frac{C_y}{z_y}\right)+3\left(\frac{A_y}{z_y}\right)\left(\frac{B_y}{z_y}\right)+2\left(\frac{A_y}{z_y}\right)^3.$$

(ii) *The fourth partial Moment.*

Consider the integral $\int_{-\infty}^{+\infty} z\,.\,x^4dx=z_y\,.\,\mu_4'(x)$.

We have $\int_{-\infty}^{+\infty} Z\,.\,x^4dx=v\,[3(1-r^2)^2+6r^2(1-r^2)\,y^2+r^4y^4]$,

$$\int_{-\infty}^{+\infty} Z_{30}\,.\,x^4dx=-24vry,\quad \int_{-\infty}^{+\infty} Z_{21}\,.\,x^4dx=-12v\,[y+r^2(y^3-3y)],$$
$$\int_{-\infty}^{+\infty} Z_{12}\,.\,x^4dx=-4\,[v_2\,\{r^3(y^3-3y)+3ry\}+6v_1r\,\{r^2(y^2-1)+1\}+6vr^3y],$$
$$\int_{-\infty}^{+\infty} Z_{03}\,.\,x^4dx=v_3\,[3(1-r^2)^2+6r^2(1-r^2)\,y^2+r^4y^4]+12v_2r\,[r^3(y^3-3y)+3ry]$$
$$+36v_1r^2\,[r^2(y^2-1)+1]+24vr^4y,$$
$$\int_{-\infty}^{+\infty} Z_{40}\,.\,x^4dx=24v,\quad \int_{-\infty}^{+\infty} Z_{31}\,.\,x^4dx=24vr\,(y^2-1),$$
$$\int_{-\infty}^{+\infty} Z_{22}\,.\,x^4dx=12v\,[r^2(y^4-6y^2+3)+(y^2-1)],$$
$$\int_{-\infty}^{+\infty} Z_{13}\,.\,x^4dx=-4\,[v_3\,\{r^3(y^3-3y)+3ry\}+9v_2r\,\{r^2(y^2-1)+1\}-18vr^3y^2+6vr^3],$$
$$\int_{-\infty}^{+\infty} Z_{04}\,.\,x^4dx=v_4\,[3(1-r^2)^2+6r^2(1-r^2)\,y^2+r^4y^4]+16v_3r\,[r^3(y^3-3y)+3ry]$$
$$+72v_2r^2\,[r^2(y^2-1)+1]-96vr^4y^2+24vr^4.$$

Combining these expressions, we get

$$z_y\,.\,\mu_4'(x)=[3(1-r^2)^2+6r^2(1-r^2)\,y^2+r^4y^4]\,z_y+4\,[r^3(y^3-3y)+3ry]\,A_y$$
$$-6\,[r^2(y^2-1)+1]\,B_y+4ry\,.\,C_y+[(\beta_{20}-3)(1-4r^2)-3r^4(\beta_{02}-3)$$
$$+6r^2(q_{22}-1-2r^2)-4r\,(q_{31}-r\beta_{20})-4r^3(q_{13}-r\beta_{02})].$$

Hence, if the regression curve be the origin:

$$\mu_4(x)=3(1-r^2)^2+\frac{\left\{\begin{array}{r}[(\beta_{20}-3)(1-4r^2)-3r^4(\beta_{02}-3)+6r^2(q_{22}-1-2r^2)\\ -4r\,(q_{31}-r\beta_{20})-4r^3(q_{13}-r\beta_{02})]\,\tau_1\end{array}\right\}}{z_y}$$
$$-6(1-r^2)\left[\frac{B_y}{z_y}+\left(\frac{A_y}{z_y}\right)^2\right]-4\left(\frac{A_y}{z_y}\right)\left(\frac{C_y}{z_y}\right)-6\left(\frac{A_y}{z_y}\right)^2\left(\frac{B_y}{z_y}\right)-3\left(\frac{A_y}{z_y}\right)^4$$
$$\equiv 3(1-r^2)^2+\frac{e_1\tau_1}{z_y}-6(1-r^2)\left[\frac{B_y}{z_y}+\left(\frac{A_y}{z_y}\right)^2\right]-4\left(\frac{A_y}{z_y}\right)\left(\frac{C_y}{z_y}\right)$$
$$-6\left(\frac{A_y}{z_y}\right)^2\left(\frac{B_y}{z_y}\right)-3\left(\frac{A_y}{z_y}\right)^4 \quad\text{...(66).}$$

This can be written as

$$\mu_4(x) = 6(1-r^2)\,.\,\mu_2(x) - 3(1-r^2)^2 + \frac{D_y}{z_y} - 4\left(\frac{A_y}{z_y}\right)\left(\frac{C_y}{z_y}\right) - 6\left(\frac{A_y}{z_y}\right)^2\left(\frac{B_y}{z_y}\right) - 3\left(\frac{A_y}{z_y}\right)^4,$$

where $D_y = e_1 r_1$.

From equations (64), (65) and (66) the clitic and kurtic curves can be obtained.

3. *The Logarithmically Transformed Normal Curve**. The object of this section is to analyse more fully the relations between the fourth and lower moment coefficients of the curve:

$$y = y_0 \,.\, \frac{1}{x} \,.\, e^{-\frac{1}{2}\left(\frac{\log_{10} x - l}{s}\right)^2} \quad \text{.........................(67).}$$

We have from p. 120:

$$\mu_1' = e^{bl + \frac{1}{2}b^2s^2},$$

$$\mu_2 = e^{2bl + b^2s^2}\,.\,[e^{b^2s^2} - 1],$$

$$\mu_3 = e^{3bl + \frac{3}{2}b^2s^2}\,.\,[e^{3b^2s^2} - 3e^{b^2s^2} + 2],$$

$$\mu_4 = e^{4bl + 2b^2s^2}\,.\,[e^{6b^2s^2} - 4e^{3b^2s^2} + 6e^{b^2s^2} - 3].$$

The mode of (67) is given by

$$x_{\text{mode}} = e^{bl - b^2s^2}.$$

Write

$$e^{b^2s^2} = \lambda,$$

then:

$$\left.\begin{aligned} \text{Skewness} = \chi = \frac{\text{Mean} - \text{Mode}}{\text{Standard Deviation}} &= \frac{1 - \lambda^{-\frac{3}{2}}}{\sqrt{\lambda - 1}} \\ \beta_1 &= \lambda^2(\lambda + 3) - 4 \\ \eta = \beta_2 - 3 &= \lambda^2(\lambda^2 + 2\lambda + 3) - 6 \end{aligned}\right\} \text{............(68).}$$

Differentiating χ, we find it has a maximum value for $\lambda = 1{\cdot}7201$ which gives $\chi_{\max} = {\cdot}6561$. The skewness of the curve therefore ranges from 0 to ·6561.

* After I had completed the analysis of this section a paper by G. R. Davies on "The Analysis of Frequency Distributions" appeared in the *Journ. Am. Stat. Ass.* December, 1929. The author refers therein to a study by himself on "The Logarithmic Curve of Distribution" in the same *Journal*, December, 1925—a study of which I had been unaware.

In the first of these papers the parameters corresponding to our l and s, are found from the mean and standard deviation computed by replacing the class marks by their logarithms. In the second paper the quartile dispersion is adopted as the basis of the method of fitting. Two illustrations of the application of this method are given; the total number of observations being 10 and 32. I have not had occasion to test the efficiency of this method in relation to the method of moments (described in Section C, 2), but it is conceivable that for such small numbers the method of quartiles might be adequate.

On p. 859 of his second paper Davies writes:

"Since [the logarithmic normal] can be varied to give any required degree of skewness...."

The author gives two equations corresponding to our (68) as well as a small β-diagram indicating the relation of the log. normal curve to the Pearson curves, yet he makes the above remark. The relation between β_1 and β_2 expressed by equations (68) will always exist, no matter by what method the parameters of the curve have been determined, and thereby the skewness of the curve is strictly defined. In particular, it cannot describe mesokurtic distributions.

DIAGRAM SHOWING THE RELATION BETWEEN β_1 AND β_2 FOR THE LOGARITHMICALLY TRANSFORMED NORMAL CURVE.

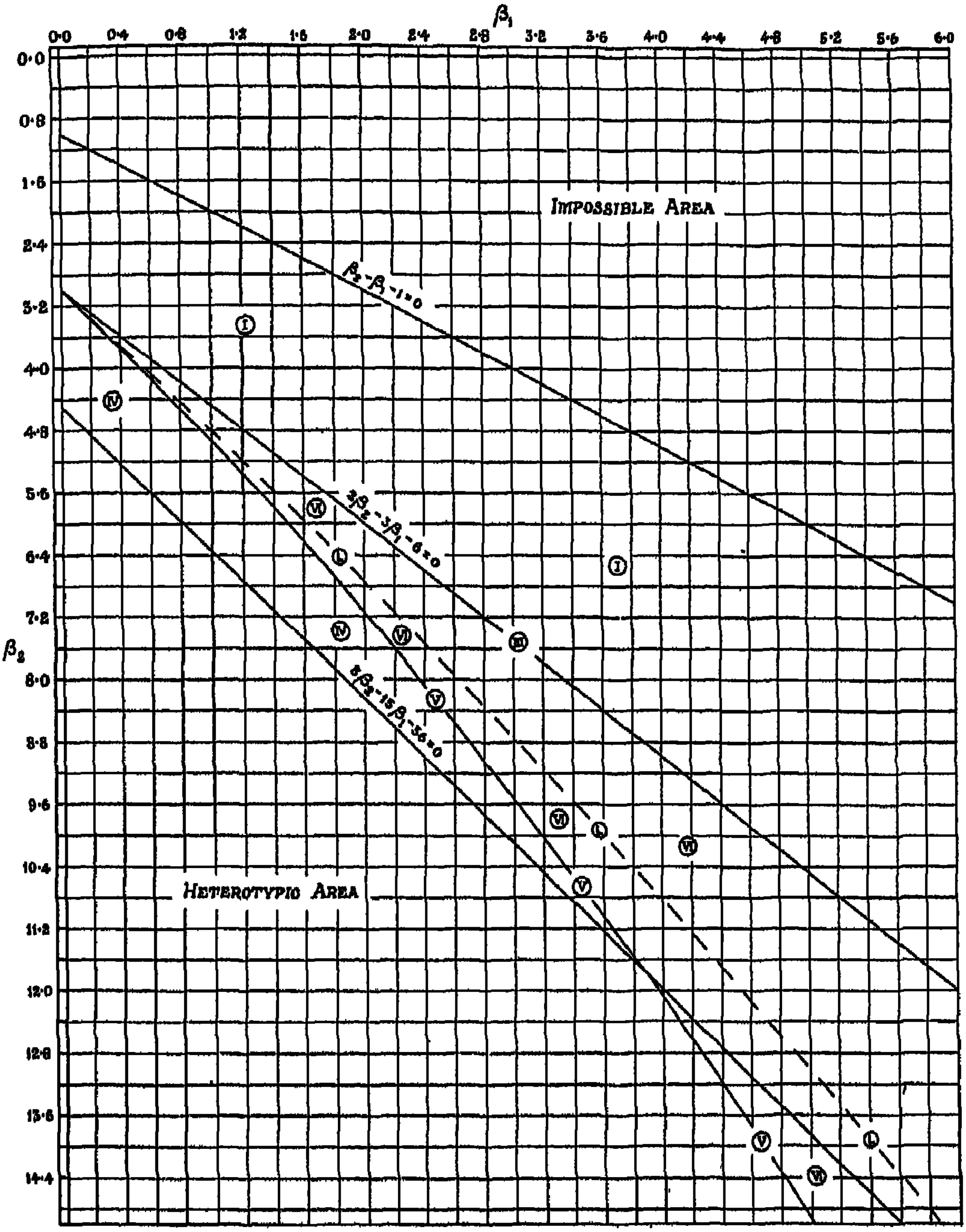


Diagram (4).

The relations between the first four moments of (67) are expressed by β_1 and β_2, both being functions of λ only. Eliminating λ between equations (68) we get*

$$\beta_1^4 + 12\beta_1^3 + 156\beta_1^2 + 64\beta_1 - \eta^3 + 12\eta^2 - 36\eta + 9\beta_1^2 . \eta - 6\beta_1 . \eta^2 - 108\beta_1 . \eta = 0 \quad \text{......(69).}$$

* The maximum value of χ and the equation corresponding to our (69) are incorrect as given on p. 195, *Biometrika*, Vol. IV.

This curve, plotted in relation to the Pearson Types, is shown as the broken line LL in the accompanying β_1, β_2 diagram; it passes about midway through the Type VI area. The diagram will be of use in ascertaining from the β's of an observed distribution whether the data follow a law of the form (67), or not.

We proceed to a comparison of the log. normal curve with the corresponding Type VI by fitting both curves to a distribution whose β's satisfy approximately relations (68). The outcome of such a comparison is of practical importance in so far as the log. normal curve is easier to apply than the Type VI; the particular advantage of the former curve being that the cell frequencies are directly obtainable.

The second column of Table (1) shows the distribution of 1951 readings of the height of the barometer, at Greenwich, on the first day for a reading of 30·1″—30·2″ on the third day (see Table III, p. 154). The constants are

$$\text{Mean height of barometer} = 30{\cdot}0049'',$$

$$\sigma = 2{\cdot}419{,}847 \text{ (unit} = \tfrac{1}{10}''), \quad \beta_1 = {\cdot}712{,}997, \quad \beta_2 = 4{\cdot}307{,}638.$$

TABLE (1).

Distribution of Barometric Heights, represented by a Log. Normal and a Type VI Curve.

Barometric Height	Observed Frequency	Theor. Freq. Log. Normal	Theor. Freq. Type VI
30·75	1		
30·65	1	3·4	3·4
30·55	10		
30·45	32	34·0	34·2
30·35	111	127·2	127·1
30·25	214	252·6	252·6
30·15	386	336·8	336·3
30·05	365	342·1	342·0
29·95	288	288·3	288·6
29·85	199	212·8	213·2
29·75	129	143·0	143·1
29·65	86	89·6	89·6
29·55	62	53·4	53·3
29·45	26	30·0	30·5
29·35	17	17·1	17·0
29·25	10	9·3	9·3
29·15	9	5·1	5·1
29·05	2	2·7	2·7
28·95	2	1·4	1·4
28·85	1	1·6	1·6
Totals	1951	1951	1951
χ^2	—	23·334	23·484
P	—	·038	·037

The β-diagram shows that the condition (69) is approximately fulfilled. The observed value of β_1 in equations (68) gives $\beta_2 = 4{\cdot}2908$.

The constants of the log. normal curve are, in tenths of inches as units:

$$\xi_1 = \text{mean-start} = 8{\cdot}813{,}403,$$
$$l = {\cdot}929{,}362, \quad s = {\cdot}117{,}082.$$

The constants of the Type VI curve

$$y = y_0 (x - a)^{q_2} x^{-q_1}$$

are

$$q_1 = 50{\cdot}369{,}290, \qquad q_2 = 15{\cdot}683{,}288,$$
$$a = 15{\cdot}513{,}656, \quad \log y_0 = 57{\cdot}414{,}365,$$
$$\text{Mean-start} = 7{\cdot}918{,}337.$$

In columns three and four of Table (1), the theoretical frequencies are exhibited; a very close agreement between the two theories is manifest.

The question now arises, how accurately will the one curve reproduce the other for β's satisfying exactly the relation (69)? Suppose we take

$$\beta_1 = {\cdot}961{,}000, \quad \sigma = 2{\cdot}400{,}000,$$
$$\beta_2 = 4{\cdot}756{,}100, \quad N = 1000.$$

The log. normal curve has for its constants:

$$\xi_1 = 7{\cdot}589{,}466, \quad l = {\cdot}859{,}515, \quad s = {\cdot}134{,}077.$$

The Type VI, with origin at the mean:

$$y = y_0 \left(1 + \frac{x}{a_2}\right)^{q_2} \left(1 + \frac{x}{a_1}\right)^{-q_1}$$

has

$$a_1 = 22{\cdot}407{,}476, \quad a_2 = 6{\cdot}594{,}502,$$
$$q_1 = 38{\cdot}769{,}330, \quad q_2 = 10{\cdot}115{,}484, \quad \log y_0 = 2{\cdot}225{,}343.$$

Corresponding ordinates of the two curves at unit intervals of the argument are given in Table (2). The argument is measured from the start of the log. normal curve and the correspondence is about the means of the curves*.

The close agreement obtained between the two theories in this example, as in the previous one, indicates that for all practical purposes the Type VI curve may be replaced by the log. normal when β_1 and β_2 satisfy relation (69). The examples suggest too that for fairly high values of the β's, a small deviation from (69) hardly affects the form of this curve. However, it remains to be investigated generally within what range of deviation from (69) the two curves will still give equally reliable results.

The high contact the log. normal curve has at its start—a theoretical disadvantage—is brought out clearly in these illustrations by the distance from start to mean.

* [The Log. Normal Curve can only be looked upon as a possibly easier means of determining sub-range frequencies in such a case. Its form is deduced from the Weber-Fechner Law in psychology, which can have no application to meteorological phenomena. The agreement is really only established with a Type VI curve, which lies on an infinitesimal portion of the area to which this curve applies. ED.]

TABLE (2).

Corresponding Ordinates of the Log. Normal and Type VI Curves (Special Case).

Log. Normal	Type VI
·11	·06
7·38	7·14
51·12	51·53
126·19	126·54
179·15	178·80
183·54	183·17
153·22	153·10
111·87	111·95
74·63	74·76
46·81	46·90
28·14	28·17
16·42	16·42
9·40	9·38
5·30	5·29
2·97	2·96
1·66	1·65
·92	·92
·51	·51
·29	·28

E. *An Examination of the Adequacy of the Mathematical Surfaces. Graphical Analysis and Specification of Observed Data.*

1. *Data.* When I began investigating the present problem, Professor Pearson kindly placed at my disposal a number of correlation tables showing the distribution of contemporaneous barometric heights at various meteorological stations. The total number of observations, in the tables, varied from about 1800 to about 3000. I tested Narumi's surfaces on one of these distributions by fitting the theoretical regression and scedastic curves, but found the surfaces to be inadequate. Eventually, the 15-Constant Surface (Type AaAa) was resorted to. The process of fitting and of constructing the contours was arduous enough; in addition, however, the result did not repay the labour. The scantiness of the material made it impossible to judge the accuracy of the graduation. A similar insufficiency of observations is found in the examples on which the surfaces have been tested in earlier papers. As a first requisite, therefore, for obtaining results that would be of some value, distributions had to be found in which the irregularities of sampling would be less pronounced.

Table I shows the number of marriages contracted in Australia, 1907—14, arranged according to the ages of bride and bridegroom in 3-year groups. It was formed from Table LIV of Knibbs' work: *The Mathematical Theory of Population, of its Character and Fluctuations, and of the Factors which influence them*, Melbourne, 1917, pp. 190—191, where the ages are given by single years. Unspecified cases, brides over 85 and bridegrooms over 90 were rejected. In these data, as in

practically all marriage statistics, there is unquestionably a misstatement of ages by persons under 21 years of age, the chief motive of such a misstatement being to avoid legal requirements. No attempt was made to adjust the numbers.

In Table II the number of single births (male and female) in Australia, 1922—26, is tabulated according to the ages of father and mother in 3-year and 2-year groups respectively. The table was compiled from the corresponding tables in the *Australian Demography Bulletins*, Nos. 40, 41, 42, 43, and 44, the unspecified cases being again omitted.

The distribution of barometric heights on alternate days at Greenwich, 1848—1926, for the whole year, summer months (March 21—September 21) and winter months, is shown in Tables III, IV, and V respectively. These tables were drawn up from the barometric readings published in *Astronomical and Meteorological and Magnetical Observations made at the Royal Observatory, Greenwich*, for each of the 79 years. The marginal totals of Table IV, and also those of Table V, are not identical because the last reading in the summer or winter period was not correlated with the first reading in the period for the next year.

Finally, Table VI exhibits the distribution of a set of measurements made by W. Johannsen on the length and breadth of beans. The table is reproduced from Wicksell's study, *The Correlation Function of Type A and the Regression of its Characteristics*, p. 40.

The choice of one or two of the distributions might be regarded by some readers as ill advised. There were, however, no alternatives; other data, with equally large numbers, that would be better suited for illustrations, could not be found. Looked at from the skewness of the distributions, which varies from slightly abnormal to considerably abnormal, the data are representative of statistics of common occurrence.

In each of the tables only the central values of the groups are recorded.

2. *Regression, Scedastic, Clitic and Kurtic Curves.* The problem of skew correlation has repeatedly been approached from a consideration of the form of the regression curves. But we are concerned not so much with these discussions as with testing the regression, scedastic, clitic, and kurtic curves associated with the theoretical surfaces. In particular, an examination of the clisy and kurtosis of the arrays will provide us with a practical test as to the generality of Narumi's hypothesis, namely, that the array distributions reduced to a common origin and scale are similar and similarly situated curves.

The statistical measures computed from the distributions and to be used for specifying them, are given in Tables I (*a*), I (*b*), I (*c*) to VI (*c*). The higher moments of the extreme arrays, where the observations are relatively few, were not calculated. The standard deviations of the arrays, as well as those of the marginal totals, are expressed in terms of the grouping units. Sheppard's corrections have been applied to all the momental constants. Often they reduced the values of the correlation ratios below that of the corresponding correlation coefficient. Corrections for

TABLE I.

Number of Marriages arranged according to the Ages of the Contracting Parties, Australia, 1907—14.

Age of Bride. (Central Values of 3-year groups.) → $+x$

Age of Bridegroom. (Central Values of 3-year groups.) ← $+y$

	12·5	15·5	18·5	21·5	24·5	27·5	30·5	33·5	36·5	39·5	42·5	45·5	48·5	51·5	54·5	57·5	60·5	63·5	66·5	69·5	72·5	75·5	78·5	81·5	84·5	Totals
16·5	—	69	167	47	7	4	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	294
19·5	2	651	5577	3619	836	201	67	26	10	4	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	10995
22·5	2	1066	15064	27394	12677	3313	920	331	124	67	21	13	6	2	—	1	—	—	—	—	—	—	—	—	—	61001
25·5	—	615	9024	24028	24110	10684	3018	951	380	157	55	26	16	6	3	1	—	—	—	—	—	—	—	—	—	73054
28·5	—	284	4619	13619	16407	12865	5502	1930	765	321	98	52	24	8	5	—	—	1	1	—	—	—	—	—	—	56501
31·5	—	125	1941	6141	7942	7540	5515	2551	1055	407	136	73	34	12	2	3	—	1	—	—	—	—	—	—	—	33478
34·5	—	66	922	2952	4196	4130	3389	2538	1349	590	241	125	42	24	2	1	1	—	—	1	—	—	—	—	—	20569
37·5	—	41	506	1536	2347	2658	2282	1918	1494	914	308	146	88	21	12	8	—	1	—	1	—	—	—	—	—	14281
40·5	—	29	214	716	1135	1410	1469	1248	1138	991	506	256	110	50	16	8	4	—	—	—	—	—	—	—	—	9320
43·5	—	10	100	352	585	763	829	912	828	749	554	326	144	55	23	9	2	4	—	1	—	—	—	—	—	6238
46·5	—	9	73	217	337	443	548	556	691	561	511	438	203	95	47	22	8	7	4	—	—	—	—	—	—	4770
49·5	1	4	49	101	228	264	310	342	454	540	354	350	313	172	55	41	21	13	8	—	—	—	—	—	—	3620
52·5	—	3	17	56	76	111	168	170	230	286	262	240	233	174	81	45	19	7	8	3	1	—	—	—	—	2190
55·5	—	—	10	30	49	71	99	129	155	182	170	200	192	144	96	76	20	15	13	2	—	1	1	—	—	1655
58·5	—	1	2	17	28	44	45	53	81	95	94	134	137	103	85	91	43	25	14	4	2	2	—	—	—	1100
61·5	—	1	4	6	19	27	28	31	53	82	55	77	88	99	65	60	43	29	24	13	4	1	—	1	—	810
64·5	—	1	—	4	14	20	18	34	31	43	42	69	64	60	56	40	44	43	34	20	7	2	1	—	—	649
67·5	—	—	1	7	6	8	13	13	21	24	36	35	50	45	42	44	32	39	39	23	3	2	3	1	—	467
70·5	—	—	—	—	4	8	13	9	15	22	16	20	22	36	24	26	23	29	22	25	6	4	2	—	—	328
73·5	—	—	—	1	5	5	1	5	5	17	10	12	18	16	15	12	14	16	21	24	10	1	3	—	—	211
76·5	—	—	—	2	—	1	4	3	2	5	3	3	9	9	6	16	7	7	13	6	12	7	1	—	1	119
79·5	—	—	1	2	2	—	2	2	2	2	3	6	4	3	3	8	3	3	3	6	10	3	2	3	—	73
82·5	—	—	—	—	—	—	—	—	—	3	1	2	4	2	3	—	5	2	1	1	—	—	3	—	—	27
85·5	—	—	—	—	—	—	1	—	—	—	1	1	1	3	—	1	2	—	—	—	1	2	—	1	—	14
88·5	—	—	—	—	—	1	—	—	—	—	—	—	3	—	—	—	—	—	1	—	—	—	—	—	—	5
Totals	5	2975	38291	80847	71010	44541	24261	13752	8883	6062	3478	2605	1805	1139	645	513	291	242	206	130	56	25	16	6	1	301785

TABLE II.

Number of Single Births arranged according to the Ages of the Parents, Australia, 1922—26.

→ +x Age of Mother. (Central Values of 2-year groups.)

	13·0	15·0	17·0	19·0	21·0	23·0	25·0	27·0	29·0	31·0	33·0	35·0	37·0	39·0	41·0	43·0	45·0	47·0	49·0	51·0	53·0	55·0	Totals
16·5	1	15	93	52	16	2	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	181
19·5	—	58	1268	3308	2208	719	232	82	36	16	3	4	2	—	—	—	—	—	—	—	—	—	7936
22·5	—	55	1666	7999	12949	10808	4506	1569	691	275	162	50	32	15	5	7	—	—	—	—	—	—	40789
25·5	1	30	787	5224	12977	20682	20814	11711	4452	1860	769	374	168	79	27	7	2	—	—	—	—	—	79964
28·5	—	20	411	2528	7321	14618	21087	22889	17712	7519	2998	1293	587	298	100	37	7	2	1	—	—	—	99328
31·5	1	4	154	1142	3881	8061	12974	17886	20951	19603	10770	4124	1675	677	267	108	20	4	1	—	—	—	102303
34·5	—	3	89	538	1780	4189	7265	9967	13381	15719	17551	12336	5013	1984	606	207	58	2	2	—	—	—	90670
37·5	—	—	49	275	826	1886	3397	5792	7699	9546	11917	12730	11449	5713	1655	532	113	26	4	—	—	—	73609
40·5	—	1	27	127	394	886	1608	2498	3983	5518	6385	8088	8788	8600	4423	1326	236	35	6	—	1	—	52930
43·5	—	—	15	57	195	422	761	1161	1863	2510	4193	4601	5514	5775	4651	2958	734	86	11	—	—	—	35507
46·5	—	2	5	32	112	206	367	644	860	1318	1710	2637	3286	3460	3180	2516	1224	239	16	2	1	—	21817
49·5	—	2	4	21	62	114	197	311	551	615	905	1281	1679	2284	1920	1637	844	282	68	4	—	—	12781
52·5	—	1	—	7	21	60	102	157	209	348	526	615	849	1016	1115	997	483	177	32	2	—	—	6717
55·5	—	—	2	8	18	32	54	78	108	168	246	358	500	525	519	490	354	104	18	1	2	2	3587
58·5	—	—	1	2	9	18	26	42	69	76	131	156	217	345	280	233	155	59	21	1	—	—	1821
61·5	—	—	—	—	6	11	14	26	36	55	70	102	80	143	134	114	75	32	10	3	—	—	911
64·5	—	—	—	1	2	2	12	10	24	22	38	44	56	75	70	72	40	15	6	—	—	—	489
67·5	—	—	—	1	—	3	2	6	4	7	18	16	20	39	24	25	12	4	2	—	—	—	183
70·5	—	—	1	—	1	1	1	2	5	4	6	16	9	11	10	9	5	3	1	—	—	—	85
73·5	—	—	1	—	—	—	—	3	2	2	5	6	2	5	3	5	3	1	—	—	—	—	38
76·5	—	—	—	—	—	—	—	—	2	—	2	3	4	6	5	3	—	—	—	—	—	—	25
79·5	—	—	—	—	—	—	2	—	2	—	1	—	2	—	1	—	—	1	—	—	—	—	9
82·5	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	2
Totals	3	191	4573	21322	42758	62620	73423	74834	72640	65182	58407	48834	39932	31050	18975	11283	4365	1072	199	13	4	2	631682

TABLE III.

Barometric Heights at Greenwich on Alternate Days. Whole Year, 1848—1926.

First day. Height in Inches (Central Values). → $+x$

Third Day. Height in Inches (Central Values). ↓ $+y$

	30·75	30·65	30·55	30·45	30·35	30·25	30·15	30·05	29·95	29·85	29·75	29·65	29·55	29·45	29·35	29·25	29·15	29·05	28·95	28·85	28·75	28·65	28·55	28·45	28·35	Totals
30·75	2	1	2	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
30·65	—	1	3	1	3	2	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	13
30·55	2	1	6	14	17	17	3	8	2	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	73
30·45	1	4	10	51	53	48	31	23	15	13	1	5	2	1	—	—	—	—	—	—	—	—	—	—	—	258
30·35	1	5	21	48	91	113	84	70	52	31	29	7	2	6	3	—	—	—	—	—	—	—	—	—	—	563
30·25	—	—	16	53	107	177	214	191	142	82	75	39	23	17	7	2	1	1	—	—	1	—	—	—	—	1148
30·15	1	1	10	32	111	214	386	365	288	199	129	86	62	26	17	10	9	2	2	1	—	—	—	—	—	1951
30·05	—	—	1	25	75	217	363	554	543	436	284	193	108	63	40	21	11	6	8	2	—	1	—	—	—	2951
29·95	—	—	1	20	51	152	312	589	737	670	489	297	205	119	50	29	14	8	3	1	3	—	—	—	—	3750
29·85	—	—	2	5	29	93	230	473	654	707	608	457	262	170	121	56	29	15	3	5	1	1	—	—	—	3921
29·75	—	—	1	6	14	59	132	270	545	613	659	535	345	251	125	73	39	15	11	3	3	—	—	—	—	3699
29·65	—	—	—	—	5	26	100	196	336	493	468	522	384	253	172	97	62	31	21	5	3	2	—	—	—	3176
29·55	—	—	—	1	2	11	41	102	192	272	382	374	303	244	155	103	69	38	20	6	13	3	2	—	—	2333
29·45	—	—	—	1	3	9	21	53	123	168	254	275	221	193	170	97	89	33	23	10	5	3	1	—	—	1752
29·35	—	—	—	1	2	6	19	28	49	109	152	160	165	164	129	100	63	39	24	7	6	7	1	2	—	1233
29·25	—	—	—	—	—	—	8	14	29	63	83	91	120	106	106	69	51	28	20	9	7	7	1	1	—	813
29·15	—	—	—	—	—	2	1	7	25	40	45	57	56	68	54	77	38	26	20	14	5	4	2	—	—	541
29·05	—	—	—	—	—	1	1	6	4	10	24	37	39	33	35	36	17	9	15	6	4	5	—	—	—	282
28·95	—	—	—	—	—	—	1	—	11	8	13	21	15	23	15	21	18	12	12	5	5	8	—	1	—	189
28·85	—	—	—	—	—	—	—	1	1	2	—	8	9	8	14	11	13	5	2	2	4	—	2	—	—	82
28·75	—	—	—	—	—	—	—	—	—	3	3	7	7	4	11	5	9	6	—	2	—	2	1	—	—	60
28·65	—	—	—	—	—	—	—	1	—	—	1	4	3	2	7	4	8	5	2	3	—	—	2	—	1	43
28·55	—	—	—	—	—	—	—	—	1	1	—	—	1	—	2	1	1	2	3	—	—	—	—	—	—	12
28·45	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	1	1	—	—	—	—	—	—	—	4
28·35	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	1
Totals	7	13	73	258	563	1148	1951	2951	3749	3921	3700	3176	2333	1752	1233	813	542	282	189	81	60	43	12	4	1	28855

TABLE IV.

Barometric Heights at Greenwich on Alternate Days. Summer Months, 1848—1926.

First Day. Height in Inches (Central Values). → +x

Third Day. Height in Inches (Central Values). ↓ +y

	30·45	30·35	30·25	30·15	30·05	29·95	29·85	29·75	29·65	29·55	29·45	29·35	29·25	29·15	29·05	28·95	28·85	28·75	28·65	Totals
30·45	2	2	3	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	10
30·35	—	11	20	18	13	7	5	5	—	—	—	—	—	—	—	—	—	—	—	79
30·25	3	19	54	68	67	48	28	26	11	8	2	2	—	—	—	—	—	—	—	336
30·15	1	20	79	176	194	148	103	57	33	20	6	4	1	—	—	—	—	—	—	842
30·05	2	13	79	185	340	344	256	175	99	39	26	10	7	1	—	3	1	—	—	1580
29·95	2	10	60	154	370	492	468	323	170	117	62	17	5	2	3	—	1	—	—	2256
29·85	—	3	23	128	279	456	479	416	305	163	88	55	22	2	4	—	—	—	—	2423
29·75	—	3	15	52	139	380	420	460	355	217	137	59	25	9	2	—	—	—	—	2273
29·65	—	—	4	31	109	190	325	314	319	223	146	76	40	22	5	2	1	—	—	1807
29·55	—	—	1	8	34	110	168	235	225	191	137	70	36	20	7	7	—	2	—	1251
29·45	—	—	—	6	15	53	91	142	157	123	95	74	34	25	5	4	2	1	—	827
29·35	—	—	—	2	11	18	44	77	72	65	72	54	35	16	8	4	—	—	—	478
29·25	—	—	—	—	3	6	23	25	34	43	35	39	22	12	5	2	1	—	1	251
29·15	—	—	—	—	1	4	12	14	15	15	10	12	18	7	4	3	1	2	—	118
29·05	—	—	—	—	—	—	2	6	10	9	4	7	4	2	1	—	—	—	1	46
28·95	—	—	—	—	—	—	2	4	4	1	1	3	3	3	3	—	1	—	—	25
28·85	—	—	—	—	—	—	1	—	2	1	1	1	—	—	—	1	—	—	—	7
28·75	—	—	—	—	—	—	—	—	1	1	—	1	—	1	—	—	1	—	—	5
28·65	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1
Totals	10	81	338	830	1576	2256	2427	2279	1812	1236	822	484	252	122	47	27	9	5	2	14615

TABLE V.

Barometric Heights at Greenwich on Alternate Days. Winter Months, 1848—1926.

First Day. Height in Inches (Central Values). → + x

Third Day. Height in Inches (Central Values). → + y

	30·75	30·65	30·55	30·45	30·35	30·25	30·15	30·05	29·95	29·85	29·75	29·65	29·55	29·45	29·35	29·25	29·15	29·05	28·95	28·85	28·75	28·65	28·55	28·45	28·35	Totals
30·75	2	1	2	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
30·65	—	1	3	1	3	2	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	13
30·55	2	1	6	14	17	17	3	8	2	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	73
30·45	1	4	10	49	51	45	29	22	15	13	1	5	2	1	—	—	—	—	—	—	—	—	—	—	—	248
30·35	1	5	21	48	80	93	66	57	45	26	24	7	2	6	3	—	—	—	—	—	—	—	—	—	—	484
30·25	—	—	16	50	86	123	146	124	94	54	49	28	15	15	5	2	1	1	—	—	1	—	—	—	—	812
30·15	1	1	10	31	91	135	210	171	140	96	72	53	42	20	13	9	9	2	2	1	—	—	—	—	—	1109
30·05	—	—	1	23	62	138	178	214	199	180	109	94	69	37	30	14	10	6	5	1	—	1	—	—	—	1371
29·95	—	—	1	18	41	92	158	219	245	202	166	127	88	57	33	24	12	5	3	—	3	—	—	—	—	1494
29·85	—	—	2	5	26	70	102	194	198	228	192	152	99	82	66	34	27	11	3	5	1	1	—	—	—	1498
29·75	—	—	1	6	11	44	80	131	165	193	199	180	128	114	66	48	30	13	11	3	3	—	—	—	—	1426
29·65	—	—	—	—	5	22	69	87	146	168	154	203	161	107	96	57	40	26	19	4	3	2	—	—	—	1369
29·55	—	—	—	1	2	10	33	68	82	104	147	149	112	107	85	67	49	31	13	6	11	3	2	—	—	1082
29·45	—	—	—	1	3	9	15	38	70	77	112	118	98	98	96	63	64	28	19	8	4	3	1	—	—	925
29·35	—	—	—	1	2	6	17	17	31	65	75	88	100	92	75	65	47	31	20	7	6	7	1	2	—	755
29·25	—	—	—	—	—	—	8	11	23	40	58	37	77	71	67	47	39	23	18	8	7	6	1	1	—	562
29·15	—	—	—	—	—	2	1	6	21	28	31	42	41	58	42	59	31	22	17	13	3	4	2	—	—	423
29·05	—	—	—	—	—	1	1	6	4	8	18	27	30	29	28	32	15	8	15	6	4	4	—	—	—	236
28·95	—	—	—	—	—	—	1	—	11	6	9	17	14	22	12	16	15	9	12	4	5	8	—	1	—	164
28·85	—	—	—	—	—	—	—	1	1	1	—	6	8	7	13	11	13	5	1	2	4	—	2	—	—	75
28·75	—	—	—	—	—	—	—	—	—	3	3	6	6	4	10	5	8	8	—	1	—	2	1	—	—	55
28·65	—	—	—	—	—	—	—	1	—	—	1	4	3	2	7	4	8	5	1	3	—	—	2	—	1	42
28·55	—	—	—	—	—	—	—	—	1	1	—	—	1	—	2	1	1	2	3	—	—	—	—	—	—	12
28·45	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	1	1	—	—	—	—	—	—	—	4
28·35	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	1
Totals	7	13	73	248	482	810	1121	1375	1493	1494	1421	1364	1097	930	749	561	420	235	162	72	55	41	12	4	1	14240

TABLE VI.

Correlation of Length and Breadth of Beans.

Length in mm. (Central Values). → $+x$

Breadth in mm. (Central Values). → $+y$

	17·0	*16·5*	*16·0*	*15·5*	*15·0*	*14·5*	*14·0*	*13·5*	*13·0*	*12·5*	*12·0*	*11·5*	*11·0*	*10·5*	*10·0*	*9·5*	Totals
9·125	—	2	—	—	3	—	—	—	—	—	—	—	—	—	—	—	5
8·875	4	8	17	19	—	—	—	—	—	—	—	—	—	—	—	—	48
8·625	2	23	101	156	93	23	2	—	—	—	—	—	—	—	—	—	400
8·375	—	18	105	494	574	227	56	9	—	—	—	—	—	—	—	—	1483
8·125	—	4	44	375	956	913	362	73	12	3	—	—	—	—	—	—	2742
7·875	—	—	7	81	385	871	794	330	89	19	3	—	—	—	—	—	2579
7·625	—	—	1	4	65	236	469	361	175	55	27	4	—	—	—	—	1397
7·375	—	—	—	—	6	23	91	137	124	78	37	22	11	—	1	—	530
7·125	—	—	—	—	—	1	13	18	28	35	25	32	11	6	1	—	170
6·875	—	—	—	—	—	—	—	1	9	8	21	12	13	7	1	—	72
6·625	—	—	—	—	—	—	—	—	—	—	2	—	1	4	3	—	10
6·375	—	—	—	—	—	—	—	—	—	·1	—	—	—	1	1	1	4
Totals	6	55	275	1129	2082	2294	1787	929	437	199	115	70	36	18	7	1	9440

TABLE I (*a*).

Constants of the Distribution of Ages at Marriage of Bride and Bridegroo

Age of Bride	Age of Bridegroom	—	—
—	—	—	$q_{21} = 1{\cdot}576{,}067$
$\bar{x} = 25{\cdot}721{,}621$ yrs.	$\bar{y} = 29{\cdot}383{,}065$ yrs.	—	$q_{12} = 1{\cdot}477{,}487$
$\sigma_1 = 2{\cdot}239{,}567$*	$\sigma_2 = 2{\cdot}640{,}766$*	$r = {\cdot}708{,}164$	$q_{31} = 7{\cdot}069{,}828$
$\sqrt{\beta_{10}} = +2{\cdot}013{,}900$	$\sqrt{\beta_{01}} = +1{\cdot}963{,}079$	$\eta_{yx} = {\cdot}710{,}278$	$q_{22} = 6{\cdot}345{,}730$
$\beta_{20} = 9{\cdot}290{,}441$	$\beta_{02} = 8{\cdot}332{,}812$	$\eta_{xy} = {\cdot}707{,}139$	$q_{13} = 6{\cdot}297{,}500$

* Unit = three years.

TABLE I (*b*).

Constants of the Distribution of Age of Bride for a given Age of Bridegroo (Central Values).

Age of Bridegroom in years	Number of Observations	Mean Age of Bride in years	$\sigma(x)$ in three year units	$\sqrt{\beta_1}\,(\bar{x})$	$\beta_2\,(x)$
16·5	294	18·540,818	—	—	—
19·5	10,995	20·006,711	·849,742	1·769,638	9·808,637
22·5	61,001	21·871,010	1·000,300	1·455,142	8·579,920
25·5	73,054	23·624,156	1·168,982	1·061,578	6·392,154
28·5	56,501	25·132,271	1·389,366	·887,181	5·152,802
31·5	33,478	26·637,494	1·614,018	·697,360	4·091,808
34·5	20,560	28·117,823	1·878,609	·624,843	3·550,400
37·5	14,281	29·720,222	2·107,620	·512,228	3·237,101
40·5	9,320	31·859,012	2·351,849	·340,680	2·787,481
43·5	6,236	33·782,303	2·502,022	·216,977	2·719,178
46·5	4,770	35·790,566	2·761,511	·138,784	2·717,721
49·5	3,620	38·067,125	2·904,725	·072,250	2·691,692
52·5	2,190	40·509,500	3·089,507	− ·059,573	2·738,178
55·5	1,655	42·383,987	3·248,947	− ·038,079	2·715,627
58·5	1,100	45·246,362	3·404,342	− ·186,099	2·627,812
61·5	810	47·014,814	3·721,307	− ·142,120	2·673,850
64·5	649	49·110,170	3·964,999	− ·152,912	2·410,205
67·5	487	51·450,719	4·016,722	− ·326,509	2·647,233
70·5	326	52·374,233	4·172,709	− ·248,391	2·220,272
73·5	211	54·372,038	4·368,691	− ·402,175	2·352,104
76·5	119	57·096,638	4·441,906	—	—
79·5	73	56·226,026	—	—	—
82·5	27	56·055,557	—	—	—
85·5	14	54·973,682 }	—	—	—
88·5	5		—	—	—

TABLE I (c).

Constants of the Distribution of Age of Bridegroom for a given Age of Bride (Central Values).

Age of Bride in years	Number of Observations	Mean Age of Bridegroom in years	$\sigma(y)$ in three year units	$\sqrt{\beta_1(y)}$	$\beta_2(y)$
12·5	5	24·177,180	—	—	—
15·5	2,975		1·635,113	2·080,477	10·440,077
18·5	38,291	24·677,352	1·454,074	1·888,116	9·515,703
21·5	80,847	26·073,156	1·506,919	1·888,895	9·604,855
24·5	71,010	27·841,233	1·655,005	1·829,298	8·990,077
27·5	44,541	30·022,665	1·877,199	1·633,245	7·751,172
30·5	24,261	32·621,553	2·174,789	1·337,310	6·214,616
33·5	13,752	35·311,518	2·437,164	1·018,662	4·932,579
36·5	8,883	38·367,276	2·705,296	·710,997	3·932,594
39·5	6,062	41·474,925	2·942,219	·585,660	3·921,733
42·5	3,478	44·399,655	2·944,590	·405,820	3·821,951
45·5	2,605	47·263,533	3·114,894	·228,121	3·537,179
48·5	1,805	50·459,004	3·273,171	·138,528	3·698,322
51·5	1,139	53·537,751	3·216,616	− ·032,434	3·528,247
54·5	645	56·397,675	3·112,739	− ·221,998	3·515,073
57·5	513	58·576,023	3·056,301	− ·205,697	3·729,017
60·5	291	61·747,422	2·878,866	− ·016,835	3·213,237
63·5	242	63·024,792	2·856,280	− ·789,109	4·195,793
66·5	206	64·907,766	2·690,352	− ·620,610	4·391,585
69·5	130	67·730,769	2·456,385	− 1·491,384	7·436,024
72·5	56	71·892,858	2·280,592	—	—
75·5	25	71·940,000	—	—	—
78·5	16		—	—	—
81·5	6	73·760,868	—	—	—
84·5	1		—	—	—

TABLE II (a).

Constants of the Distribution of Ages of Parents at Birth of Child.

Age of Mother	Age of Father	—	—
—	—	—	$q_{21}=$ ·275,142
$\bar{x}=$ 29·529,067 yrs.	$\bar{y}=$ 33·500,298 yrs.	—	$q_{12}=$ ·302,215
$\sigma_1=$ 3·083,148*	$\sigma_2=$ 2·495,111†	$r=$ ·734,944	$q_{31}=$ 1·823,465
$\sqrt{\beta_{10}}=+$ ·317,180	$\sqrt{\beta_{01}}=+$ ·724,331	$\eta_{yx}=$ ·732,763	$q_{22}=$ 1·837,166
$\beta_{20}=$ 2·430,327	$\beta_{02}=$ 3·624,169	$\eta_{xy}=$ ·747,517	$q_{13}=$ 2·065,256

* Unit = two years. † Unit = three years.

TABLE II (*b*).

Constants of the Distribution of Age of Mother for a given Age of Father (Central Values).

Age of Father in years	Number of Observations	Mean Age of Mother in years	$\sigma(x)$ in two year units	$\sqrt{\beta_1(x)}$	$\beta_2(x)$
16·5	181	17·895,028	·859,357	—	—
19·5	7,936	19·915,574	1·105,597	1·420,909	7·540,025
22·5	40,789	21·932,114	1·371,058	1·167,144	6·493,849
25·5	79,964	24·184,358	1·554,972	·692,084	4·622,581
28·5	99,328	26·362,838	1·758,487	·357,841	3·802,815
31·5	102,303	28·529,886	1·939,264	·041,134	3·182,722
34·5	90,670	30·677,270	2·134,633	− ·226,993	2·863,438
37·5	73,609	32·737,206	2·296,582	− ·399,857	2·831,788
40·5	52,930	34·594,068	2·424,931	− ·552,244	2·922,367
43·5	35,507	36·091,222	2·514,318	− ·601,512	3·074,234
46·5	21,817	37·183,894	2·605,658	− ·666,868	3·222,510
49·5	12,781	37·756,592	2·668,569	− ·710,871	3·367,968
52·5	6,717	38·087,986	2·647,144	− ·708,986	3·260,171
55·5	3,587	38·203,234	2·728,736	− ·719,828	3·480,831
58·5	1,821	38·205,930	2·733,062	− ·703,866	3·393,736
61·5	911	37·886,938	2·866,767	− ·532,490	2·904,552
64·5	489	38·059,304	2·796,094	− ·623,135	3·011,293
67·5	183	37·885,246	2·703,871	—	—
70·5	85	36·952,942	—	—	—
73·5	38	36·368,422	—	—	—
76·5	25	36·444,444	—	—	—
79·5	9		—	—	—
82·5	2		—	—	—

TABLE II (*c*).

Constants of the Distribution of Age of Father for a given Age of Mother (Central Values).

Age of Mother in years	Number of Observations	Mean Age of Father in years	$\sigma(y)$ in three year units	$\sqrt{\beta_1(y)}$	$\beta_2(y)$
13·0	3	23·396,907	—	—	—
15·0	191		—	—	—
17·0	4,573	23·568,009	1·510,032	2·295,506	13·785,722
19·0	21,322	24·700,263	1·450,460	1·786,585	8·683,177
21·0	42,758	26·289,678	1·526,565	1·646,047	7·835,439
23·0	62,620	27·851,709	1·569,772	1·519,692	7·189,398
25·0	73,423	29·436,369	1·591,399	1·474,353	7·112,446
27·0	74,834	31·100,715	1·637,989	1·371,596	6·605,027
29·0	72,640	32·769,054	1·685,861	1·328,398	6·592,213
31·0	65,182	34·491,348	1·710,695	1·205,339	6·013,309
33·0	58,407	36·360,651	1·773,938	1·165,181	6·101,650
35·0	48,834	38·241,246	1·823,740	1·042,839	5·597,504
37·0	39,932	40·184,538	1·848,826	·854,808	5·055,700
39·0	31,050	42·152,754	1·920,485	·788,422	4·917,428
41·0	18,975	44·147,904	1·912,000	·610,390	4·662,052
43·0	11,283	46·089,471	1·923,478	·451,864	4·638,673
45·0	4,365	48·097,938	1·901,278	·350,883	4·230,476
47·0	1,072	49·947,762	1·921,691	·365,142	4·707,391
49·0	199	51·509,175	—	—	—
51·0	13		—	—	—
53·0	4		—	—	—
55·0	2		—	—	—

TABLE III (a).

Constants of the Distribution of Barometric Heights (Whole Year) on Alternate Days.

First Day	Third Day	—	—
—	—	—	$q_{21}=\cdot 152{,}059$
$\bar{x}=29\cdot 780{,}934''$	$\bar{y}=29\cdot 780{,}931''$	—	$q_{12}=\cdot 169{,}381$
$\sigma_1=3\cdot 079{,}720$*	$\sigma_2=3\cdot 080{,}000$*	$r=\cdot 580{,}721$	$q_{31}=1\cdot 881{,}406$
$\sqrt{\beta_{10}}=+\cdot 450{,}793$	$\sqrt{\beta_{01}}=+\cdot 451{,}289$	$\eta_{yx}=\cdot 583{,}316$	$q_{22}=1\cdot 817{,}042$
$\beta_{20}=3\cdot 397{,}763$	$\beta_{02}=3\cdot 398{,}787$	$\eta_{xy}=\cdot 581{,}703$	$q_{13}=1\cdot 865{,}340$

* In $\frac{1}{10}$ inches.

TABLE III (b).

Constants of the Distribution of Barometric Heights (Whole Year) on First Day for a given Reading on Third Day (Central Values).

Reading on Third Day in inches	Number of Observations	Mean Height on First Day in inches	$\sigma(x)$ in $\frac{1}{10}$ inches	$\sqrt{\beta_1(x)}$	$\beta_2(x)$
30·75	7	30·420,000 (30·75–30·65)	—	—	—
30·65	13		—	—	—
30·55	73	30·300,685	2·020,996	—	—
30·45	258	30·240,310	2·196,790	·868,163	3·770,511
30·35	563	30·160,302	2·349,221	·648,474	3·495,585
30·25	1148	30·065,679	2·431,305	·760,496	4·057,606
30·15	1951	30·004,946	2·419,847	·844,391	4·307,638
30·05	2951	29·929,261	2·395,152	·819,870	4·445,142
29·95	3750	29·876,960	2·275,430	·588,481	4·032,045
29·85	3921	29·811,719	2·376,758	·556,345	3·686,299
29·75	3699	29·753,866	2·368,077	·414,211	3·579,364
29·65	3176	29·694,490	2·530,774	·458,017	3·329,710
29·55	2333	29·633,069	2·661,476	·533,459	3·594,689
29·45	1752	29·586,301	2·741,230	·318,388	3·112,579
29·35	1233	29·534,023	2·897,096	·278,185	3·357,779
29·25	813	29·487,023	2·884,793	·363,886	2·782,546
29·15	541	29·442,606	3·076,942	·169,269	2·676,531
29·05	282	29·422,695	2·995,550	·275,610	3·027,310
28·95	189	29·361,111	3·483,078	—	—
28·85	82	29·304,878	2·899,690	—	—
28·75	60	29·338,333	2·941,605	—	—
28·65	43	29·226,744	—	—	—
28·55	12	29·285,294 (28·55–28·35)	—	—	—
28·45	4		—	—	—
28·35	1		—	—	—

TABLE III (*c*).

Constants of the Distribution of Barometric Heights (Whole Year) on Third Day for a given Reading on First Day (Central Values).

Reading on First Day in inches	Number of Observations	Mean Height on Third Day in inches	$\sigma(y)$ in $\frac{1}{10}$ inches	$\sqrt{\beta_1(y)}$	$\beta_2(y)$
30·75	7	30·460,000	—	—	—
30·65	13		—	—	—
30·55	73	30·322,603	1·844,786	—	—
30·45	258	30·242,248	2·071,353	·988,036	4·441,460
30·35	563	30·174,334	2·086,386	·659,948	3·860,262
30·25	1148	30·085,279	2·216,594	·596,379	4·064,244
30·15	1951	30·004,075	2·229,517	·645,631	3·865,614
30·05	2951	29·937,326	2·225,411	·616,284	4·236,583
29·95	3749	29·865,631	2·298,247	·606,296	4·192,548
29·85	3921	29·804,068	2·387,431	·549,445	3·883,501
29·75	3700	29·747,784	2·479,312	·371,584	3·389,310
29·65	3176	29·690,460	2·588,024	·457,062	3·733,761
29·55	2333	29·644,299	2·715,802	·390,583	3·426,421
29·45	1752	29·598,858	2·769,057	·223,362	3·224,785
29·35	1233	29·538,970	2·941,199	·282,386	3·266,638
29·25	813	29·477,921	2·988,265	·121,441	2·842,953
29·15	542	29·444,834	3·118,031	·275,851	3·201,995
29·05	282	29·412,411	3·227,022	·284,802	3·142,534
28·95	189	29·389,153	3·165,014	—	—
28·85	81	29·330,247	3·348,281	—	—
28·75	60	29·375,000	3·109,796	—	—
28·65	43	29·240,698	—	—	—
28·55	12		—	—	—
28;45	4	28·852,941	—	—	—
28·35	1		—	—	—

TABLE IV (*a*).

Constants of the Distribution of Barometric Heights (Summer Months) on Alternate Days.

First Day	Third Day	—	—
—	—	—	$q_{21}=$ ·151,949
$\bar{x}=29·790,075''$	$\bar{y}=29·790,753''$	—	$q_{12}=$ ·169,116
$\sigma_1=$ 2·415,849″	$\sigma_2=$ 2·407,294″	$r=$ ·534,545	$q_{31}=1·687,358$
$\sqrt{\beta_{10}}=+$·448,990	$\sqrt{\beta_{01}}=+$·432,215	$\eta_{yx}=$·534,897	$q_{22}=1·649,758$
$\beta_{20}=$ 3·274,413	$\beta_{02}=$ 3·214,970	$\eta_{xy}=$·533,048	$q_{13}=1·630,514$

TABLE IV (*b*).

Constants of the Distribution of Barometric Heights (Summer Months) on First Day for a given Reading on Third Day (Central Values).

Reading on Third Day in inches	Number of Observations	Mean Height on First Day in inches	$\sigma(x)$ in inches	$\sqrt{\beta_1(x)}$	$\beta_2(x)$
30·45	10	30·141,011	—	—	—
30·35	79		—	—	—
30·25	336	30·036,012	2·047,375	·689,178	3·228,771
30·15	842	29·998,812	1·879,447	·734,010	3·560,910
30·05	1580	29·928,608	1·973,564	·855,845	4·854,866
29·95	2256	29·875,310	1·923,916	·536,856	3·782,482
29·85	2423	29·816,364	2·003,191	·469·880	3·182,739
29·75	2273	29·761,835	1·959,977	·317,173	3·096,858
29·65	1807	29·709,989	2·131,423	·383,309	3·087,158
29·55	1251	29·654,396	2·148,063	·498,913	3·518,774
29·45	827	29·613,000	2·230,345	·402,282	3·191,061
29·35	478	29·567,573	2·313,959	·145,890	2·615,369
29·25	251	29·514,940	2·334,763	·235,434	3·101,169
29·15	118	29·484,746	—	—	—
29·05	46	29·497,826	—	—	—
28·95	25	29,422,000	—	—	—
28·85	7		—	—	—
28·75	5	29·380,769	—	—	—
28·65	1		—	—	—

TABLE IV (*c*).

Constants of the Distribution of Barometric Heights (Summer Months) on Third Day for a given Reading on First Day (Central Values).

Reading on First Day in inches	Number of Observations	Mean Height on Third Day in inches	$\sigma(y)$ in inches	$\sqrt{\beta_1(y)}$	$\beta_2(y)$
30·45	10	30·145,604	—	—	—
30·35	81		—	—	—
30·25	338	30·075,740	1·584,353	·349,398	2·971,343
30·15	830	30·001,084	1·741,844	·528,892	3·349,466
30·05	1576	29·937,056	1·805,284	·608,407	3·690,422
29·95	2256	29·867,509	1·850,561	·430,994	3·331,418
29·85	2427	29·808,879	2·027,150	·544,597	3·651,131
29·75	2279	29·757,635	2·122,282	·362,806	3·326,920
29·65	1812	29·706,457	2·148,778	·405,566	3·597,764
29·55	1236	29·669,498	2·209,315	·246,041	3·256,883
29·45	822	29·634,550	2·158,241	·106,433	2·854,964
29·35	484	29·570,661	2·378,887	·078,428	2·988,715
29·25	252	29·525,000	2·407,206	− ·043,209	2·529,909
29·15	122	29·469,672	2·171,182	—	—
29·05	47	29·460,638	—	—	—
28·95	27	29·442,593	—	—	—
28·85	9		—	—	—
28·75	5	29·362,500	—	—	—
28·65	2		—	—	—

TABLE V (*a*).

Constants of the Distribution of Barometric Heights (Winter Months) on alternate Days.

First Day	Third Day	—	—
—	—	—	$q_{21}=\cdot 103{,}602$
$\bar{x}=29\cdot 771{,}552''$	$\bar{y}=29\cdot 770{,}850''$	—	$q_{12}=\cdot 119{,}394$
$\sigma_1=3\cdot 634{,}793$*	$\sigma_2=3\cdot 640{,}723$*	$r=\cdot 600{,}961$	$q_{31}=1\cdot 609{,}191$
$\sqrt{\beta_{10}}=+\cdot 377{,}097$	$\sqrt{\beta_{01}}=+\cdot 378{,}687$	$\eta_{y_x}=\cdot 606{,}313$	$q_{22}=1\cdot 553{,}029$
$\beta_{20}=2\cdot 861{,}522$	$\beta_{02}=2\cdot 862{,}281$	$\eta_{x_y}=\cdot 604{,}689$	$q_{13}=1\cdot 599{,}874$

* In $\frac{1}{10}$ inches.

TABLE V (*b*).

Constants of the Distribution of Barometric Heights (Winter Months) on First for a given Reading on Third Day (Central Values).

Reading on Third Day in inches	Number of Observations	Mean Height on First Day in inches	$\sigma(x)$ in $\frac{1}{10}$ inches	$\sqrt{\beta_1(x)}$	$\beta_2(x)$
30·75	7	30·420,000	—	—	—
30·65	13		—	—	—
30·55	73	30·300,685	—	—	—
30·45	248	30·239,113	2·226,484	·855,755	3·690,966
30·35	484	30·166,116	2·438,117	·677,793	3·444,574
30·25	812	30·077,956	2·563,449	·823,536	4·153,907
30·15	1109	30·009,603	2·759,459	·852,819	3·938,848
30·05	1371	29·930,015	2·803,430	·760,497	3·728,625
29·95	1494	29·879,451	2·721,317	·590,782	3·534,137
29·85	1498	29·804,206	2·878,610	·523,148	3·233,538
29·75	1426	29·741,164	2·897,747	·358,568	3·047,418
29·65	1369	29·674,032	2·964,589	·372,890	2·910,239
29·55	1082	29·608,410	3·134,490	·393,193	3·005,976
29·45	925	29·562,432	3·108,770	·168,964	2·746,087
29·35	755	29·512,781	3·193,847	·209,155	3·166,914
29·25	562	29·474,555	3·090,935	·330,929	3·000,836
29·15	423	29·430,851	3·139,860	·115,683	2·700,373
29·05	236	29·408,051	3·093,051	·145,121	2·890,636
28·95	164	29·351,829	3·557,265	—	—
28·85	75	29·287,333	—	—	—
28·75	55	29·340,909	—	—	—
28·65	42	29·233,333	—	—	—
28·55	12		—	—	—
28·45	4	29·285,294	—	—	—
28·35	1		—	—	—

TABLE V (c).

Constants of the Distribution of Barometric Heights (Winter Months) on Third Day for a given Reading on First Day (Central Values).

Reading on First Day in inches	Number of Observations	Mean Height on Third Day in inches	$\sigma(y)$ in $\frac{1}{10}$ inches	$\sqrt{\beta_1(y)}$	$\beta_2(y)$
30·75	7	30·460,000 (30·75–30·65)	—	—	—
30·65	13		—	—	—
30·55	73	30·322,603	—	—	—
30·45	248	30·244,758	2·080,683	1·029,967	4·545,211
30·35	482	30·179,876	2·155,189	·703,291	3·864,196
30·25	810	30·089,259	2·431,190	·628,479	3·788,161
30·15	1121	30·006,289	2·530,527	·654,585	3·513,264
30·05	1375	29·937,036	2·625,525	·582,473	3·749,428
29·95	1493	29·862,793	2·843,795	·606,873	3·567,920
29·85	1494	29·796,252	2·876,403	·473,828	3·337,469
29·75	1421	29·731,984	2·956,578	·277,531	2·894,125
29·65	1364	29·669,208	3·063,120	·350,034	3·174,959
29·55	1097	29·615,907	3·167,617	·291,203	2·926,329
29·45	930	29·567,312	3·180,523	·088,028	2·843,682
29·35	749	29·518,491	3·236,809	·241,729	3·005,220
29·25	561	29·456,774	3·192,760	·068,007	2·711,597
29·15	420	29·437,619	3·339,718	·222,960	2·937,186
29·05	235	29·402,766	3·319,181	·287,181	3·120,079
28·95	162	29·380,247	3·166,841	—	—
28·85	72	29·320,833	—	—	—
28·75	55	29·375,454	—	—	—
28·65	41	29·245,122	—	—	—
28·55	12	29·102,941 (28·55–28·35)	—	—	—
28·45	4		—	—	—
28·35	1		—	—	—

TABLE VI (a).

Constants of the Distribution of Length and Breadth of Beans.

Length of Beans	Breadth of Beans	—	—
—	—	—	$q_{21} = -·653,267$
$\bar{x} = 14·404,608$ mm.	$\bar{y} = 7·975,530$ mm.	—	$q_{12} = -·502,495$
$\sigma_1 = 1·799,562$*	$\sigma_2 = 1·359,481$†	$r = ·781,142$	$q_{31} = 3·641,829$
$\sqrt{\beta_{10}} = -·910,569$	$\sqrt{\beta_{01}} = -·440,832$	$\eta_{yx} = ·771,995$	$q_{22} = 3·167,011$
$\beta_{20} = 4·862,944$	$\beta_{02} = 3·654,374$	$\eta_{xy} = ·772,270$	$q_{13} = 3·072,626$

* In ½ mm. units. † In ¼ mm. units.

TABLE VI (*b*).

Constants of the Distribution of Length of Beans for a given Breadth (Central Values).

Breadth of Beans in mm.	Number of Observations	Mean Length of Beans in mm.	$\sigma(x)$ in $\frac{1}{2}$ mm.	$\sqrt{\beta_1(x)}$	$\beta_2(x)$
9·125	5	16·047,170	—	—	—
8·875	48		·899,218	—	—
8·625	400	15·510,000	·975,329	− ·045,354	2·925,986
8·375	1483	15·132,165	·964,928	− ·238,931	3·479,772
8·125	2742	14·736,689	1·025,400	− ·261,031	3·456,580
7·875	2579	14·258,822	1·122,765	− ·246,864	3·390,668
7·625	1397	13·777,380	1·267,017	− ·447,534	3·506,273
7·375	530	13·135,849	1·599,946	− ·539,225	3·316,993
7·125	170	12·370,588	1·663,320	− ·052,792	2·593,873
6·875	72	11·763,889	1·554,314	—	—
6·625	10	11·357,143	—	—	—
6·375	4		—	—	—

TABLE VI (*c*).

Constants of the Distribution of Breadth of Beans for a given Length (Central Values).

Length of Beans in mm.	Number of Observations	Mean Breadth of Beans in mm.	$\sigma(y)$ in $\frac{1}{4}$ mm.	$\sqrt{\beta_1(y)}$	$\beta_2(y)$
17·0	6	8·584,017	—	—	—
16·5	55		·872,333	—	—
16·0	275	8·442,273	·867,956	− ·451,028	3·132,322
15·5	1129	8·296,391	·821,161	+ ·030,368	3·049,913
15·0	2082	8·153,698	·846,552	− ·105,878	3·645,700
14·5	2294	8·000,436	·834,188	− ·058,086	3·219,250
14·0	1787	7·845,621	·878,903	− ·116,996	3·274,500
13·5	929	7·712,998	·885,527	− ·065,917	3·157,425
13·0	437	7·571,224	·953,572	− ·321,257	3·187,621
12·5	199	7·434,045	1·035,346	− ·183,445	3·815,952
12·0	115	7·288,044	1·113,176	—	—
11·5	70	7·189,286	—	—	—
11·0	36	7·097,222	—	—	—
10·5	18		—	—	—
10·0	7	6·836,539	—	—	—
9·5	1		—	—	—

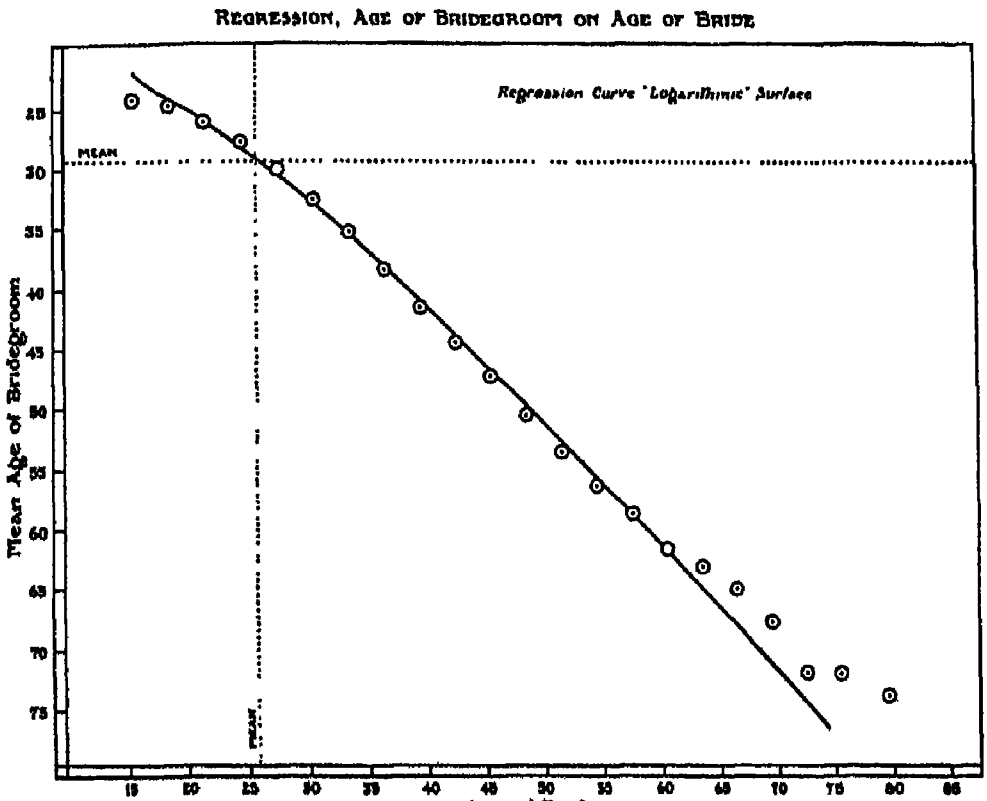


Diagram I (*a*).

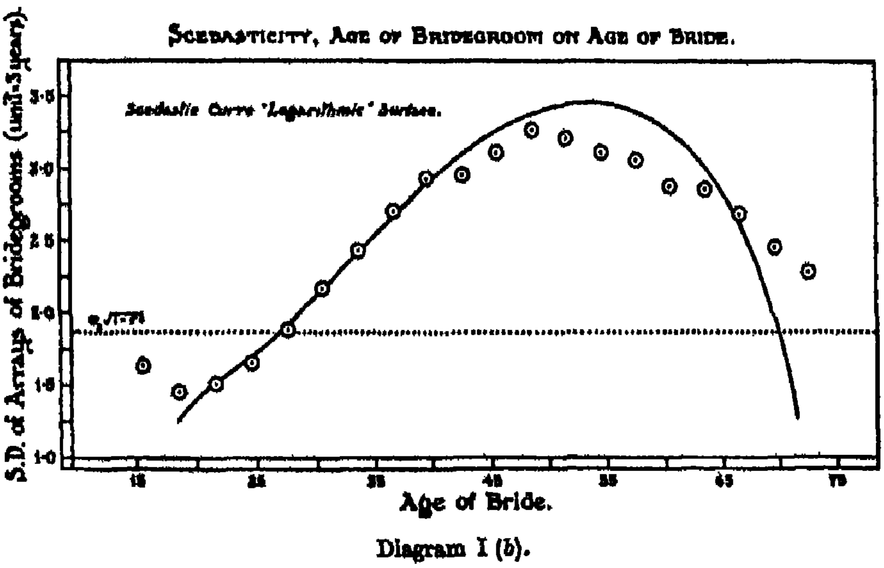


Diagram I (*b*).

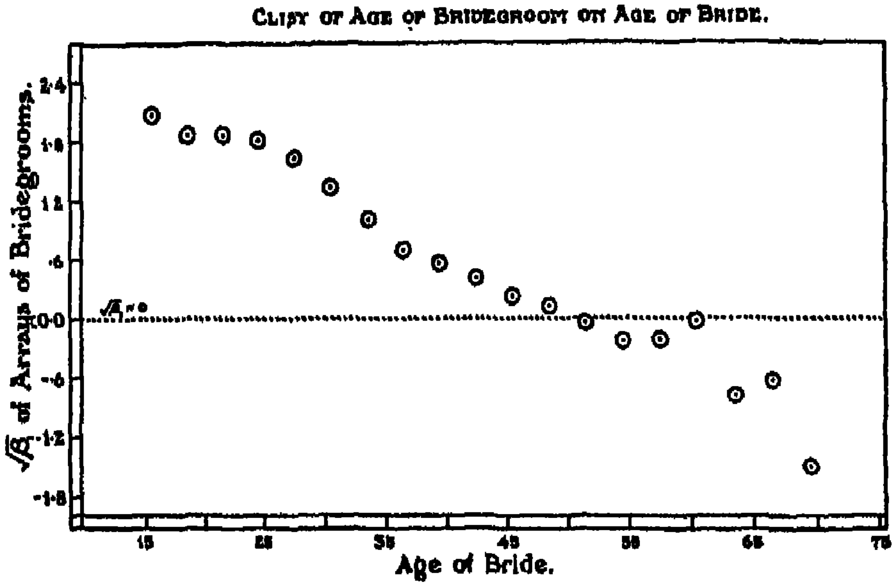


Diagram I (*c*).

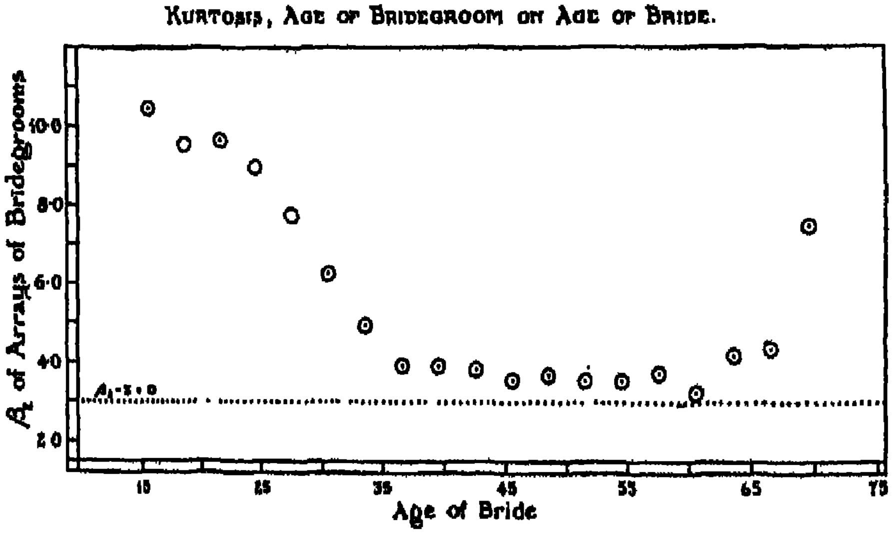


Diagram I (*d*).

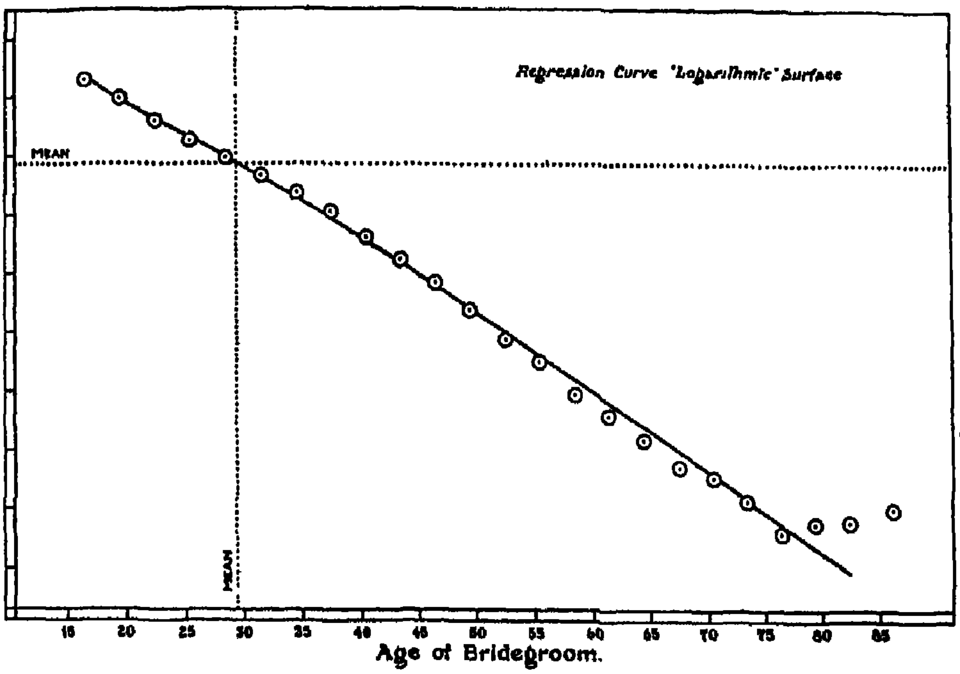


Diagram I (e).

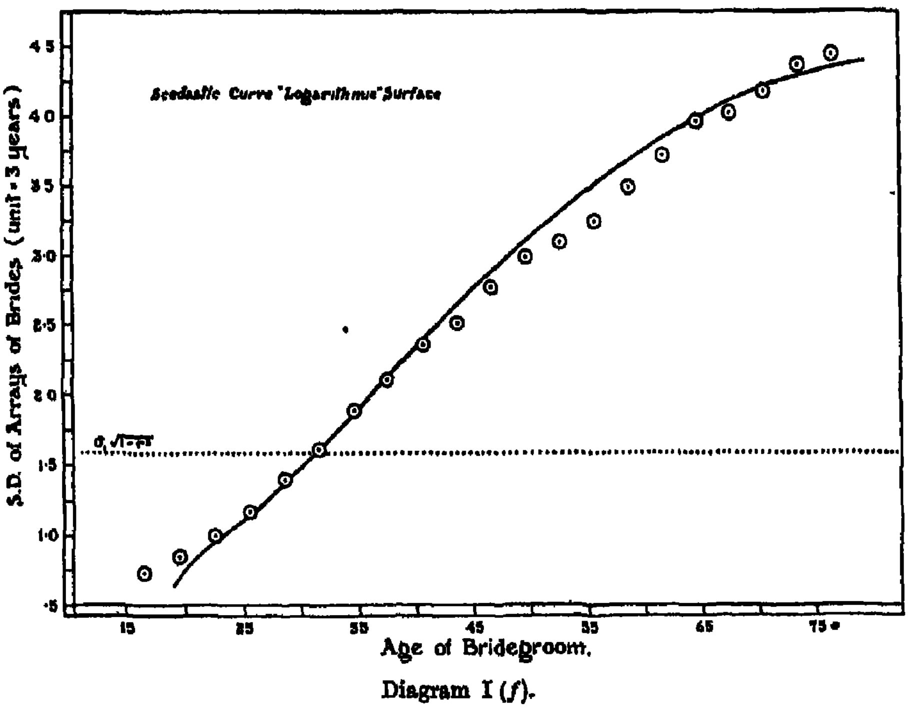


Diagram I (f).

CLISY, AGE OF BRIDE ON AGE OF BRIDEGROOM.

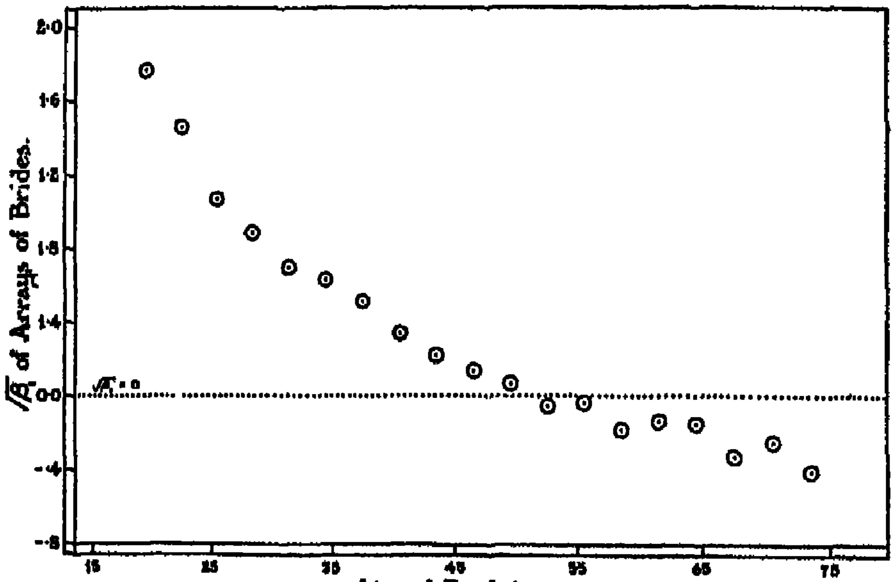


Diagram I (*g*).

KURTOSIS, AGE OF BRIDE ON AGE OF BRIDEGROOM.

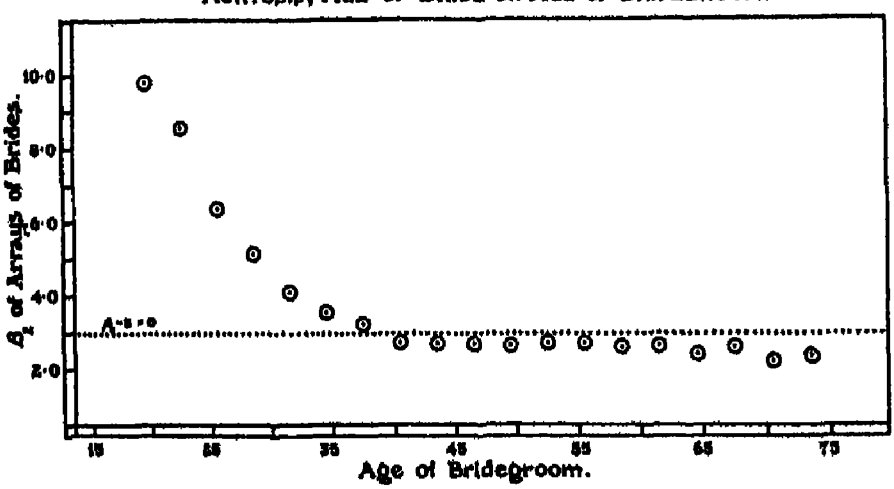


Diagram I (*h*).

REGRESSION, AGE OF FATHER ON AGE OF MOTHER.

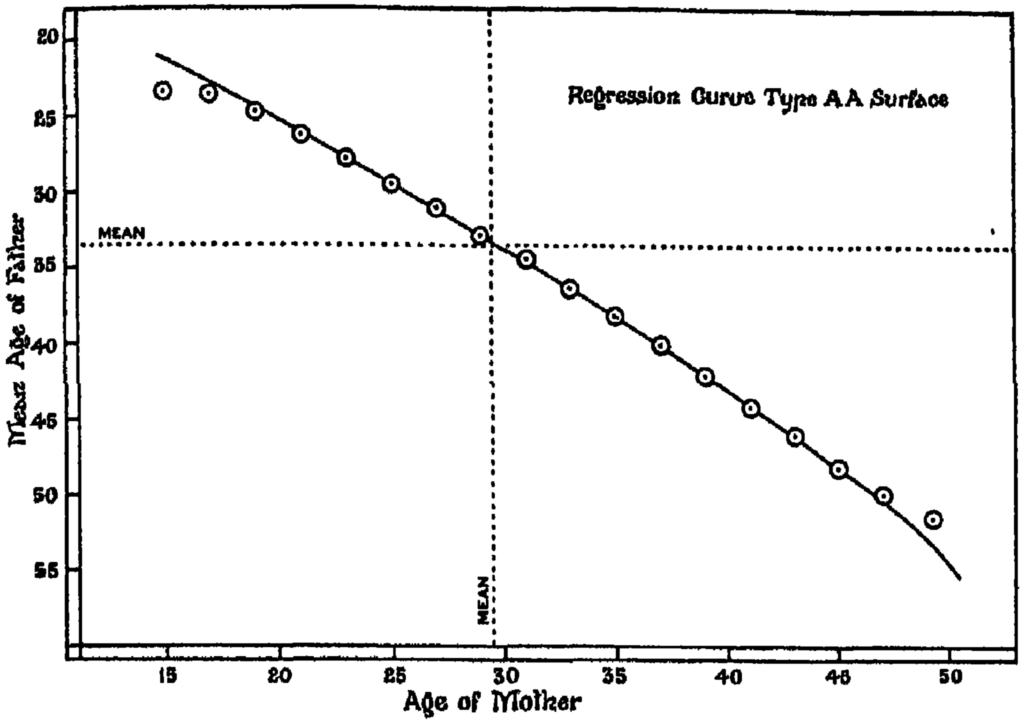


Diagram II (*a*).

SCEDASTICITY, AGE OF FATHER ON AGE OF MOTHER.

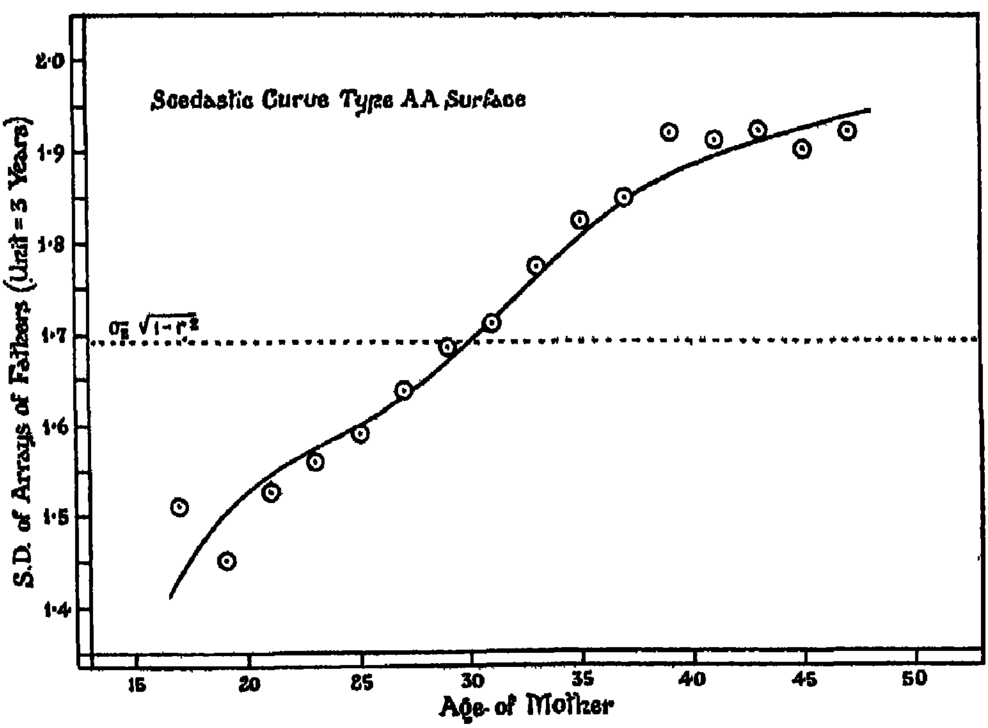


Diagram II (*b*).

CLISY, AGE OF FATHER ON AGE OF MOTHER.

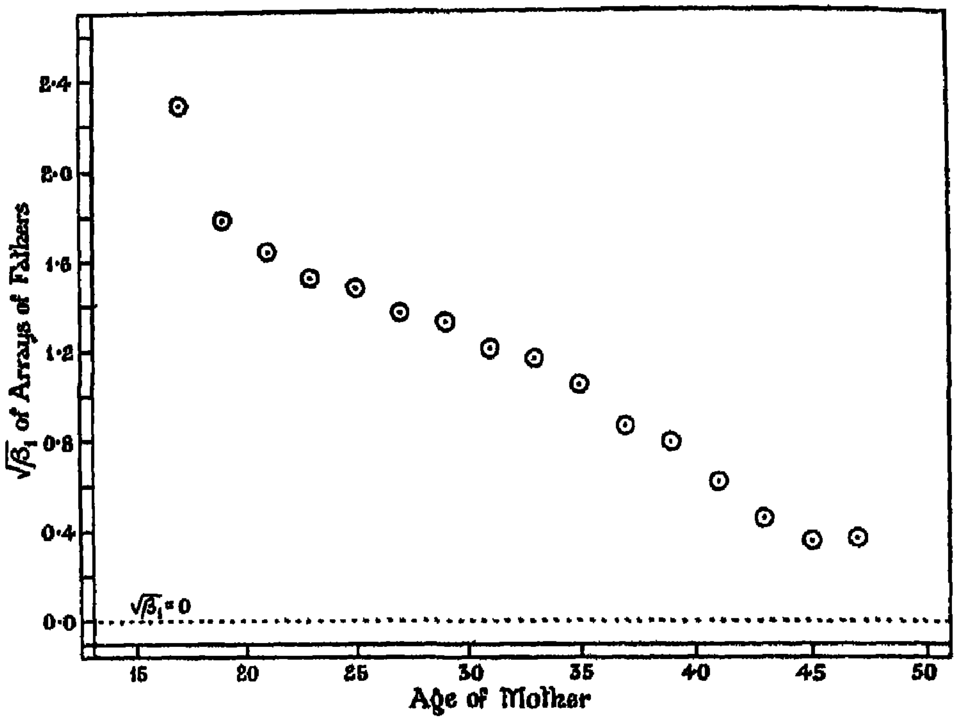


Diagram II (*c*).

KURTOSIS. AGE OF FATHER ON AGE OF MOTHER.

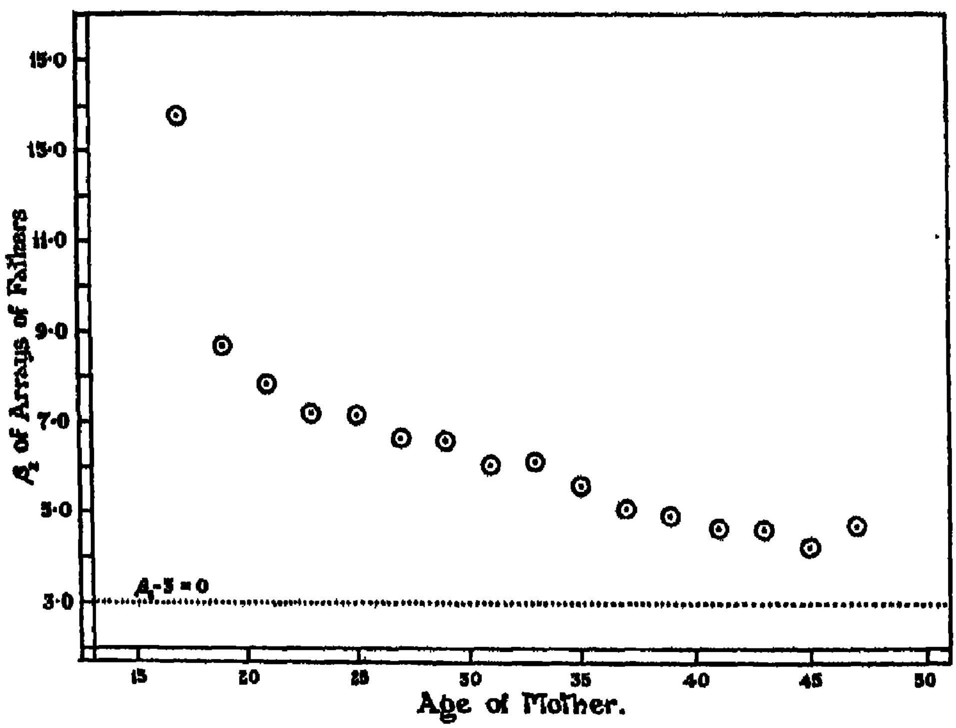


Diagram II (*d*).

REGRESSION, AGE OF MOTHER ON AGE OF FATHER.

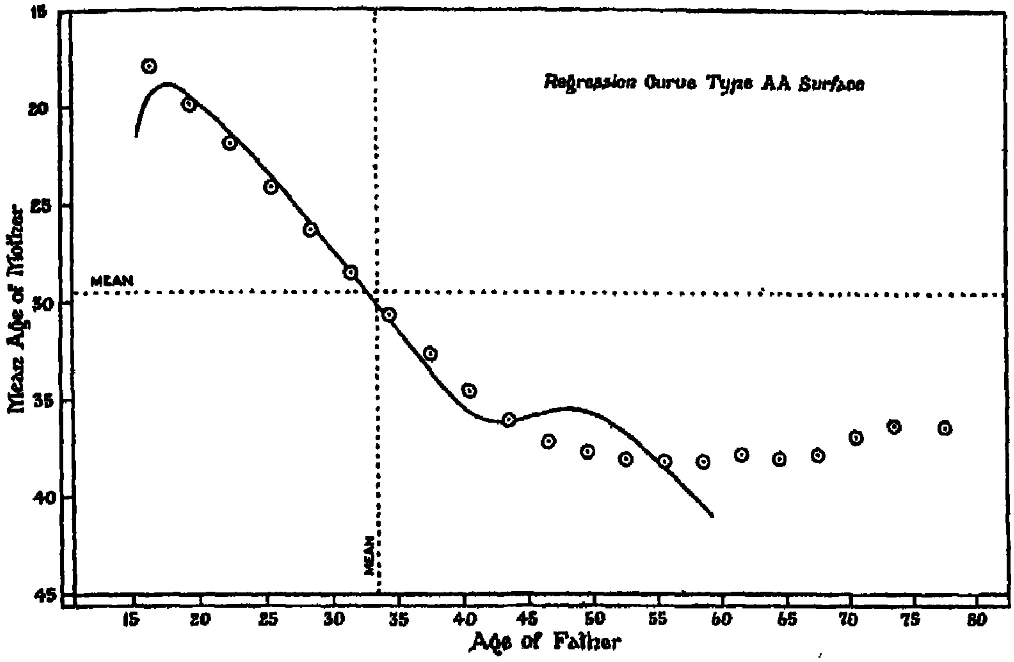


Diagram II (e).

SCEDASTICITY, AGE OF MOTHER ON AGE OF FATHER

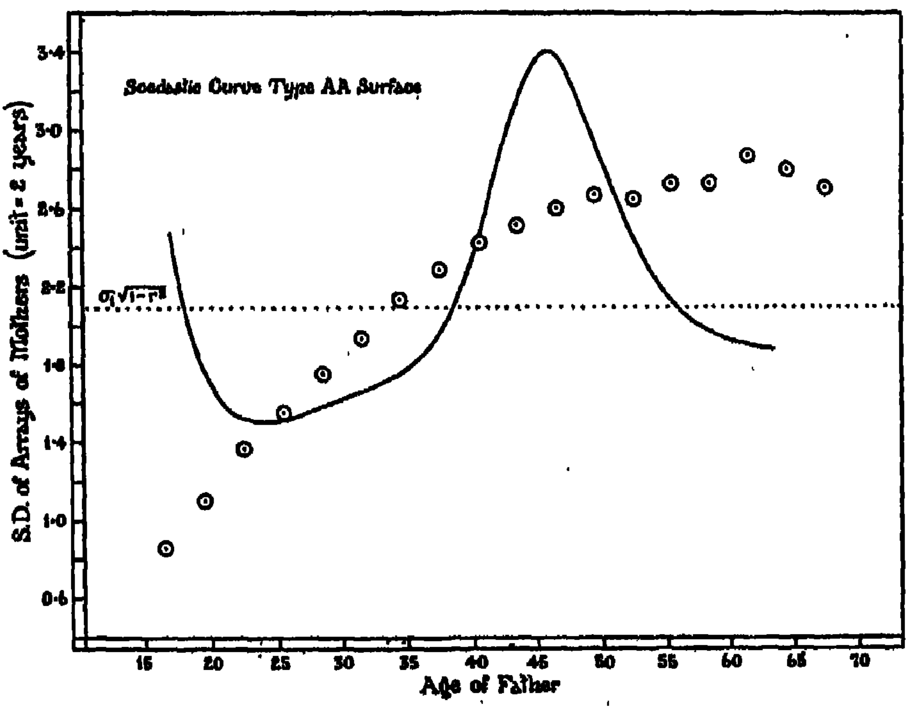


Diagram II (f).

OLISY, AGE OF MOTHER ON AGE OF FATHER.

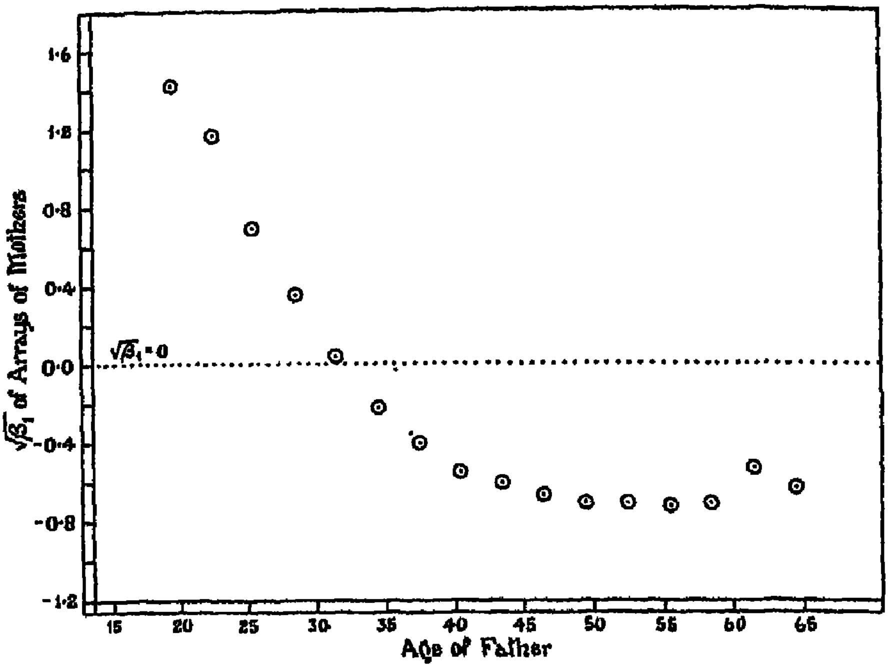


Diagram II (*g*).

KURTOSIS. AGE OF MOTHER ON AGE OF FATHER.

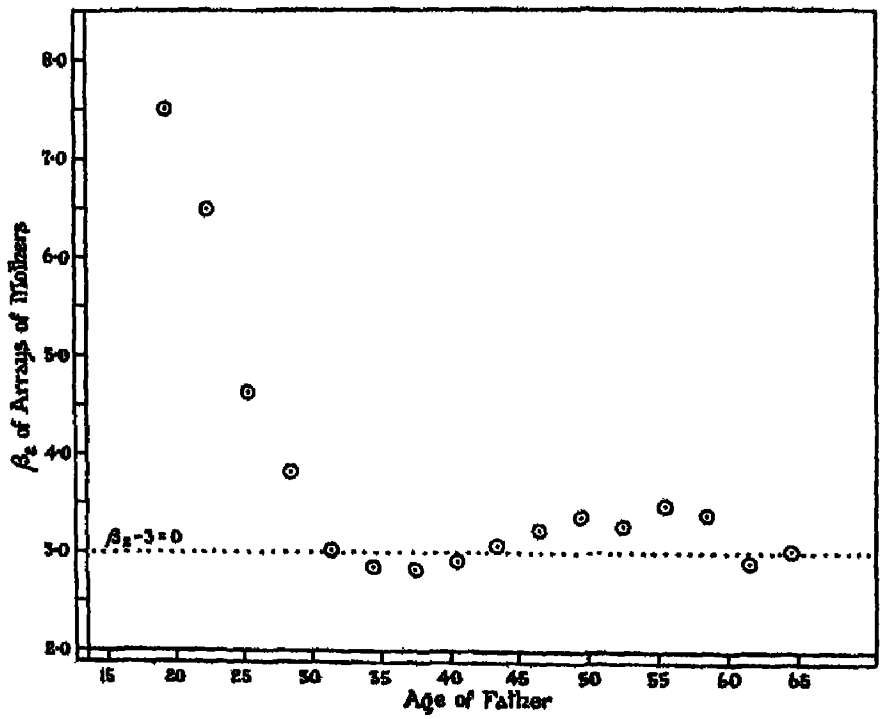


Diagram II (*h*).

REGRESSION, HEIGHT OF BAROMETER: THIRD DAY ON FIRST DAY. (WHOLE YEAR).

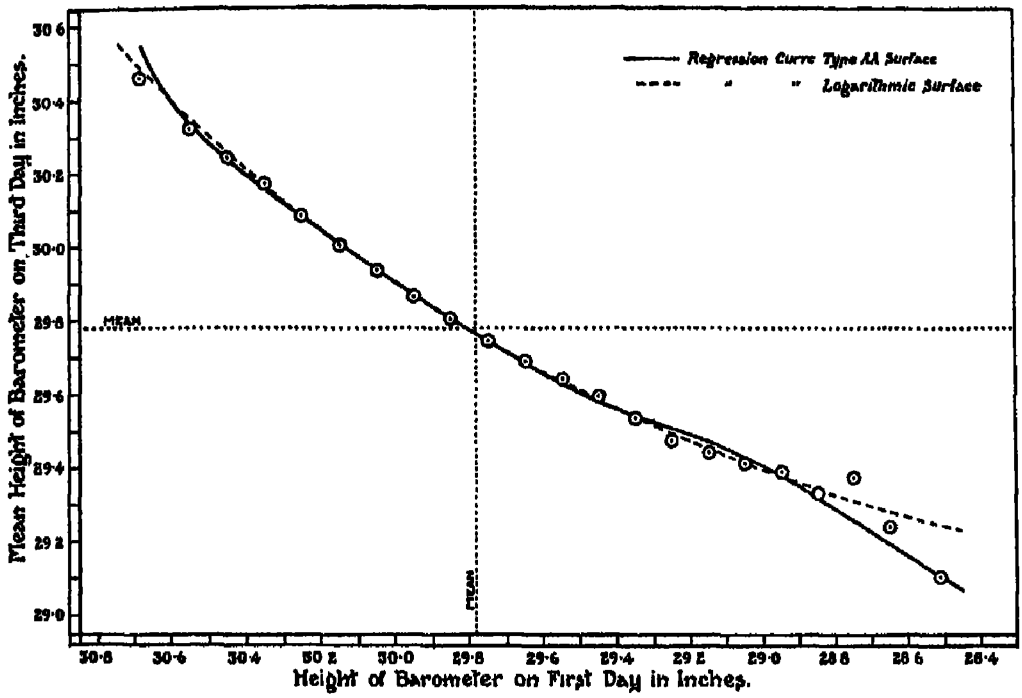


Diagram III (*a*).

SCEDASTICITY, HEIGHT OF BAROMETER: THIRD DAY ON FIRST DAY (WHOLE YEAR).

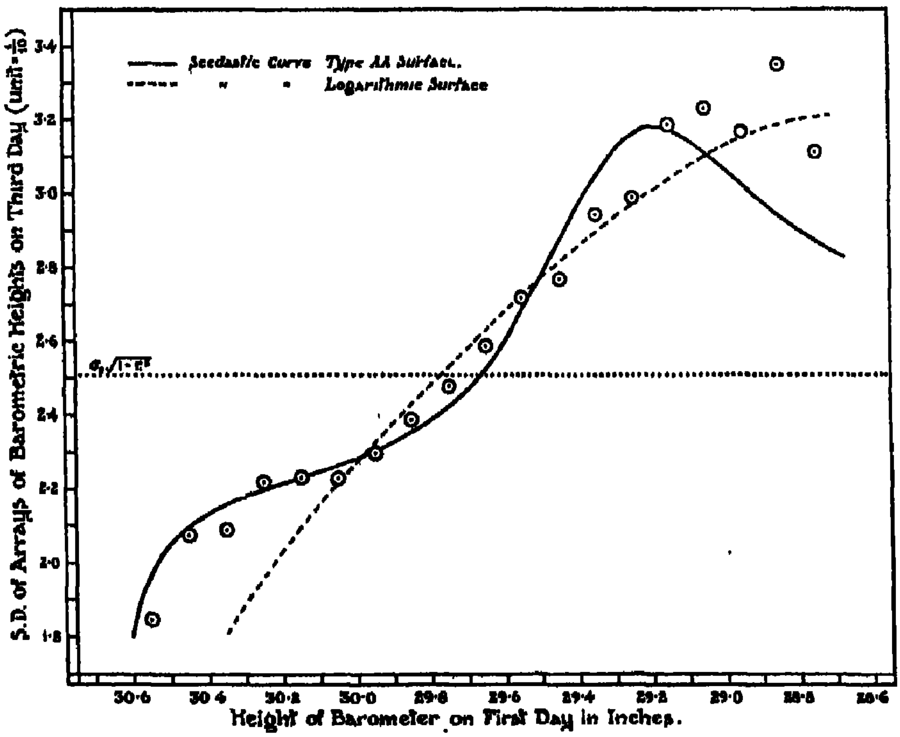


Diagram III (*b*).

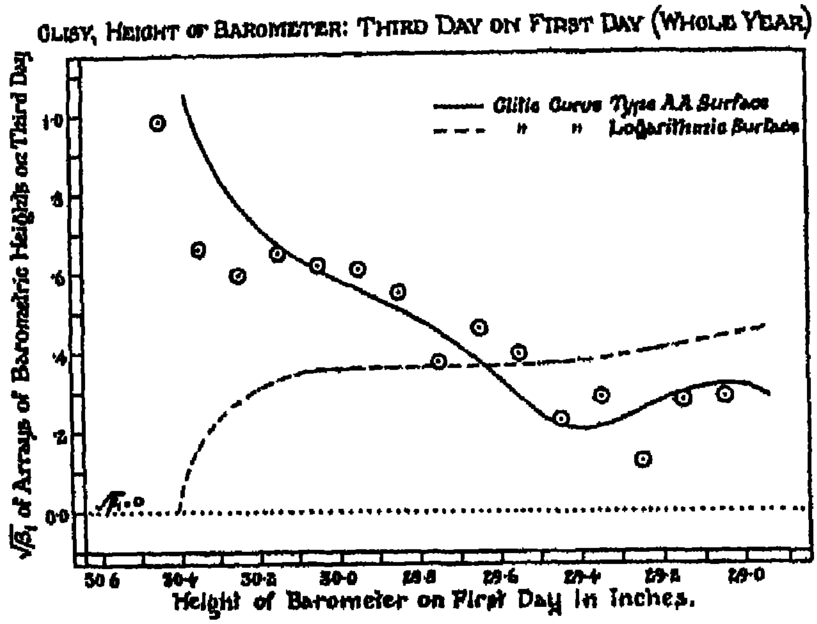


Diagram III (*c*).

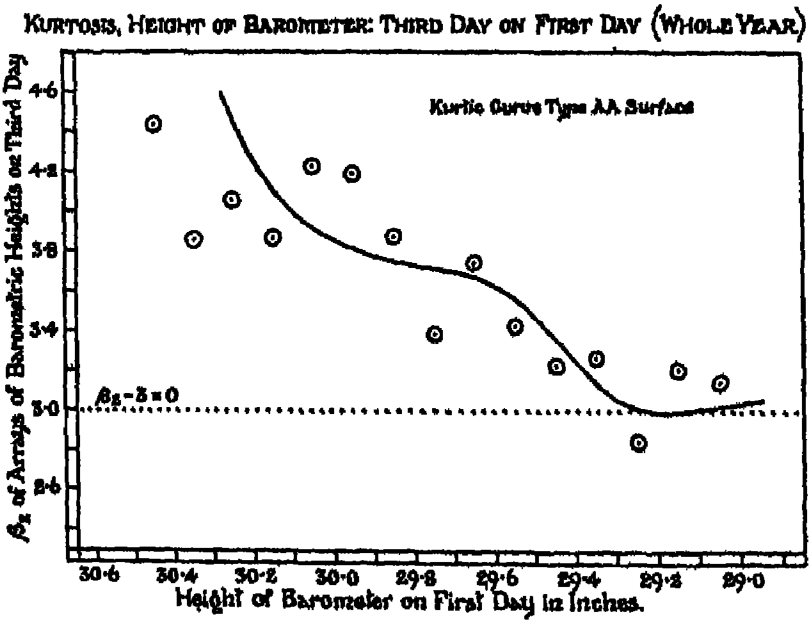


Diagram III (*d*).

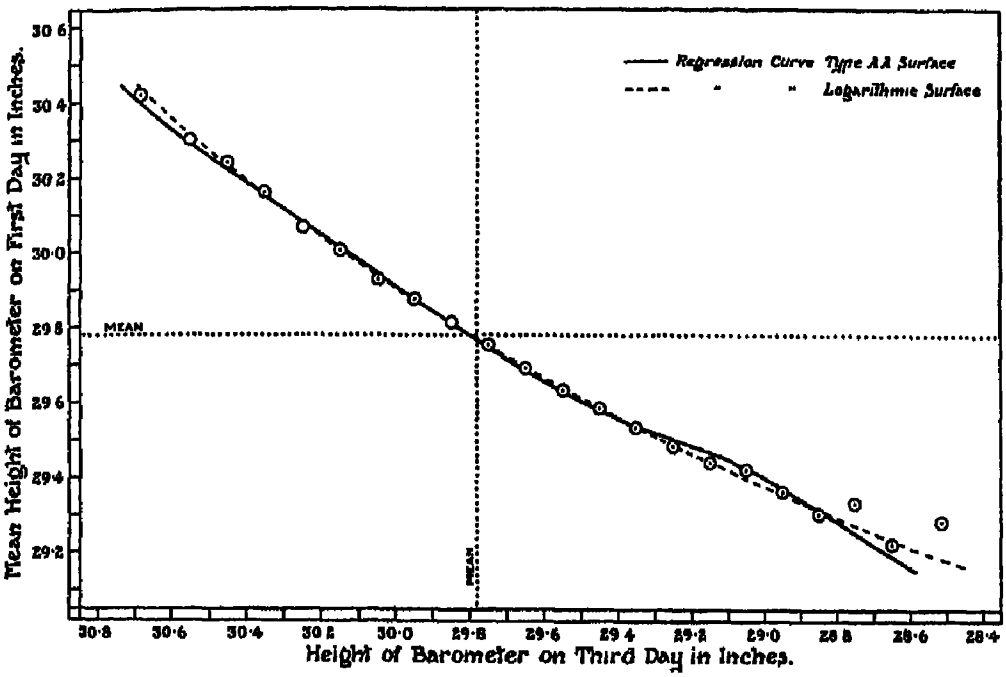


Diagram III (e).

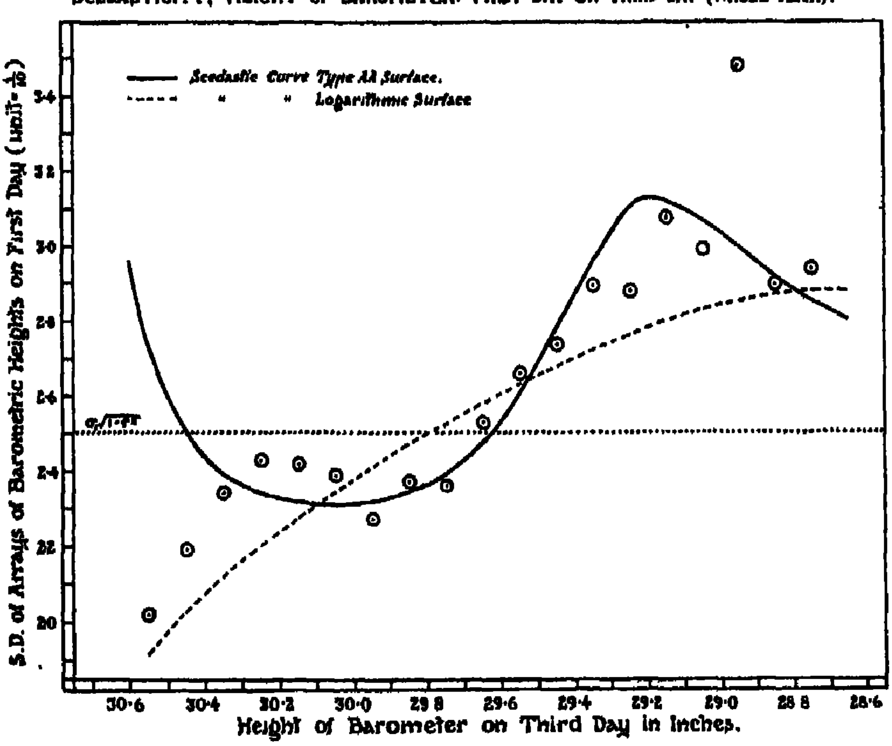


Diagram III (f).

CLISY, HEIGHT OF BAROMETER: FIRST DAY ON THIRD DAY (WHOLE YEAR.)

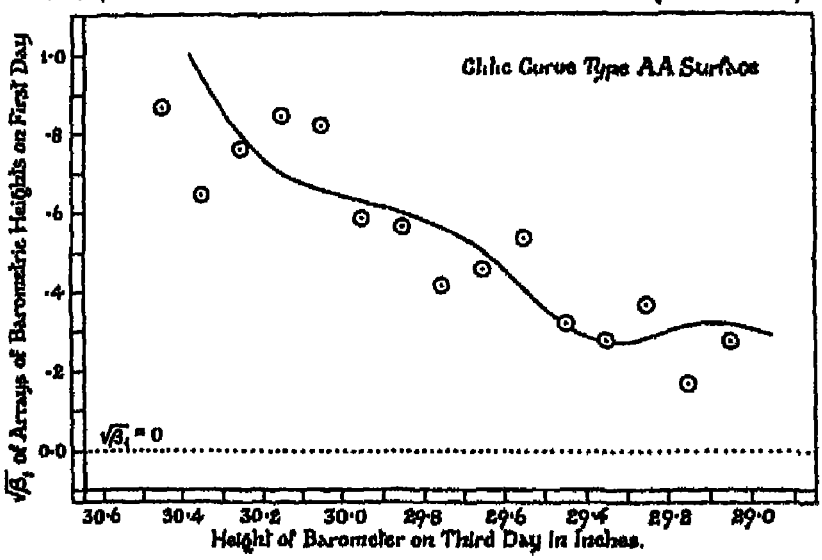


Diagram III (*g*).

KURTOSIS, HEIGHT OF BAROMETER: FIRST DAY ON THIRD DAY (WHOLE YEAR.)

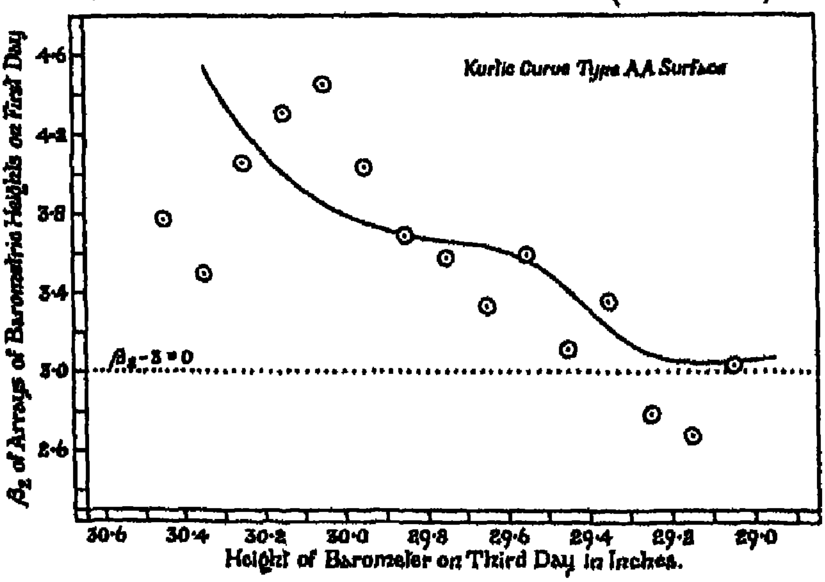


Diagram III (*h*).

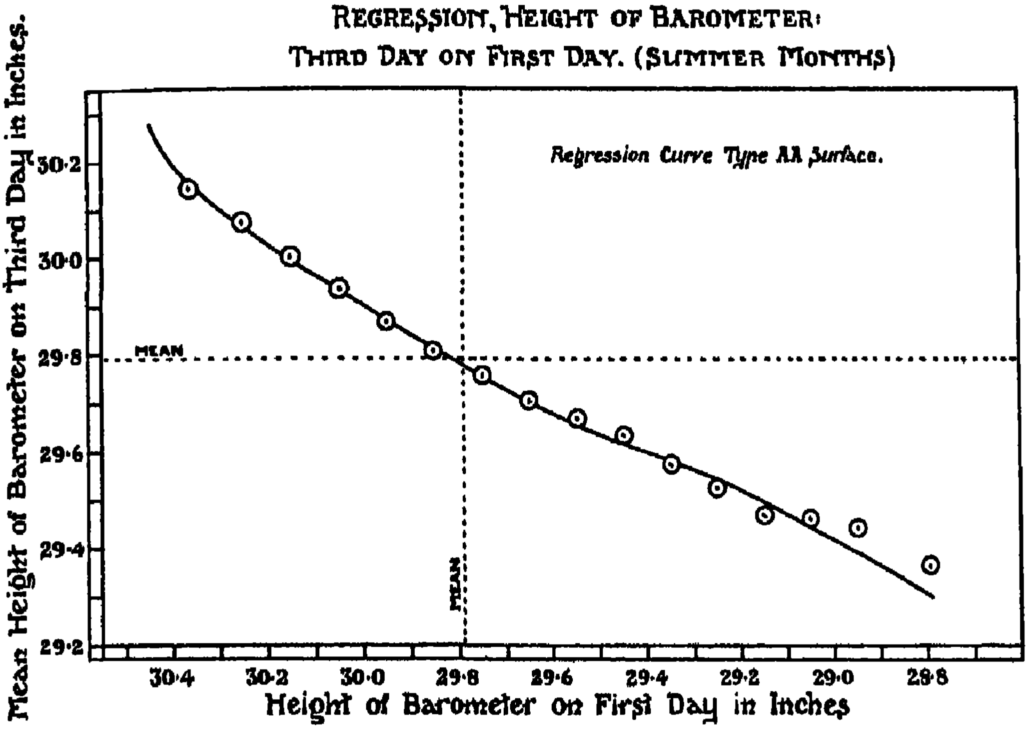


Diagram IV (*a*).

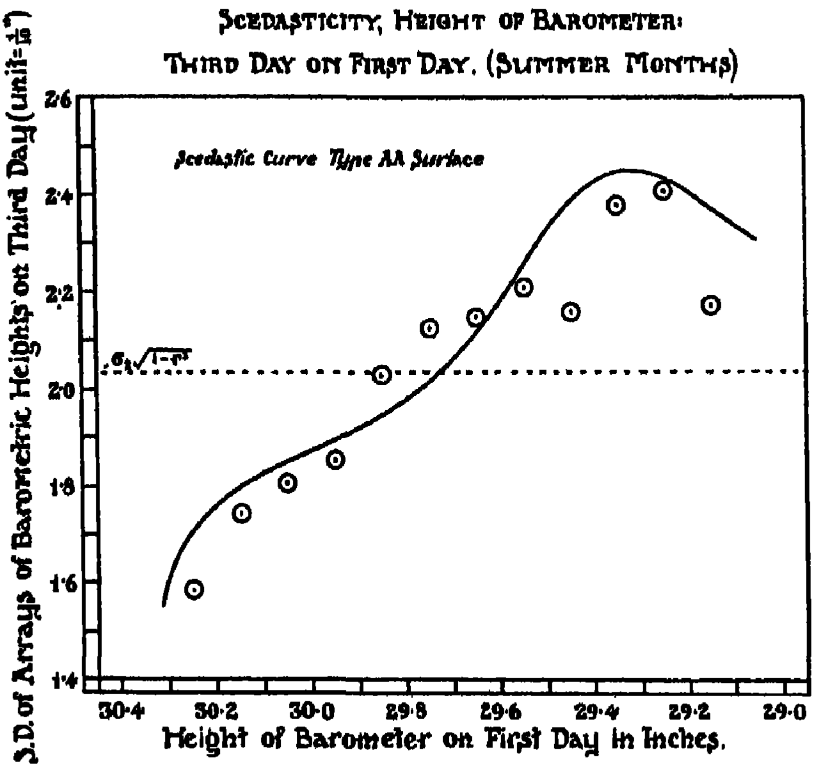


Diagram IV (*b*).

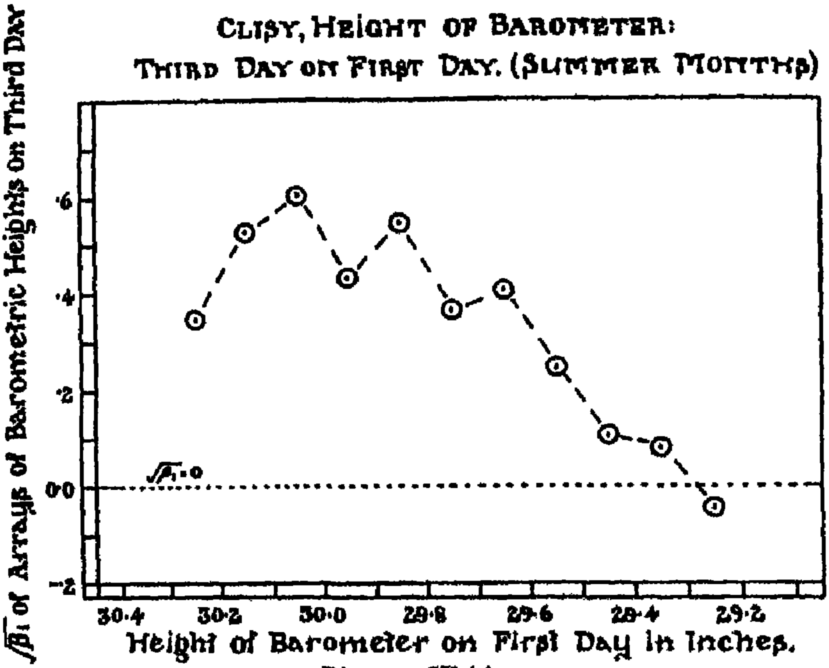


Diagram IV (*c*).

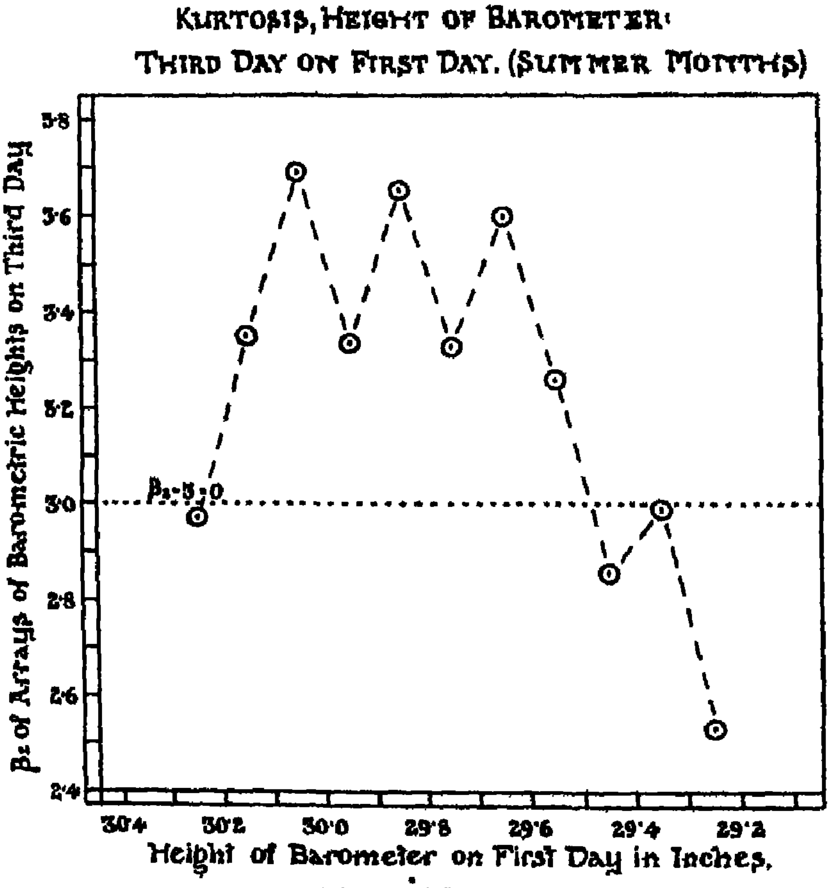


Diagram IV (*d*).

REGRESSION, HEIGHT OF BAROMETER:
FIRST DAY ON THIRD DAY (SUMMER MONTHS)

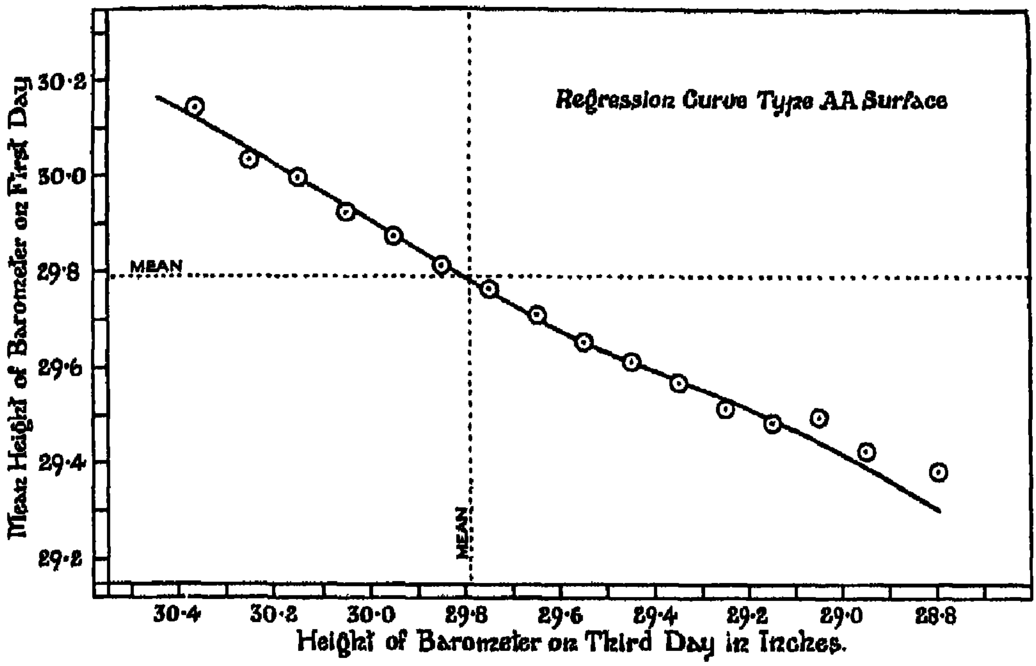


Diagram IV (e).

SCEDASTICITY, HEIGHT OF BAROMETER:
FIRST DAY ON THIRD DAY (SUMMER MONTHS)

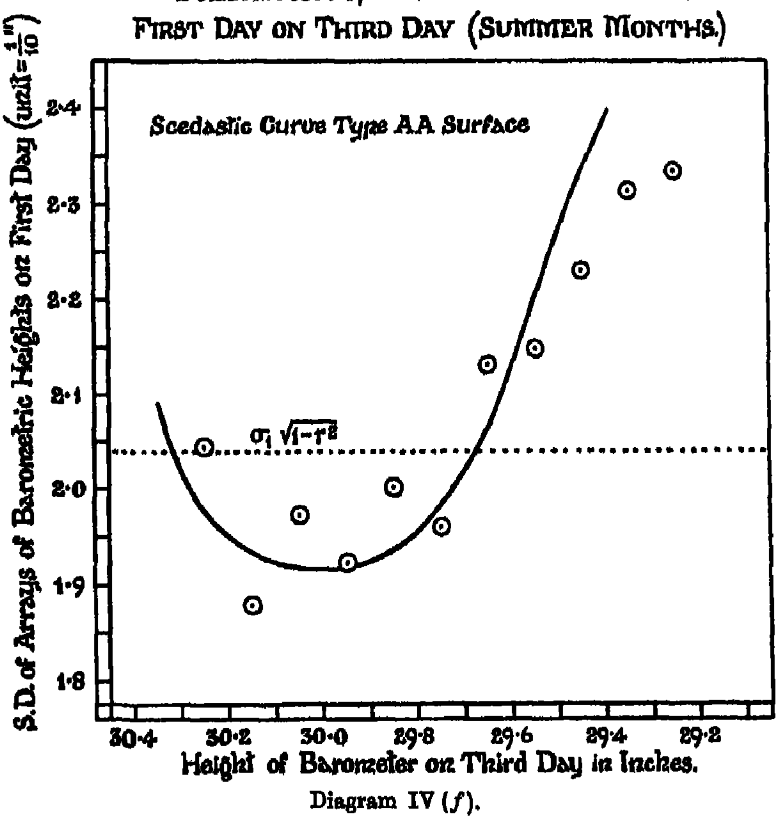


Diagram IV (f).

CLISY, HEIGHT OF BAROMETER.
FIRST DAY ON THIRD DAY (SUMMER MONTHS.)

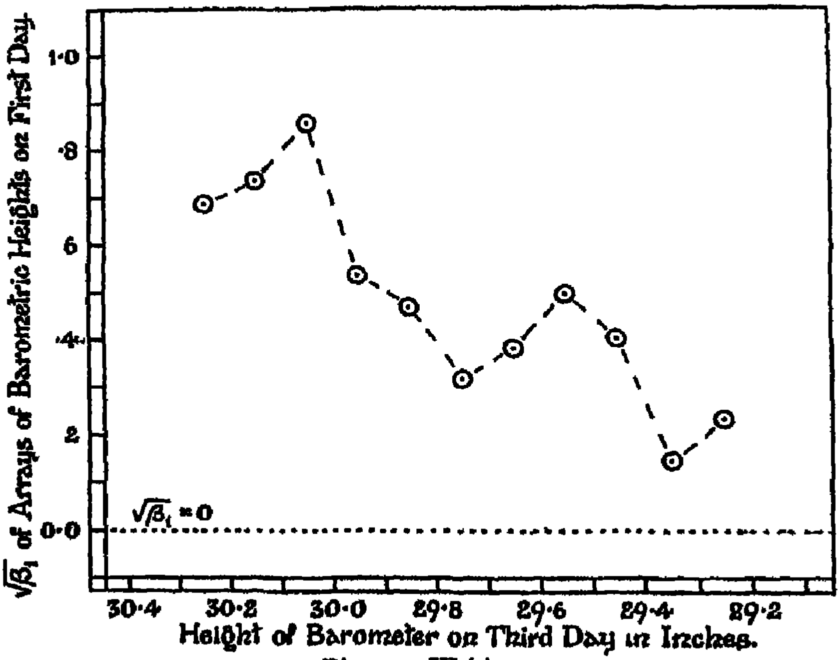


Diagram IV (*g*).

KURTOSIS, HEIGHT OF BAROMETER:
FIRST DAY ON THIRD DAY (SUMMER MONTHS)

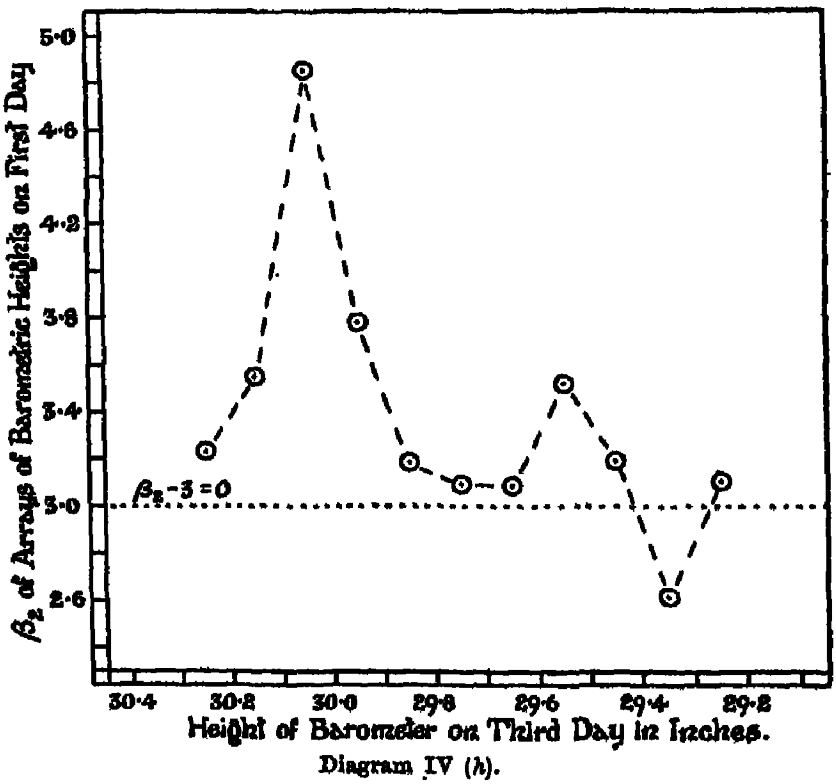


Diagram IV (*h*).

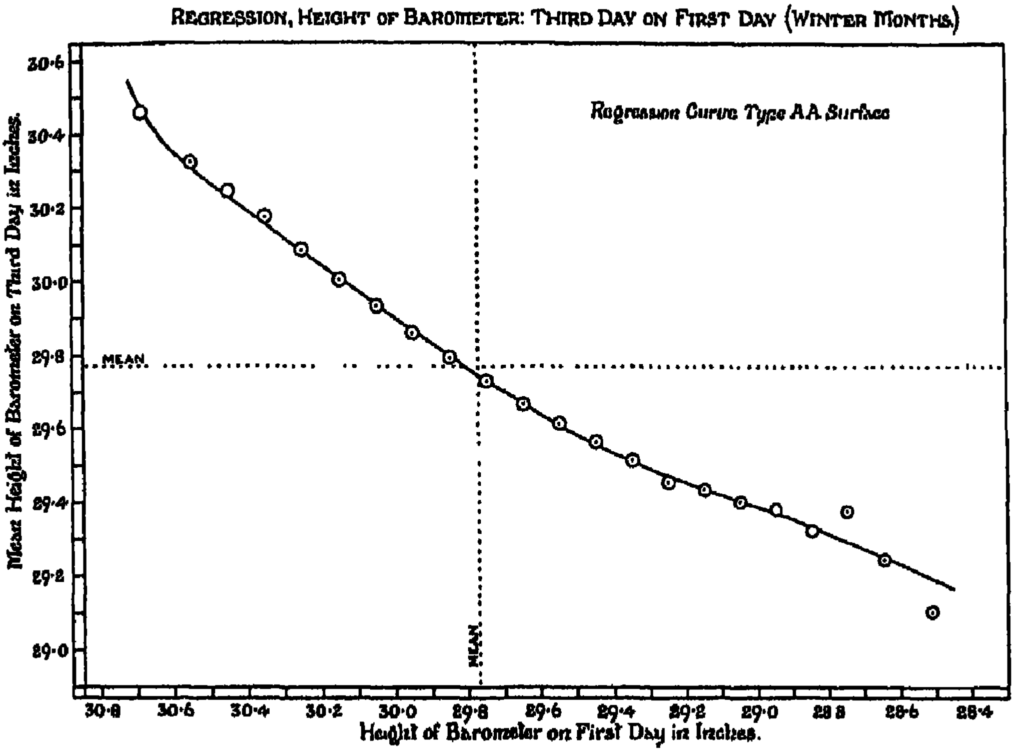


Diagram V (*a*).

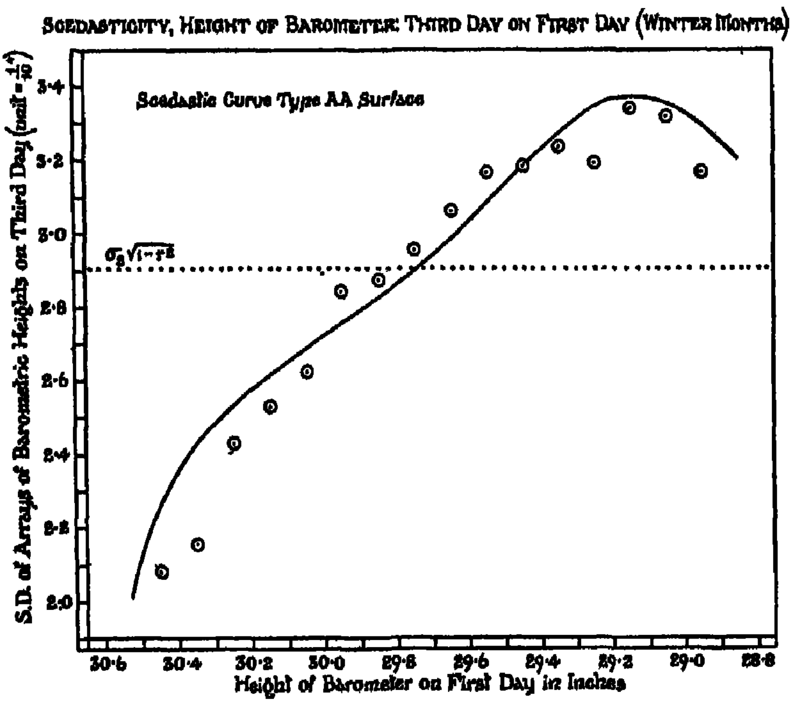


Diagram V (*b*).

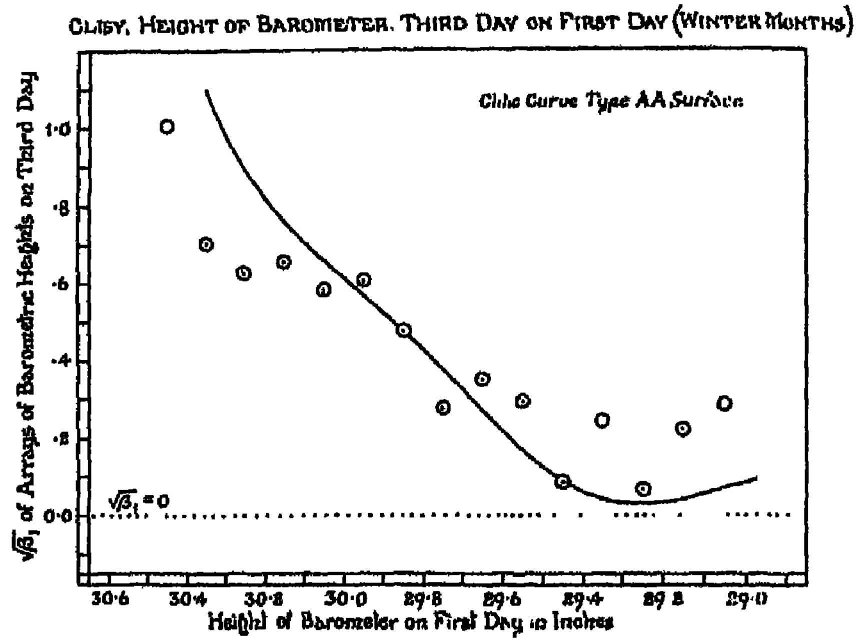


Diagram V (*c*).

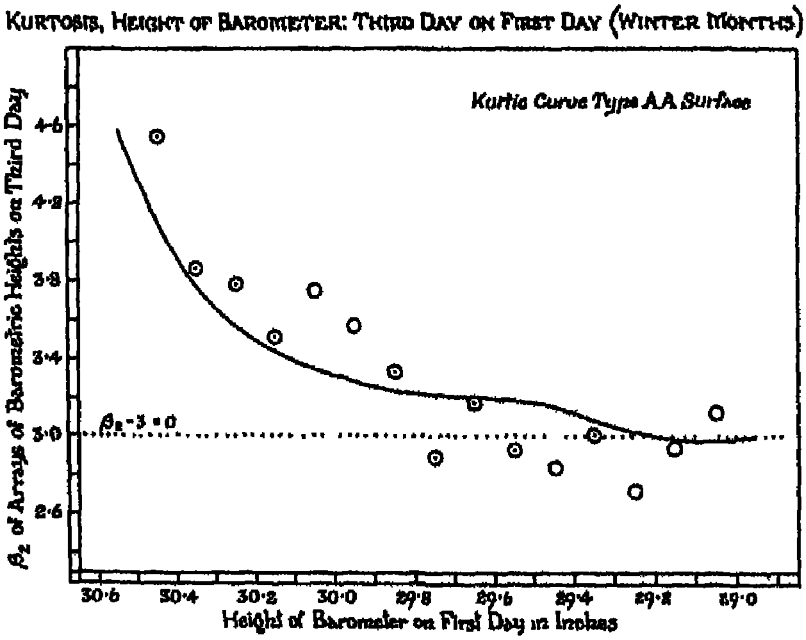


Diagram V (*d*).

REGRESSION, HEIGHT OF BAROMETER: FIRST DAY ON THIRD DAY (WINTER MONTHS)

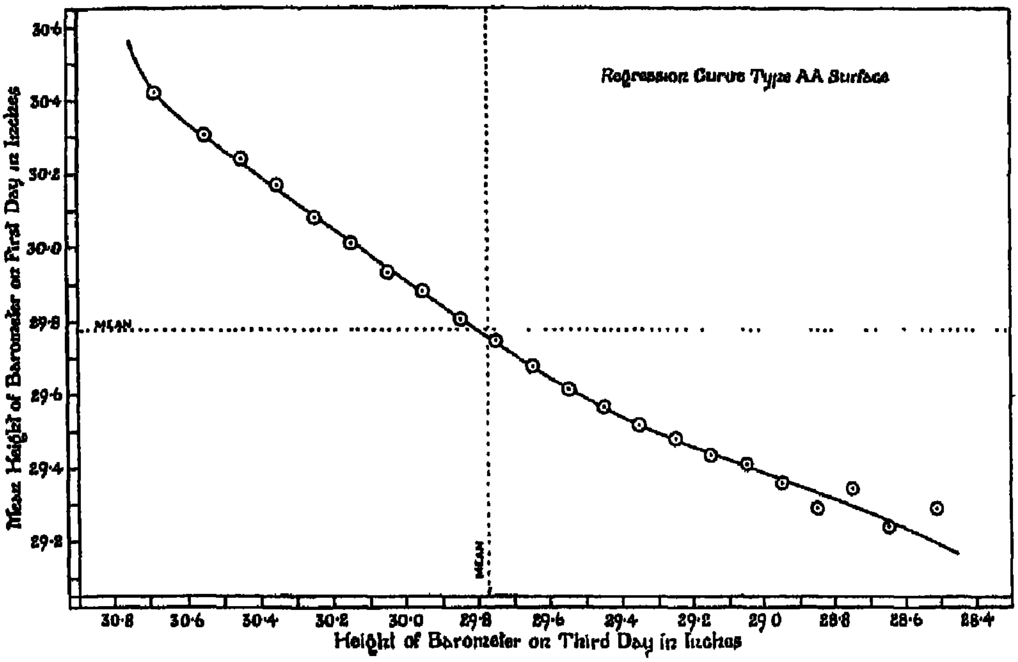


Diagram V (e).

SCEDASTICITY, HEIGHT OF BAROMETER: FIRST DAY ON THIRD DAY (WINTER MONTHS.)

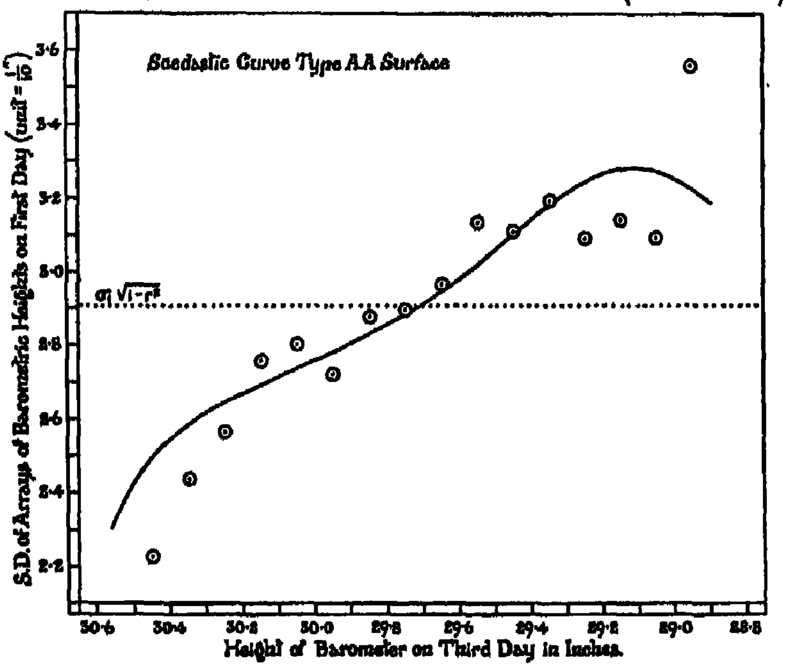


Diagram V (f).

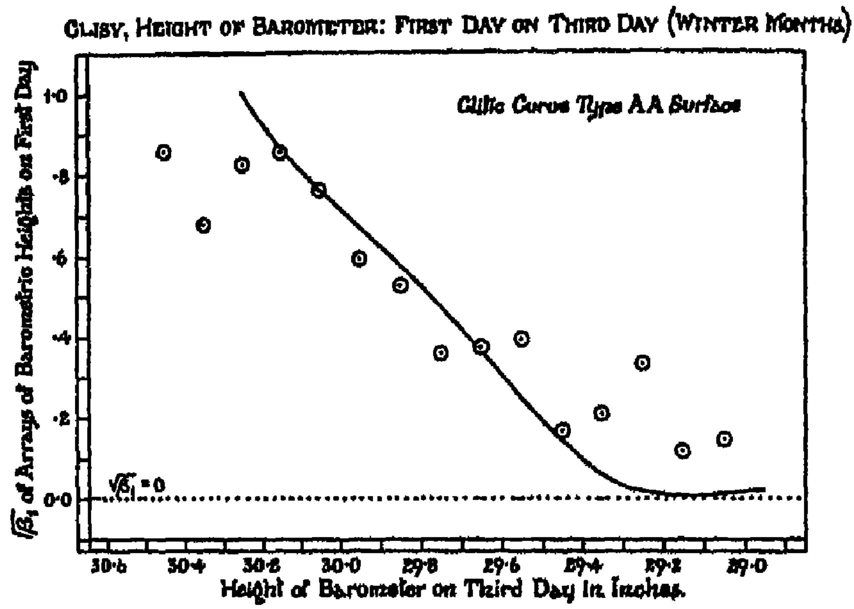


Diagram V (*g*).

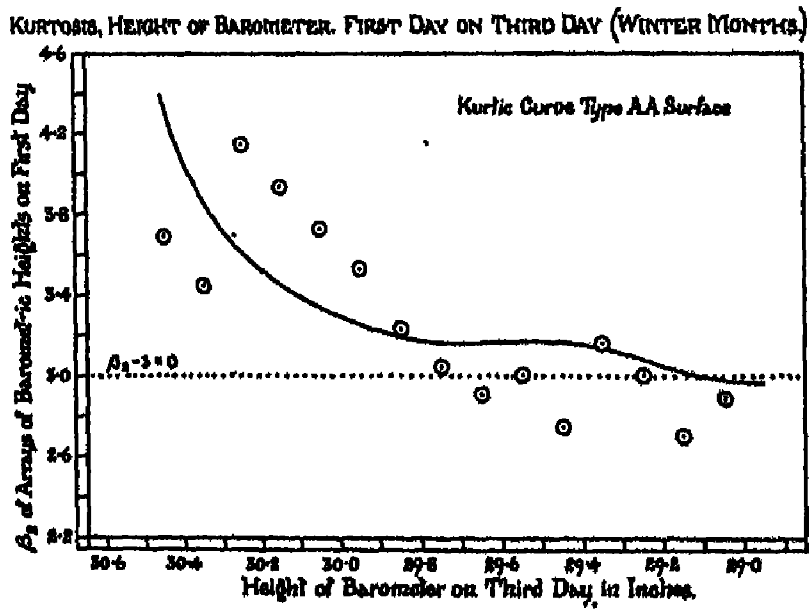


Diagram V (*h*).

REGRESSION, BREADTH ON LENGTH OF BEANS.

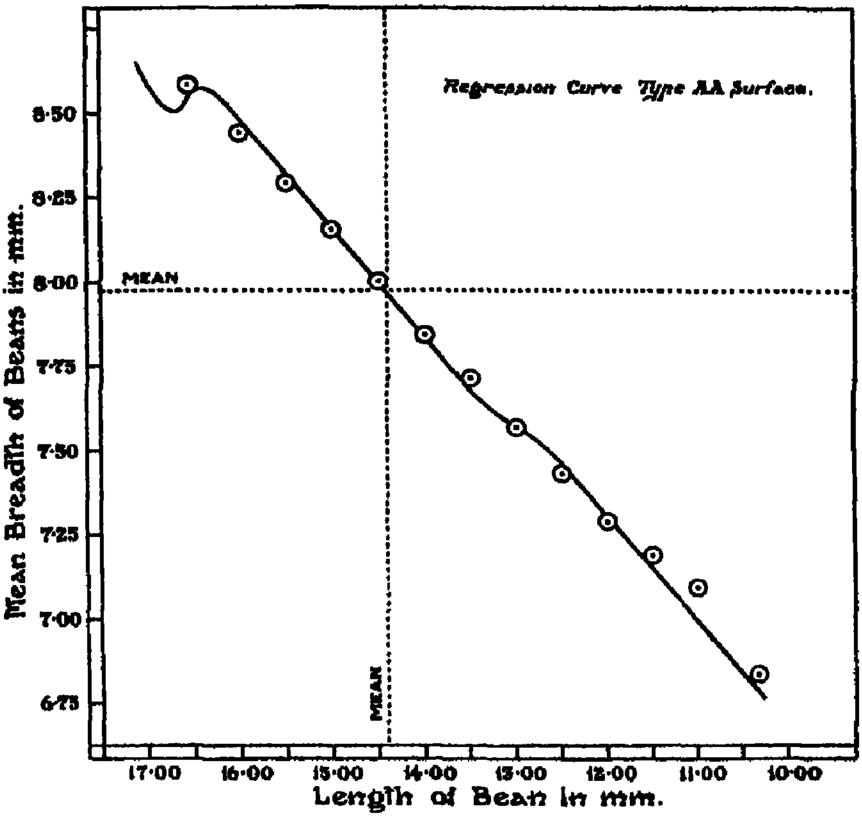


Diagram VI (*a*).

SCEDASTICITY, BREADTH ON LENGTH OF BEANS.

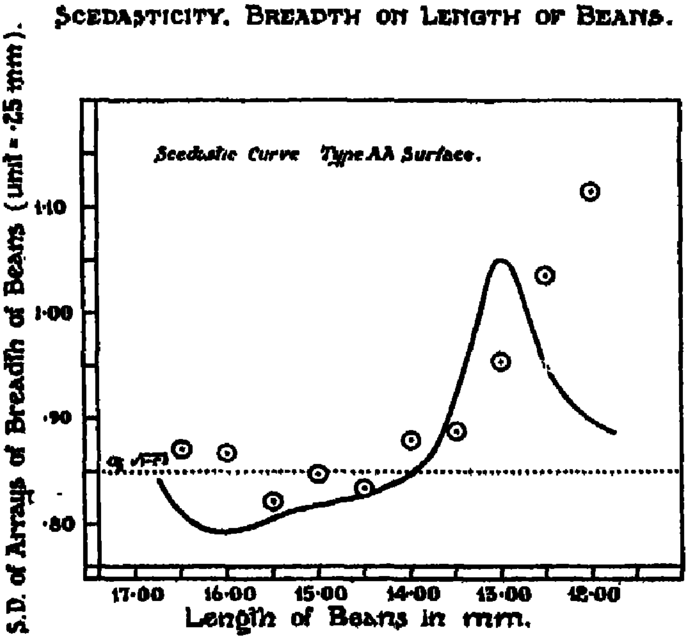


Diagram VI (*b*).

Clisy, Breadth on Length of Beans.

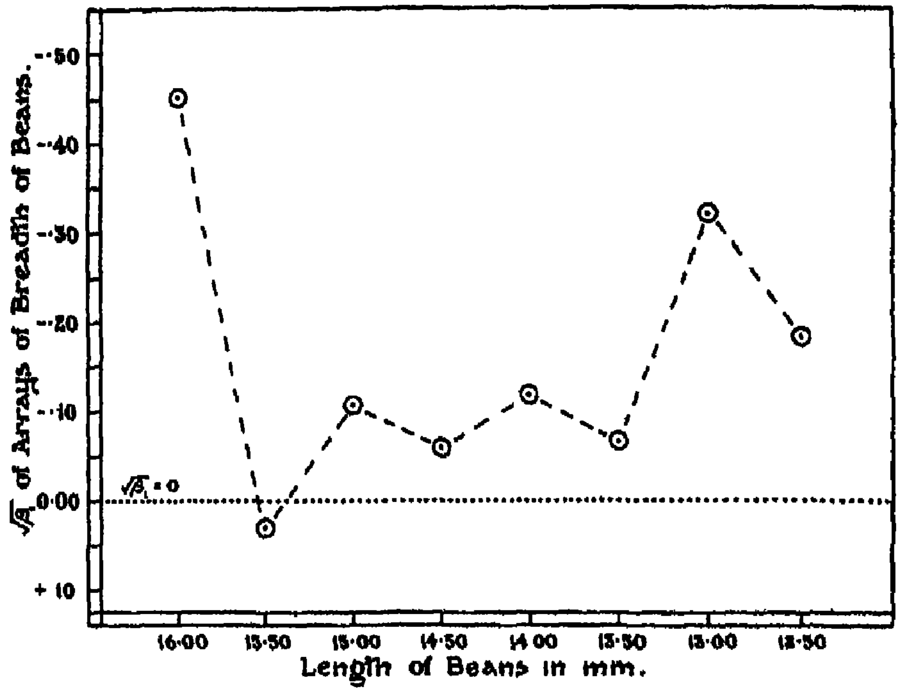


Diagram VI (*c*).

Kurtosis, Breadth on Length of Beans.

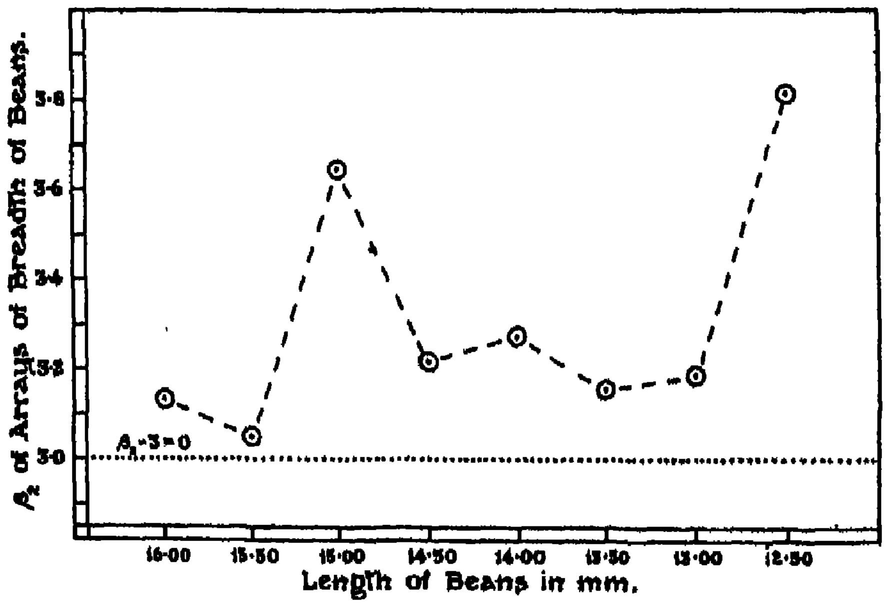


Diagram VI (*d*).

REGRESSION, LENGTH ON BREADTH OF BEANS.

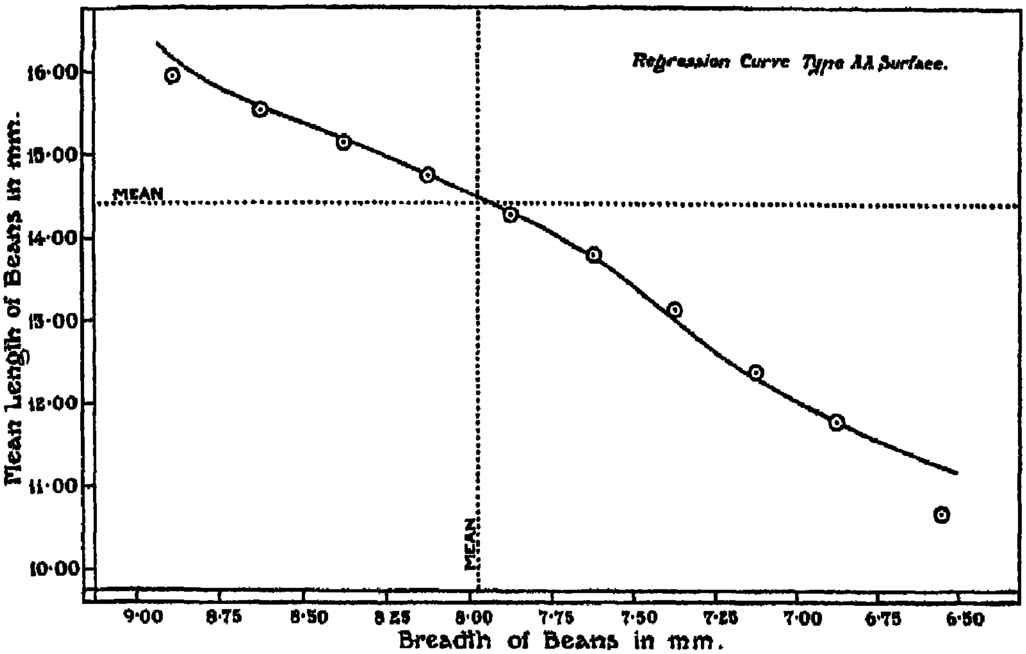


Diagram VI (*e*).

SCEDASTICITY, LENGTH ON BREADTH OF BEANS.

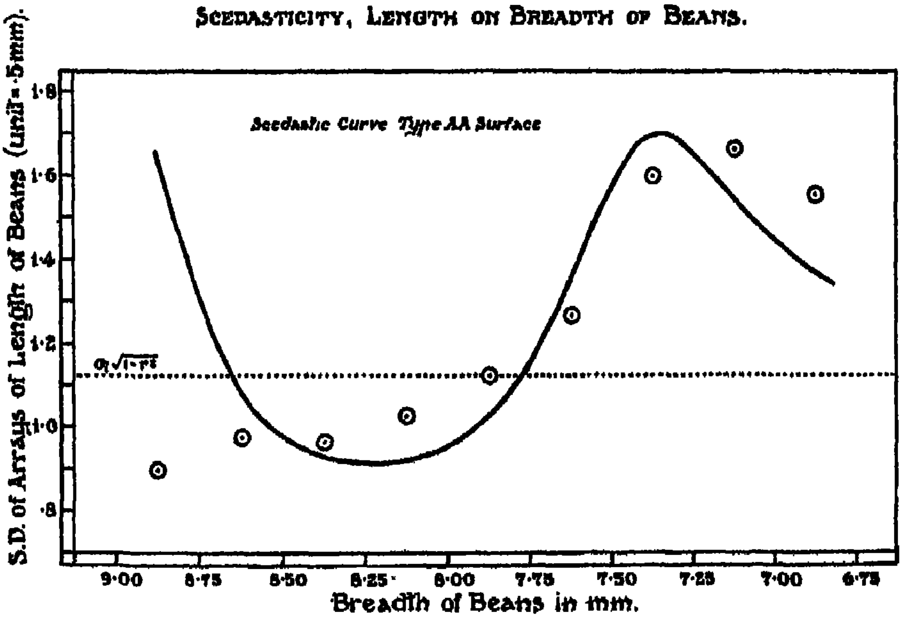


Diagram VI (*f*).

CLISY, LENGTH ON BREADTH OF BEANS.

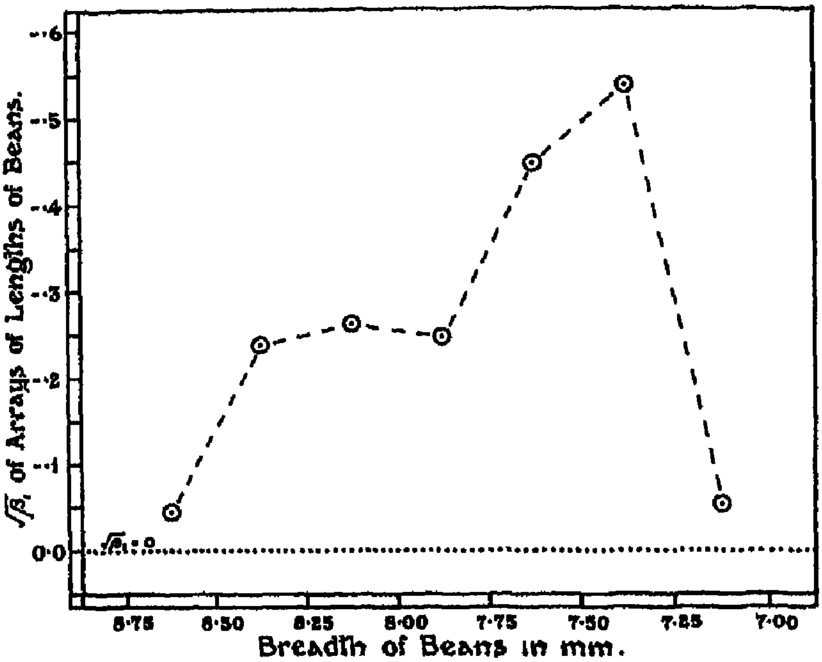


Diagram VI (g).

KURTOSIS, LENGTH ON BREADTH OF BEANS.

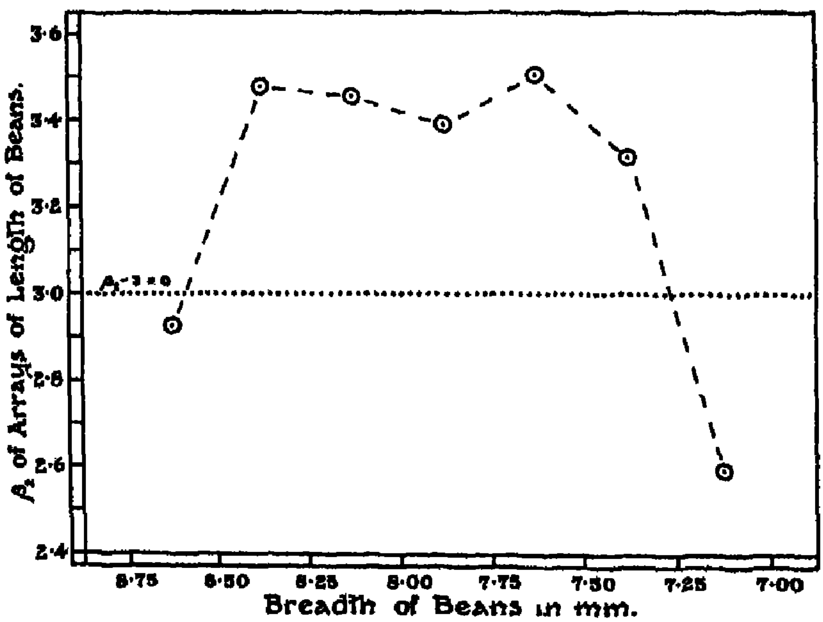


Diagram VI (h).

abruptness will be dealt with in Section 5; as a matter of fact, they were found to be negligible.

The observation points are represented by small circles in Diagrams I (*a*), I (*b*), up to VI (*h*). Except at the extremities of the distributions, the means follow fairly smooth trends. The regression of age of mother on age of father is of the rectangular hyperbolic type, while the other regressions deviate less from rectilinearity. The scedasticity reveals, not infrequently, a resemblance to the parabolic form of the double hypergeometrical series; for example, in the barometric and birth statistics. Approaching the higher moments we find, as could have been anticipated, that the observation points become more and more erratic. Yet, for the first two distributions, the variation in the clisy and kurtosis is still remarkably regular; for the barometric data a quite definite trend, other than constant, is noticeable; and it is only for the distribution of length and breadth of beans, where both the number of points on the graphs and the total number of observations are small, that these two measures seem to be scattered at random. This finding of a fairly regular variation in the shape of the array distributions disproves, beyond doubt, the generality of Narumi's hypothesis for these cases.

We proceed to fit to these points the appropriate curves of the Type AaAa* and of the Logarithmic Surface. The expressions for the partial moment curves of the former surface are given in equations (62), (63), (64), (65), and (66); those for the Logarithmic Surface in equations (39) and (40). The constants in these formulae together with the values they assume for the different distributions are shown in Tables VII and VIII. The surfaces will be considered in the order named.

Both regression curves of the Type AaAa surface were fitted to the array means of the last five distributions. The curves are geometrically somewhat quaint, but, with the exception of the regression of age of mother on age of father, the results are quite reasonable. The scedastic curves fitted for the same distributions are oscillatory to a much higher degree than the regression curves; even in the most favourable cases the fit is rather unsatisfactory. The clitic and kurtic curves were fitted for the barometric data: whole year and winter months. For the first of these distributions they do not fit badly: perhaps they fit even better than the corresponding scedastic curves do. For the latter distribution, however, they are less successful.

The regression curves of the Logarithmic Surface were tested for the barometric (whole year) and marriage distributions†. They fit very well in the former case and probably better than the curves of the Type AaAa surface do. For the marriage distribution a want of flexibility in the curves is manifest. This inefficiency is still more marked when we come to the scedastic curves. For the barometric distribution these curves, bad as they are, are to be preferred to those of the Type AaAa surface; for the marriage distribution, too, the results are

* Read *Type AaAa* instead of *Type AA* in the diagrams.

† See Section 4 for a discussion of the criteria justifying the application of this surface to these two distributions.

TABLE VII.

Coefficients of Tetrachoric Terms in Expressions for Partial Moment Curves of Type Aa Aa Surface.

		a_4	a_5	b_3	b_4	c_2	c_3	d_1	d_2	e_1	$1-r^2$
Australian Marriages	y_x	—	—	—	—	—	—	—	—	—	—
	x_y	—	—	—	—	—	—	—	—	—	
Australian Births	y_x	+·258,976	−·260,019	+·051,479	+·030,464	−·097,736	−·011,932	—	—	—	·459,857
	x_y	+·591,414	+·284,893	−·281,848	−·488,514	−·314,184	−·366,421	—	—	—	
Barometer Whole Year	y_x	+·368,070	+·181,553	−·134,386	−·074,911	−·204,774	−·140,828	+·221,757	−·087,997	+·337,295	·662,763
	x_y	+·368,476	+·182,021	−·113,524	−·068,514	−·152,062	−·164,106	+·268,864	−·088,545	+·311,655	
Barometer Summer Months	y_x	+·366,599	+·125,252	−·107,846	−·051,409	−·190,742	−·082,283	—	—	—	·714,262
	x_y	+·352,902	+·098,120	−·075,839	−·071,878	−·133,854	−·135,910	—	—	—	
Barometer Winter Months	y_x	+·307,898	−·063,206	−·150,667	−·090,200	−·185,351	−·016,549	+·193,837	−·194,781	+·133,561	·638,846
	x_y	+·309,196	−·062,860	−·132,495	−·098,180	−·136,966	−·030,599	+·237,484	−·111,105	+·118,593	
Beans	y_x	−·743,476	+·850,314	+·071,055	−·121,665	+·053,061	−·052,302	—	—	—	·389,818
	x_y	−·359,938	+·298,680	−·193,685	+·178,030	+·194,051	−·253,177	—	—	—	

rather poor. Finally the clitic curve (y on x), fitted to the barometric data, tends to a constant value.

In considering the goodness of fit of these curves we must bear in mind, as Pearson pointed out when he discussed the 15-Constant surface, that the constants in the equations of the curves are determined not directly from the moments of

TABLE VIII.

Constants in Expressions for Partial Moment Curves of Logarithmic Surface.*

		Australian Marriages		Barometer Whole Year	
		y on x	x on y	y on x	x on y
Margins	ξ	3·73516	4·49910	20·64737	20·62685
	e	·50562	·58884	1·31009	1·30965
	s	·24068	·23630	·06442	·06449
Regression	$\lambda^{(1)}$	1·440,318	1·494,276	1·365,984	1·363,189
	$\gamma^{(1)}$	·788,336	·942,229	1·793,429	1·789,158
	$d^{(1)}$	·589,051	·672,275	1·315,756	1·315,314
	$D_0^{(1)}$	·942,228	·955,639	1·005,604	1·003,222
	$D_1^{(1)}$	·098,457	·138,009	− ·007,433	− ·012,877
	$D_2^{(1)}$	·997,329	·794,485	−1·350,278	− ·774,629
Scedasticity	$\lambda^{(2)}$	2·880,636	2·988,552	2·731,967	2·726,378
	$\gamma^{(2)}$	1·696,844	2·009,130	3·594,603	3·586,043
	$d^{(2)}$	·672,485	·755,709	1·321,425	1·320,983
	$D_0^{(2)}$	·884,455	·911,279	1·011,208	1·006,443
	$D_1^{(2)}$	·393,826	·552,034	− ·029,732	− ·051,509
	$D_2^{(2)}$	1·994,657	1·588,971	−2·700,557	−1·549,257
Clisy	$\lambda^{(3)}$	—	—	4·097,951	—
	$\gamma^{(3)}$	—	—	5·403,520	—
	$d^{(3)}$	—	—	1·327,094	—
	$D_0^{(3)}$	—	—	1·016,812	—
	$D_1^{(3)}$	—	—	− ·066,896	—
	$D_2^{(3)}$	—	—	−4·050,835	—

the array distributions, but from the momental constants used in fitting the surface as a whole. The fit of the regression curves, excluding the regression of age of mother on age of father, does not leave much to be desired. The practical utility of the curves, however, is greatly reduced by their algebraic complexity. For the higher moment curves, generally speaking, the results are rather unsatisfactory. How the flexibility of a surface is affected by the number of constants in the equation is shown clearly by Diagrams III (a), III (b) and III (c); two terms

* For the marriage data $B_{21} = -\cdot001{,}453$, $B_{12} = -\cdot001{,}076$, $\rho = \cdot637{,}181$ found from equations (36). For the barometric data $B_{21} = +\cdot000{,}0101$, $B_{12} = +\cdot000{,}0058$, $\rho = \cdot592{,}632$ found from equations (35) and (37).

added to the equation of the Logarithmic Normal Surface are not enough to effect an arbitrary degree of variability in the constant clitic curve of this surface.

3. *Contours.* We turn now to the contours, or curves of equal probability, of the data to study not only their form, but also their symmetry about a set of principal axes. They were constructed directly from the observed frequencies. The labour involved in replacing the frequencies by ordinates would have been prohibitive; besides, by such a treatment of frequencies negative ordinates appear in the tails of the arrays. The method adopted in determining the contour lines is the same as that described by Pearson in *Biometrika*, Vol. XVII. pp. 296–97. The smoothing of the array curves by aid of a spline, the plotting of the contours from these curves, the smoothing of the contours and their final adjustment to

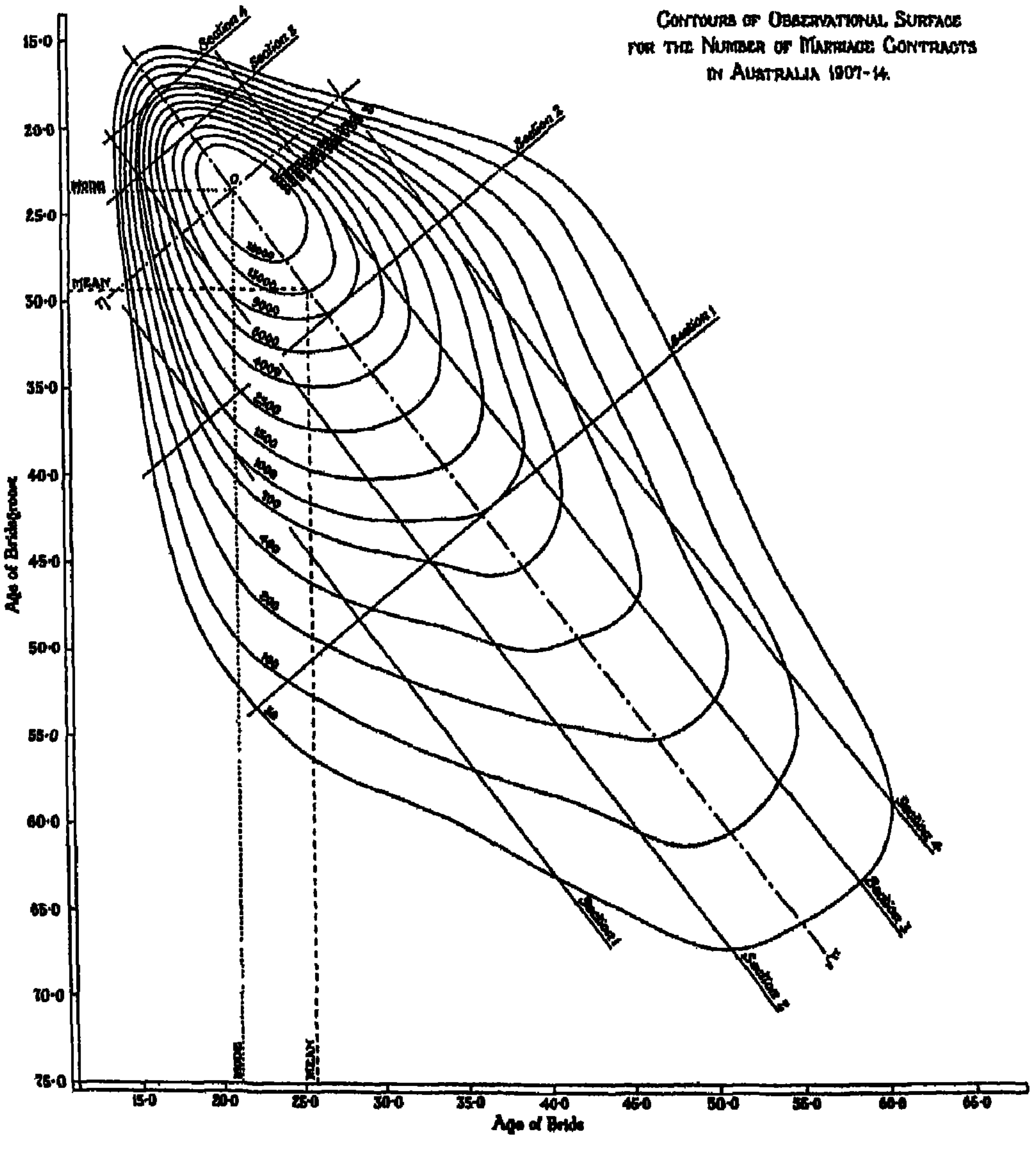


Diagram I.

one another, are the main steps in the process. Each of these steps called for much painstaking in preventing the personal equation from playing too great a part. The final solutions are shown in Diagrams I to VI.

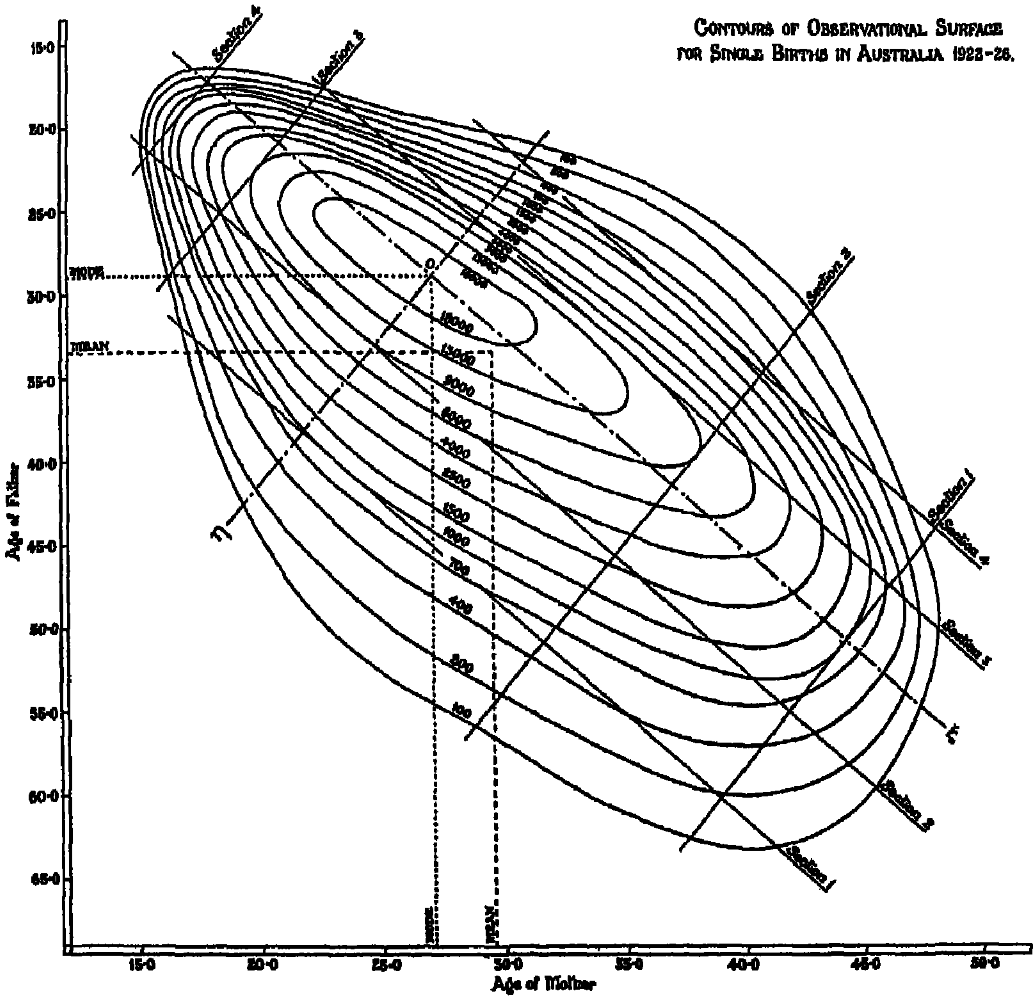


Diagram II.

In form, the contours occasionally resemble the ovals inside a Bernoulli's lemniscate; for instance, those of the three barometric height distributions.' The distribution of length and breadth of beans indicates a system of ellipses not concentrically placed; for the contours of the other two distributions no definite laws are recognisable. In fact, it is very doubtful whether any mathematical surface could reproduce the asymmetry displayed in the distribution of ages of parents at births of children.

The idea of axes of independent probability is not applicable, as is fairly evident from the diagrams, to any of the distributions. By considering a set of principal axes such as $O\xi$, $O\eta$—to be described later on—we can easily convince ourselves that whereas sections parallel to the major axis might be homoscedastic and similar, sections parallel to the minor axis, if not dissimilar, are heteroscedastic.

If these sections, reduced to homoscedasticity, were found to be similar, an extension of Narumi's hypothesis would be justified.

The set of principal axes which would be the most likely to give similar parallel sections, was that through the modes of the distributions. Accordingly it was first of all necessary to locate the positions of the modes as accurately as possible. This was done by the one or the other of the two following methods according to the skewness of the distribution. Both methods depend on the modes of the

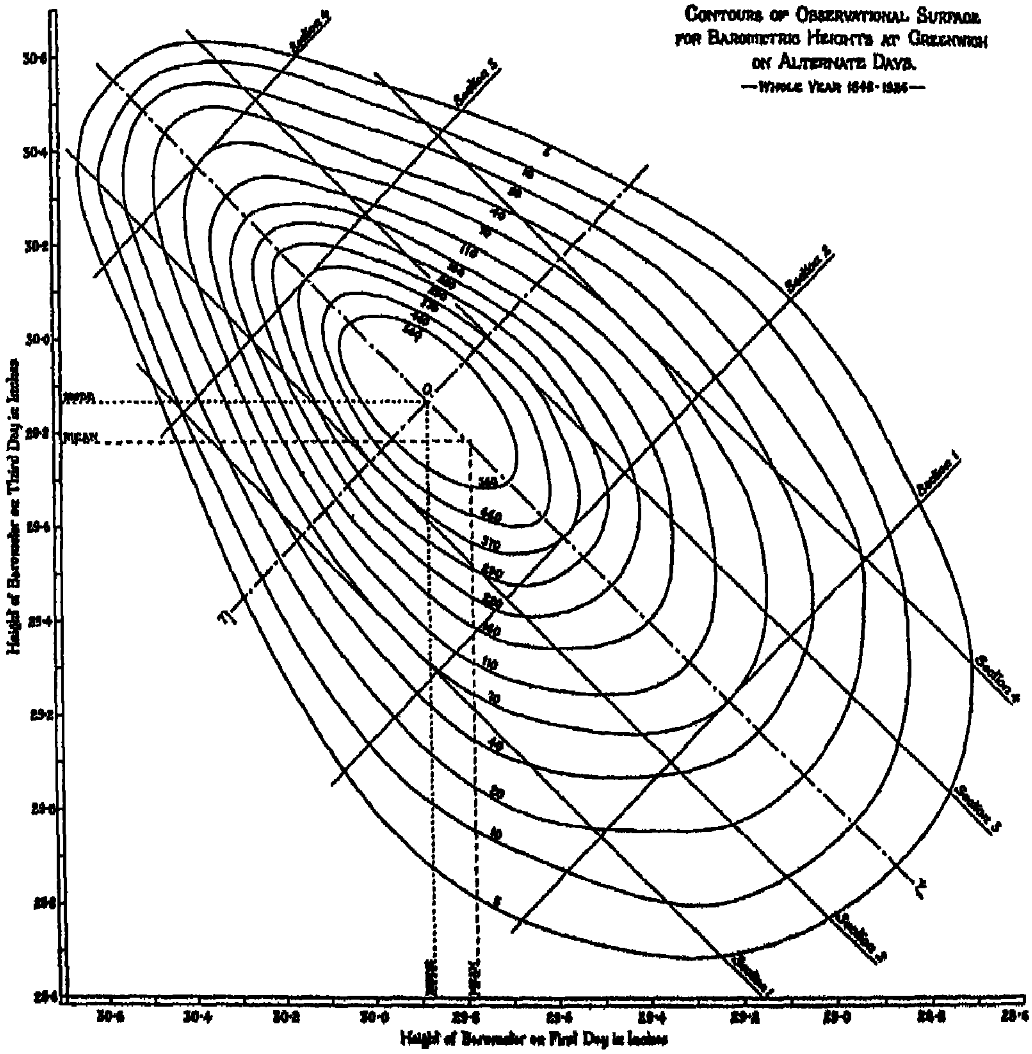


Diagram III.

arrays, centred about the mode of the surface, first being determined. Usually four rows and four columns were found to be adequate. The intersection of the curves—either straight lines or parabolas of the second or higher order—fitted to the modes of these arrays was taken to be the mode of the surface. The modes of the arrays were determined either from the β-formula expressing the distance between the mean and mode for the Pearsonian system of curves, or, by fitting a cubic by the method of least squares to the six largest frequencies in the separate arrays.

The latter method was used for the first two distributions; the former for the last four. Through the modes thus located the sets of principal axes $O\xi$, $O\eta$ were constructed. The following table gives the positions of the modes, and the angle (θ) $O\xi$ makes with the x-axis.

Table I	x mode $=21{\cdot}20$ years y mode $=23{\cdot}70$,, $\theta=51°\ 33'$	Table IV	x mode $=29{\cdot}86''$ y mode $=29{\cdot}86''$ $\theta=44°\ 28'$
Table II	x mode $=27{\cdot}08$ years y mode $=28{\cdot}95$,, $\theta=40°\ 19'$	Table V	x mode $=29{\cdot}96''$ y mode $=29{\cdot}95''$ $\theta=44°\ 29'$
Table III	x mode $=29{\cdot}87''$ y mode $=29{\cdot}86''$ $\theta=44°\ 36'$	Table VI	x mode $=14{\cdot}62$ mm. y mode $=8{\cdot}04$ mm. $\theta=34°\ 48'$

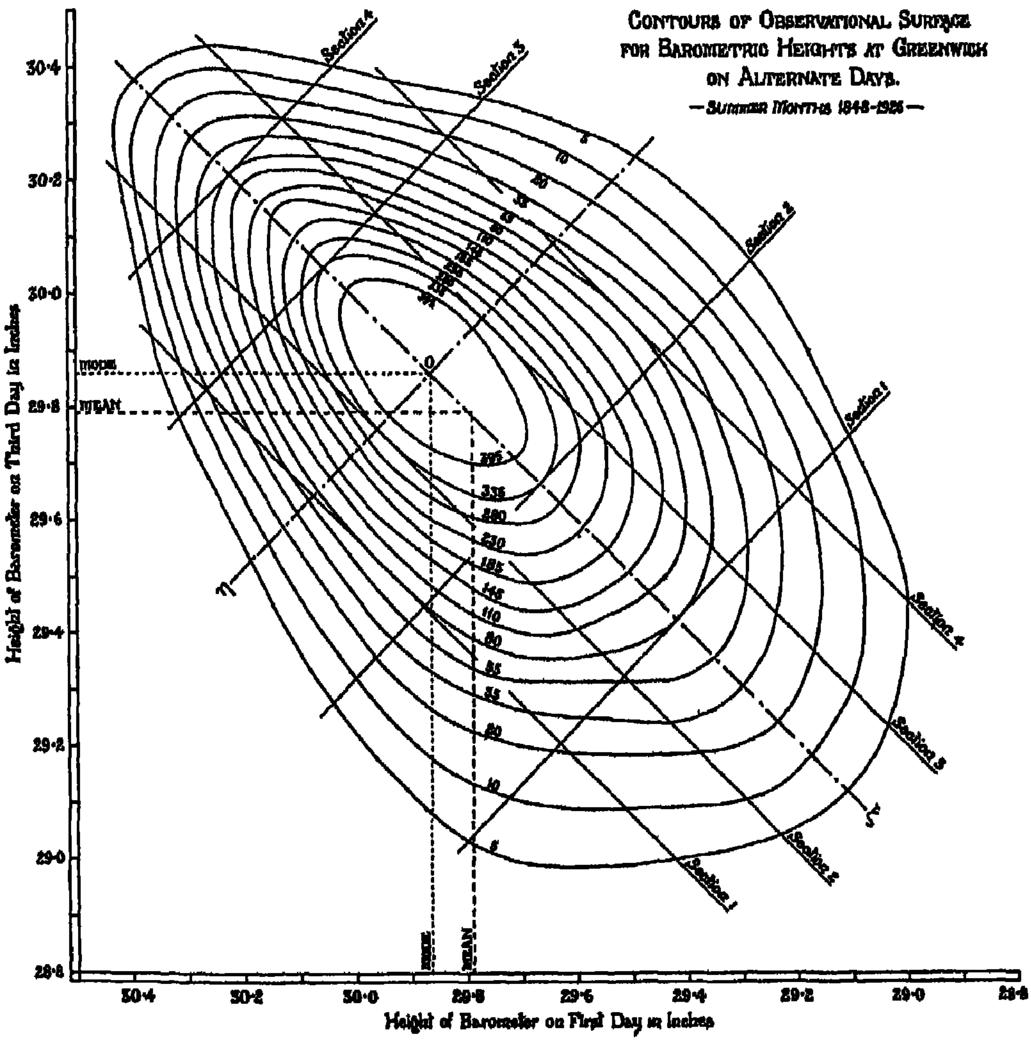


Diagram IV.

There is one interesting feature about the axes that might be pointed out. In five out of the six cases the major axis ($O\xi$) passes, if not completely then very nearly, through the observed mean. Accordingly, we could almost just as well have considered the principal axes through the mean instead of those through the mode.

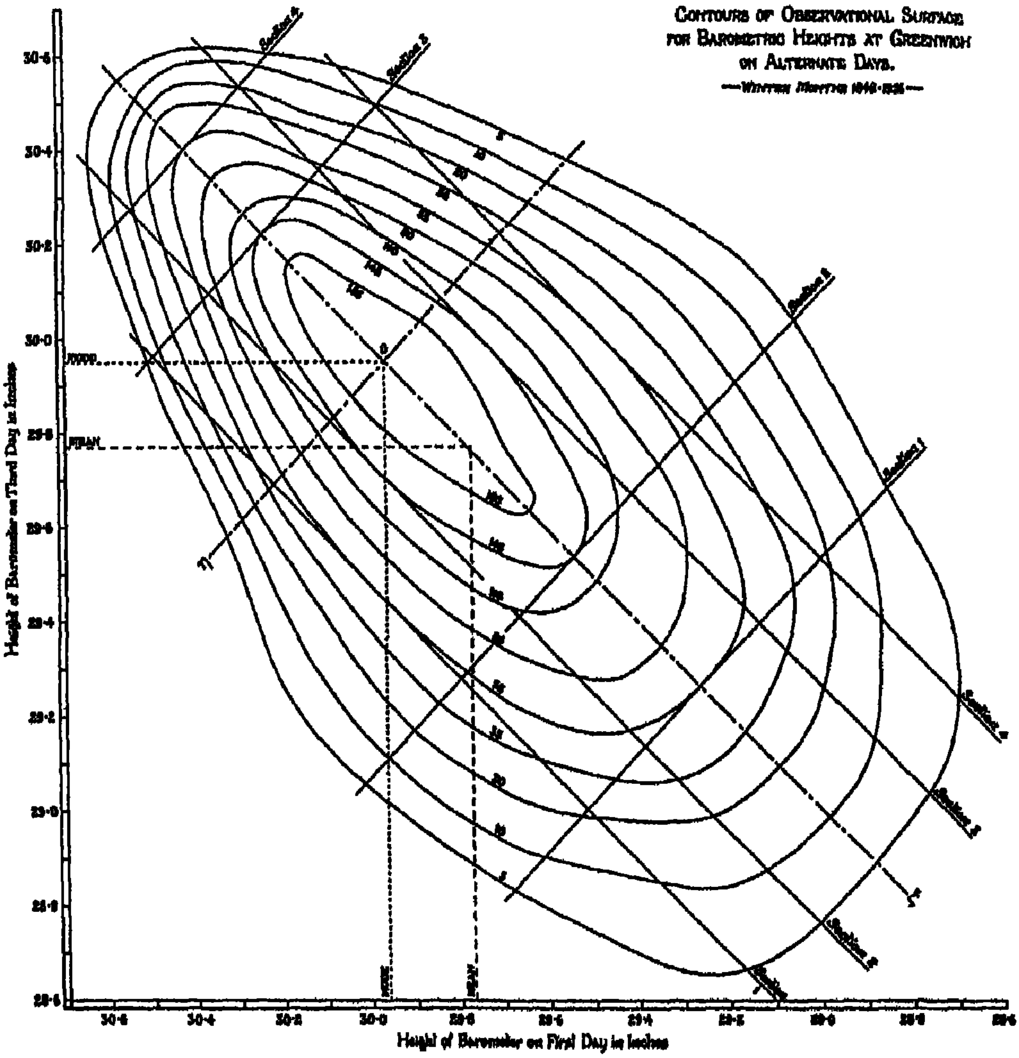


Diagram V.

We proceed to test sections parallel to the axes for similarity. Eight sections, two on each side of an axis, were examined for each of the first five contour systems. The cuts were made in such a way that they touched two of the contour lines. This at once fixed the modal ordinates and defined the vertical scale of the sections. The tails of the section-curves were determined from the array-curves originally used for plotting the contours. The area under each curve was subdivided and planimetered, whence the first two moments were computed. Next, the standard deviations and modal ordinates of each set of four sections were

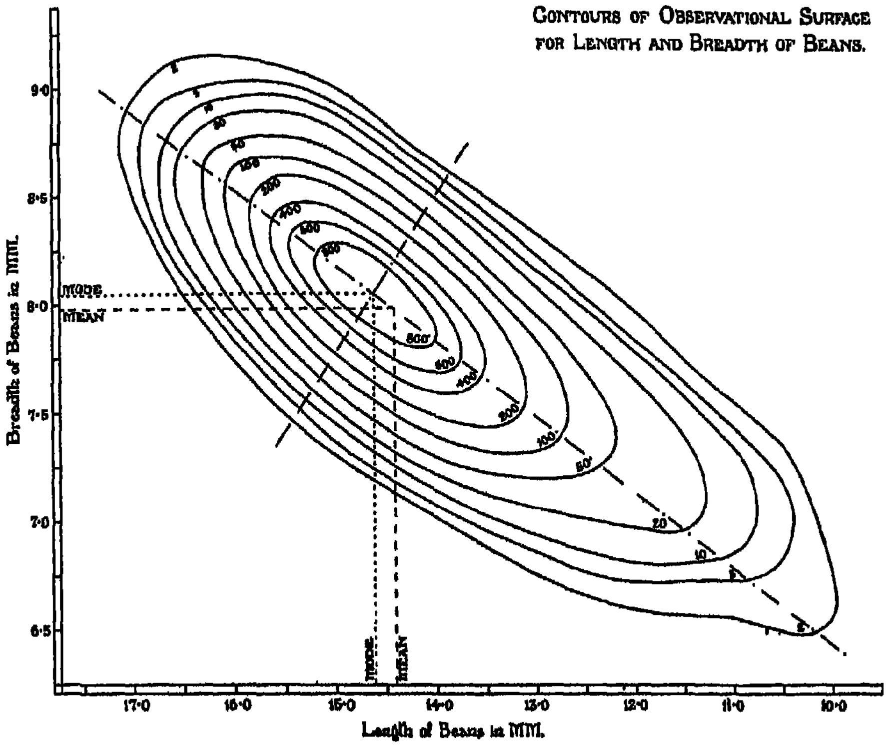


Diagram VI.

equalised, and the two sections differing most were superposed. Diagrams I (i), I (j) to V (i), V (j) show the results. As an illustration, I take the distribution of barometric heights (whole year). The modal ordinate, standard deviation, and skewness (Sk.) $= \dfrac{\text{mean} - \text{mode}}{\text{standard deviation}}$ of the sections are given below.

	Sections parallel to $O\xi$				Sections parallel to $O\eta$			
	Section 1	Section 2	Section 3	Section 4	Section 1	Section 2	Section 3	Section 4
Mod. Ord.	70	370	370	70	70	370	370	70
S. D.*	3·77	3·79	3·81	3·79	2·74	2·26	1·52	1·40
Sk.	+·11	+·20	+·17	+·24	+·02	+·06	−·02	+·03

The skewness, as measured by Sk., being not sensitive to deviations in the vertical direction, does not indicate clearly the degree of similarity between the sections. A direct comparison of corresponding ordinates of the sections was possible after

* The unit in which the standard deviations are measured corresponds in scale to the grouping unit of the observed variables.

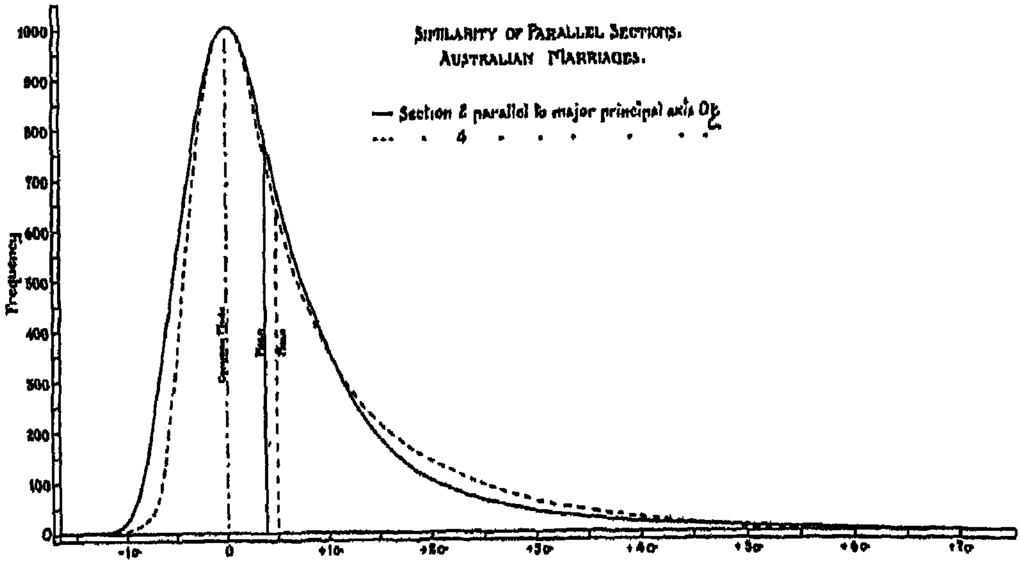


Diagram I (ξ).

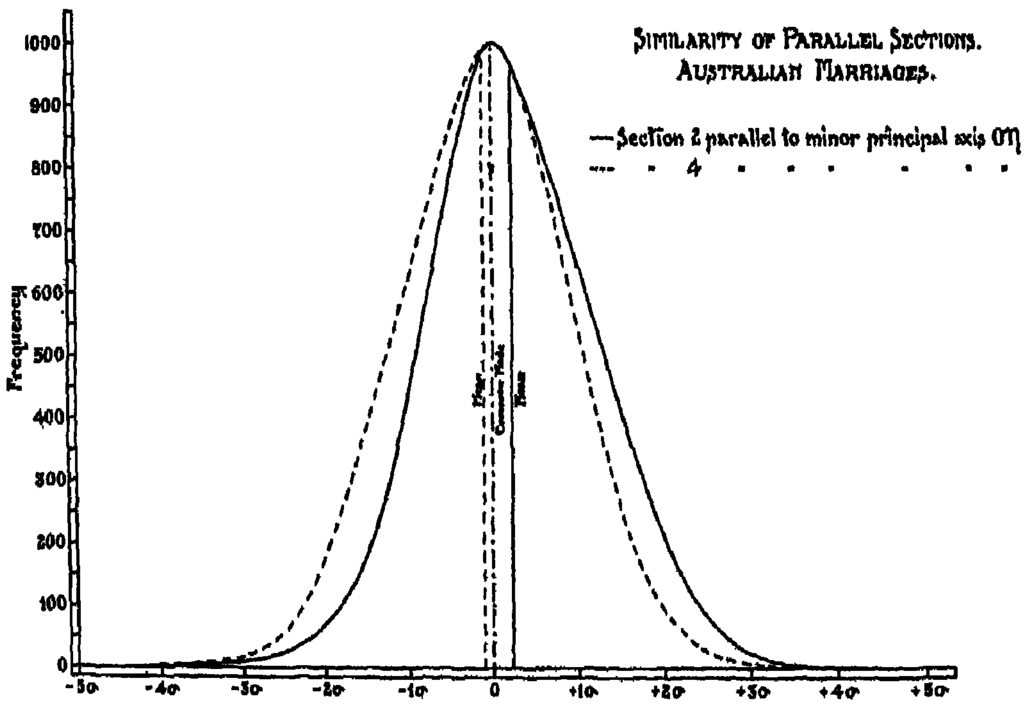


Diagram I (η).

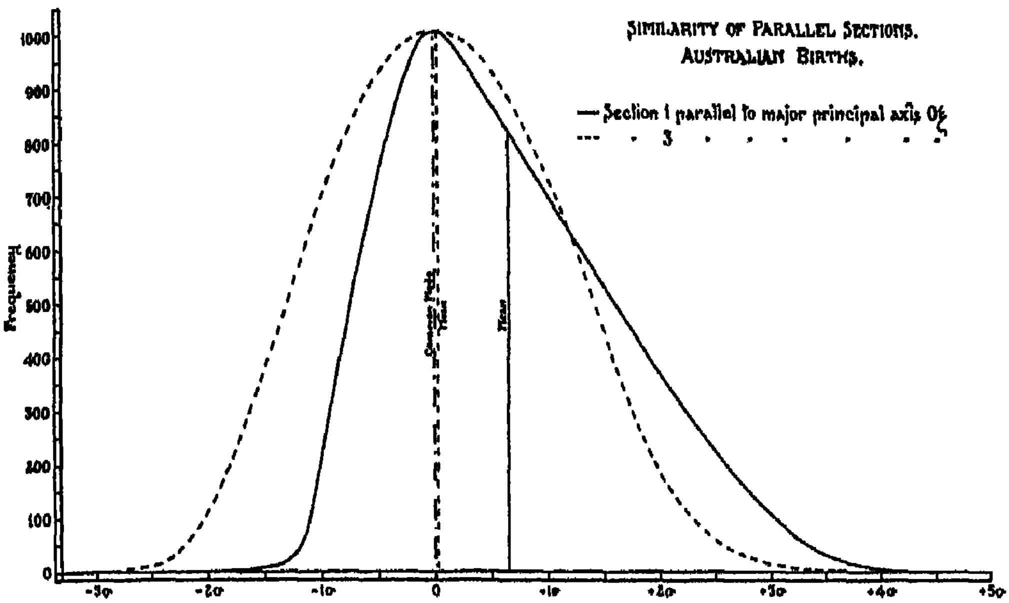


Diagram II (i).

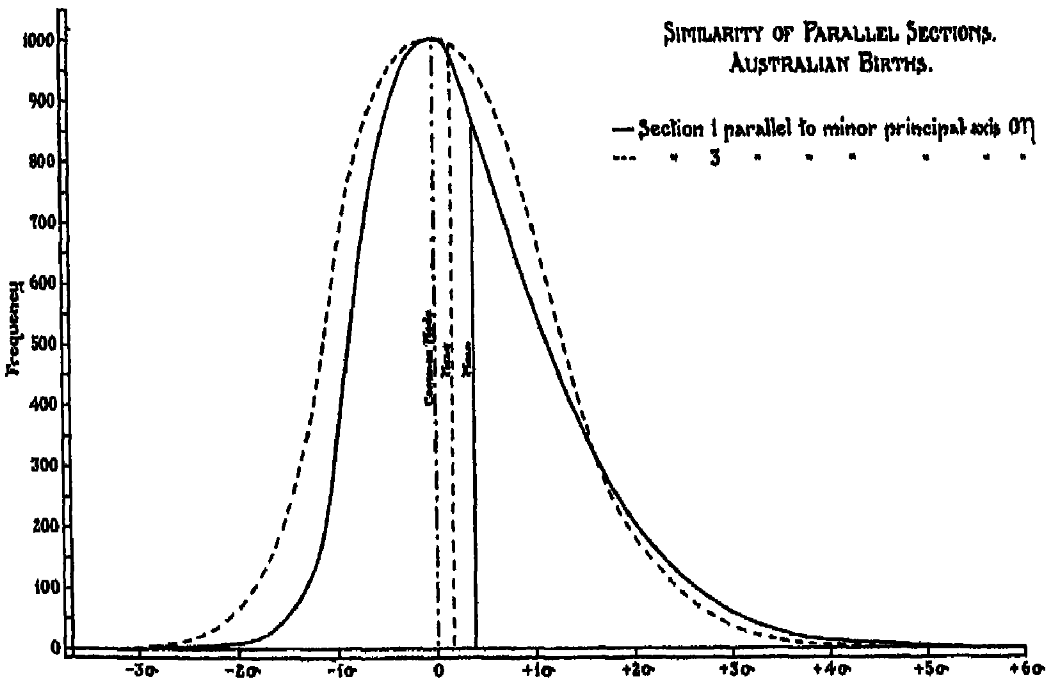


Diagram II (j).

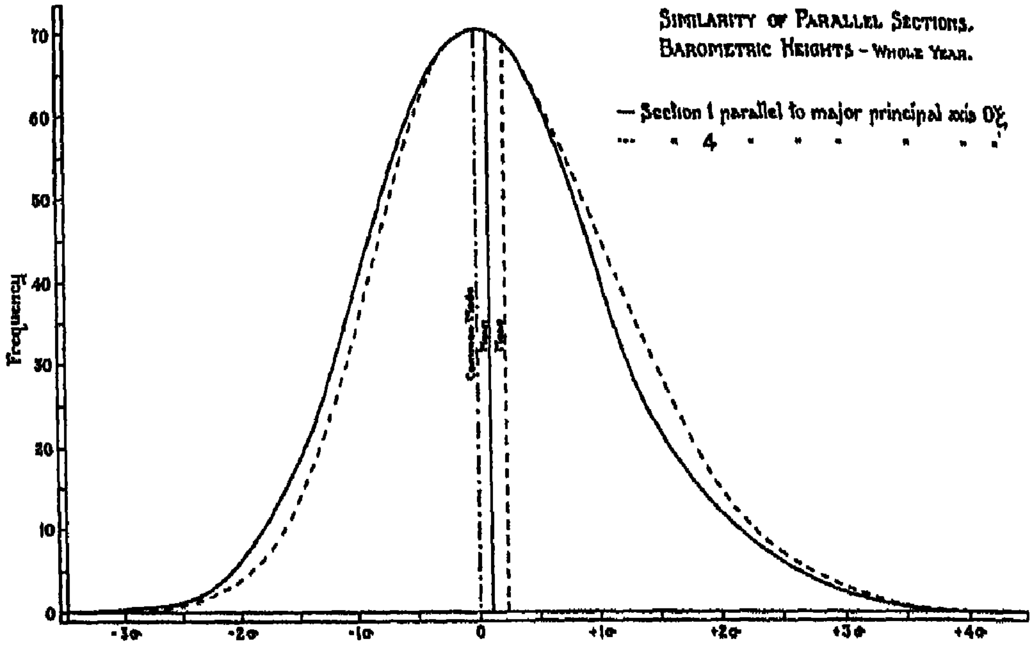


Diagram III (*i*).

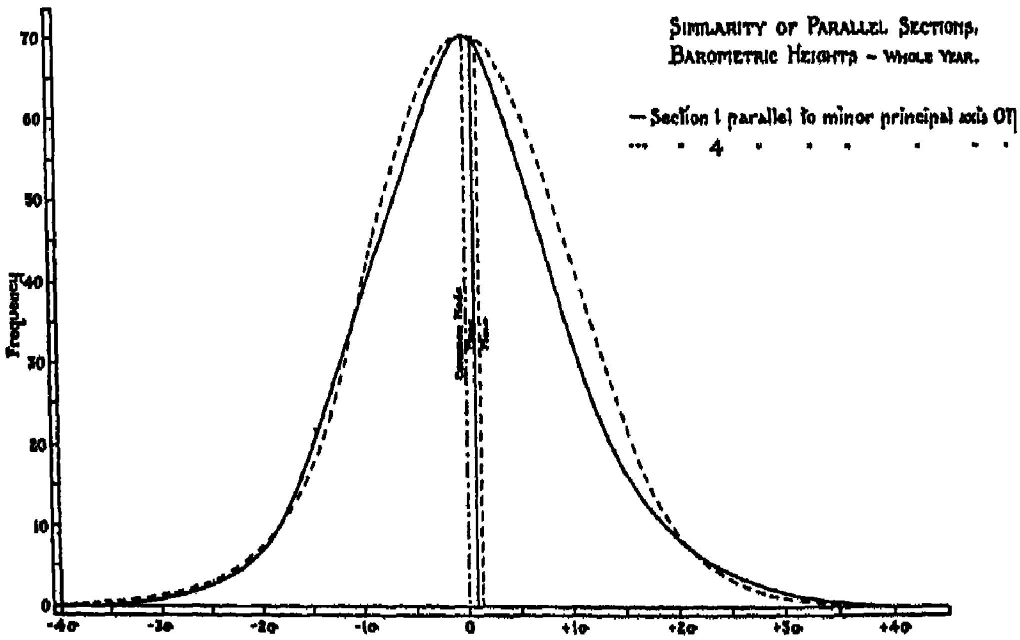


Diagram III (*j*).

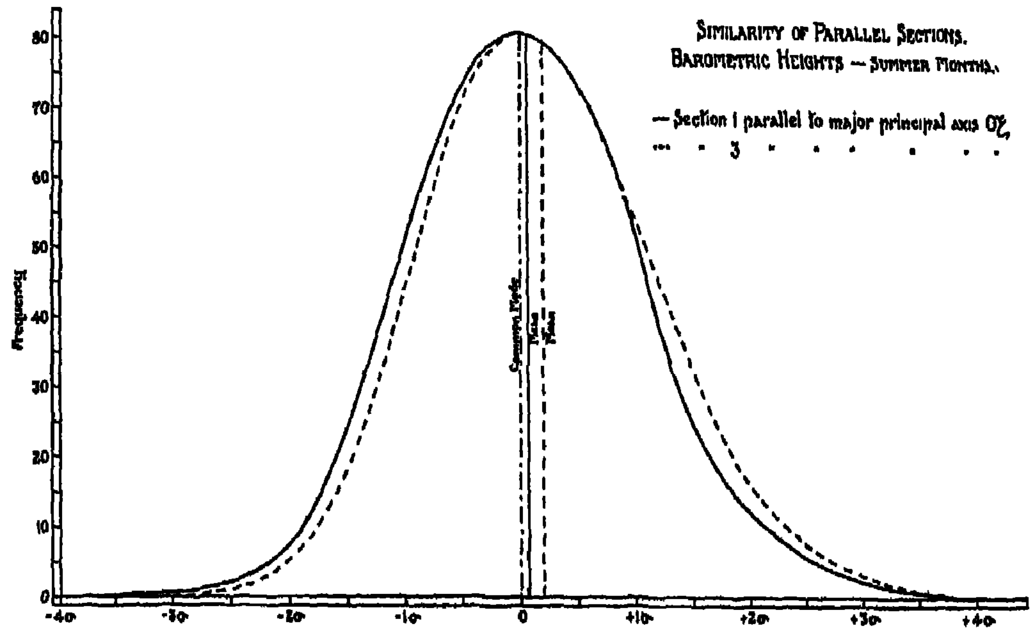


Diagram IV (*i*).

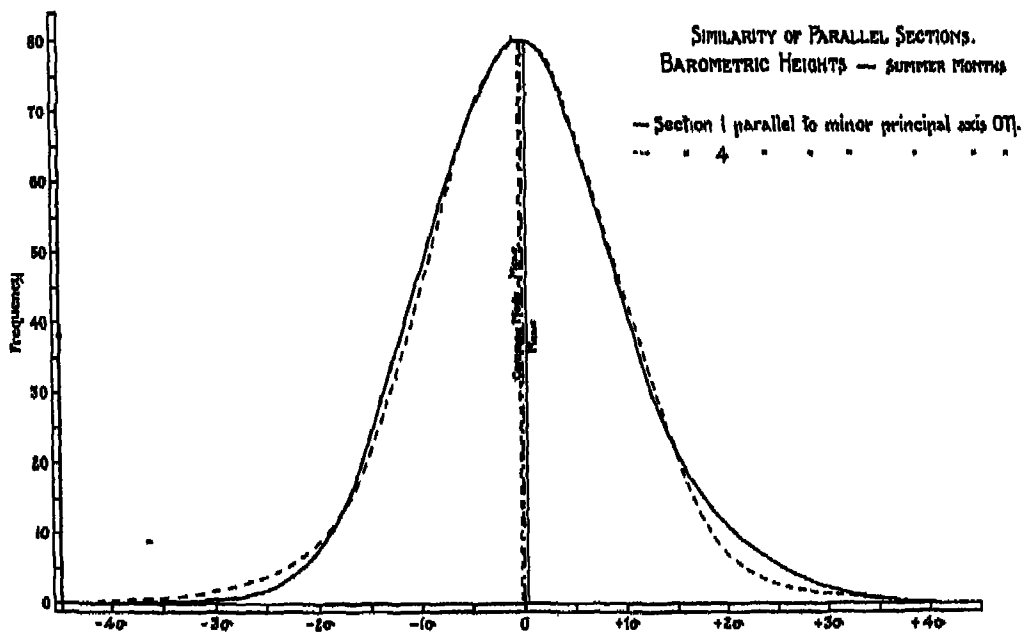


Diagram IV (*j*).

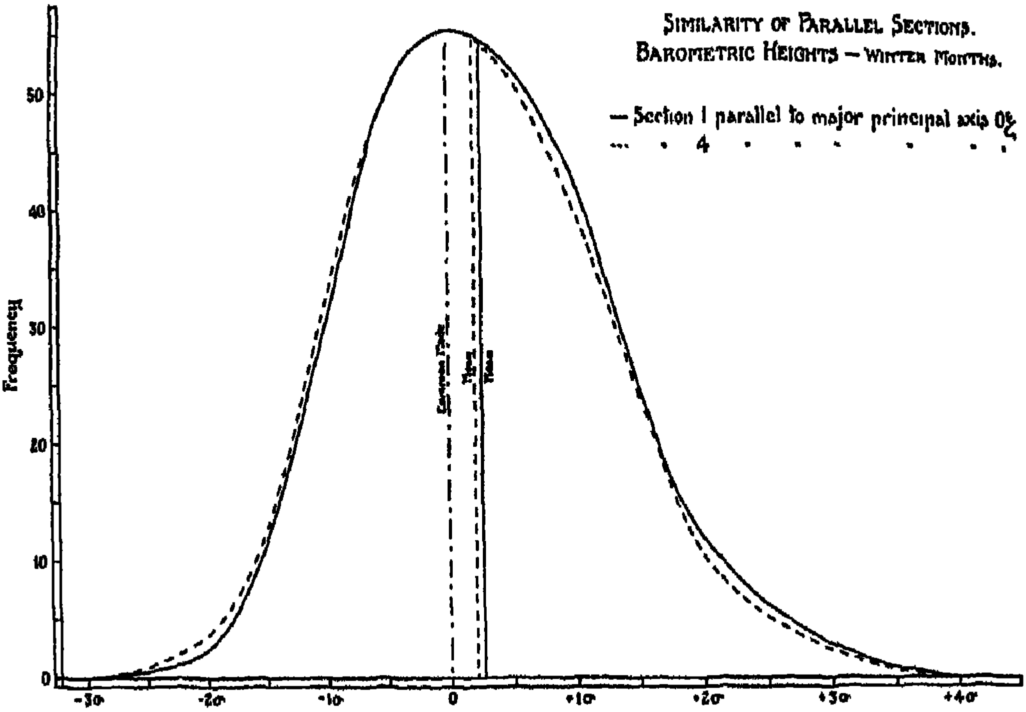


Diagram V (i).

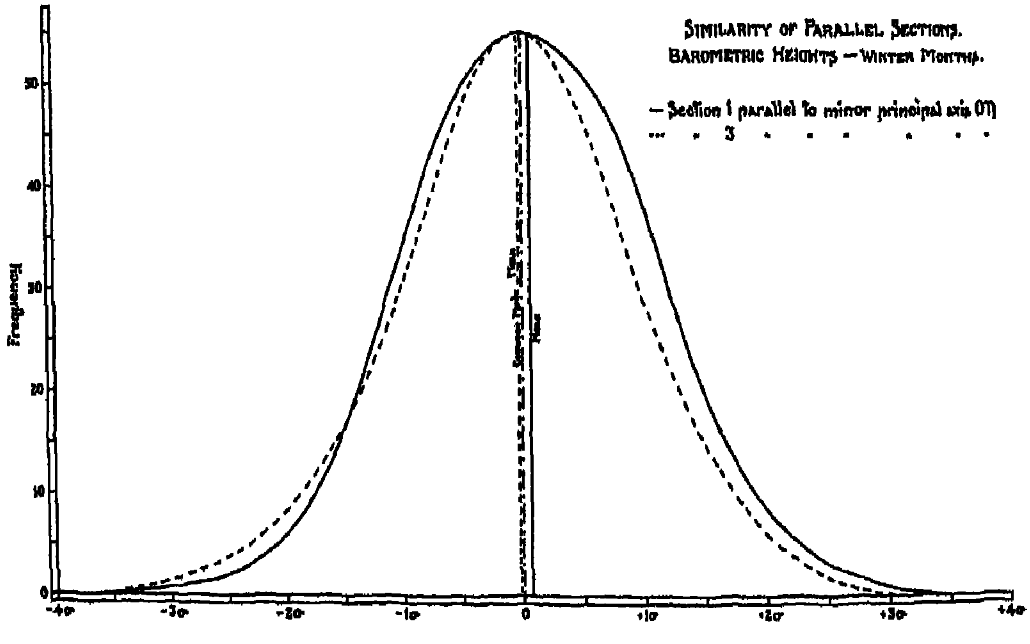


Diagram V (j).

both scales of measurement had been equalised, and on this the decision as to which two sections differ most was based. The sections parallel to $O\xi$ are, to all intents and purposes, homoscedastic; but this is the only instance where such a marked consistency was found. A rough appreciation of the heteroscedasticity of sections can be obtained from the range between the ordinates defined by, say, the lowest contour. Taking the sections parallel to $O\eta$, I find the range* of the most outlying contour (5) to have the following values:

Section 1, 12·37; Section 2, 14·04; Section 3, 9·43; Section 4, 6·12.

The ratio of the ranges of Sections 1 and 4, and of Sections 2 and 3 is ·495 and 672 respectively; while the corresponding ratios of the standard deviations are ·511 and ·673. Such a close agreement can, of course, not always be expected; but for determining only approximately the "scedasticity," I think this method will be quite adequate.

The only contour systems about whose asymmetry there could have been some doubt are those of the barometric height distributions. The superposed sections, however, are of help in this issue. It is true that they do not definitely disprove the validity of an extended Narumi's hypothesis, but they show at least that a good fit cannot be expected in these cases from surfaces based on such an assumption. The hypothesis is not disproved because of the absence of a regular variation in the shape of the sections; it is not sanctioned because of the presence of rather significant irregularities.

4. *Identical Relations between the Moments.* The application of (*a*) Edgeworth's method of simple translation, (*b*) the Logarithmic surface, and (*c*) Rhodes' surface, depends on certain relations connecting the moments of the observed distributions being fulfilled. These criteria are contained in equations (29), (32), (68), and (54) respectively. The term "conditioned" will be used to designate the value of a constant computed from these equations.

(*a*) Equations (29) can be written in forms more convenient for treatment. On combining them, we find

$$\left.\begin{aligned}\sqrt{\beta_{10}} &\equiv q_{30} = \frac{3(2q_{21} - rq_{12})}{r(4 - r^2)}\\ \sqrt{\beta_{01}} &\equiv q_{03} = \frac{3(2q_{12} - rq_{21})}{r(4 - r^2)}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(70).$$

If now, we assign to r, q_{21} and q_{12} in (70) their observed values, the "conditioned values" of β_{10} and β_{01} will be directly comparable with the observed as in the table on p. 206.

To bring out the significance of the difference between the two sets of values, I have tabulated in the last two rows the ratio

$$E = \frac{\text{Observed } \beta_1 - \text{"conditioned" } \beta_1}{\text{Standard Error† of "conditioned" } \beta_1}.$$

* Measured in the same unit as the standard deviations of the sections.

† Found from Table XXXVII in *Tables for Statisticians and Biometricians*, Part I.

The divergence testifies to Edgeworth's remark that simple translation is inadequate (p. 130).

		Table I	Table II	Table III	Table IV	Table V	Table VI
Observed	β_{10} β_{01}	4·058 3·854	·101 ·525	·203 ·204	·202 ·187	·142 ·143	·820 ·194
"Conditioned"	β_{10} β_{01}	6·502 4·958	·150 ·225	·084 ·128	·104 ·151	·035 ·059	1·072 ·314
E	β_{10} β_{01}	— —	32 71	12 6·2	6·5 2·1	23 12	2·3 3·6

(*b*) The determination of the constants of the Logarithmic Surface depends on moments and product moments up to the third order only. The relation between the fourth and lower moments is expressed in terms of the β's according to equations (68), whence the "conditioned" values of β_2 were calculated. For examining the fourth order product moments I assigned to the constants in equations (32) their observed values and then deduced the absolute product moments (q_{rs}) about the mean as origin. Only two distributions were considered, for it was clear from Diagram (4) that while the Logarithmic Normal Curve might hold for one of the marginal totals of some of the other distributions, it does not hold for both of them. The results summarised in the table below point to an approximate fulfilment of the relations between the moments. The effect, however, of this approximation on the shape of the surface had still to be investigated, and it was primarily with this end in view that I fitted the partial moment and marginal curves (pp. 208—218). The outcome, in short, is this: (i) that, if the third order moments and product moments are sufficient to give an account of the regression, they fail to do so for the scedasticity, clisy and kurtosis; (ii) that the relatively small deviation in β_{20} of the barometric data is sufficient to rank the Logarithmic Normal Curve as last, for goodness of fit, amongst four theoretical

		Table I	Table III
Observed	β_{20} β_{02}	9·290 8·333	3·398 3·399
"Conditioned"	β_{20} β_{02}	10·990 10·554	3·363 3·364
E	β_{20} β_{02}	— —	·59 ·59

		Table I	Table III
Observed	q_{13} q_{22} q_{31}	6·297 6·346 7·070	1·865 1·817 1·881
"Conditioned"	q_{13} q_{22} q_{31}	8·022 7·580 8·722	2·111 1·866 1·777

curves, the other three curves having all four moments equal to those observed; (iii) that notwithstanding the large deviations in the β_2's of the marriage data the Logarithmic Normal Curve gives as good a representation of the margins as the best-fitting Pearson curves. The equality of the fits in this case can be explained on the ground that when the β's are very high, the form of the frequency curve is not very sensitive to a change in them. Otherwise, we are bound to conclude that the agreement between the observed and "conditioned" values of the fourth order moments is not close enough for justifying the application of the logarithmic surface.

(c) The criteria justifying the use of Dr Rhodes' surface

$$\left.\begin{aligned} q_{21} - r\sqrt{\beta_{10}} &= \sqrt{\tfrac{1}{3}(1-r^2)(2\beta_{20}-3\beta_{10}-6)} \\ q_{12} - r\sqrt{\beta_{01}} &= \sqrt{\tfrac{1}{3}(1-r^2)(2\beta_{02}-3\beta_{01}-6)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(71)^*$$

were examined for all six distributions. In each case the "conditioned" β_2 was found by assuming for the other constants their observed values. In the table below E has the same meaning as in (a).

		Table I	Table II	Table III	Table IV	Table V	Table VI
Observed	β_{20}	9·290	2·430	3·398	3·274	2·862	4·863
	β_{02}	8·333	3·624	3·390	3·215	2·862	3·654
"Conditioned"	β_{20}	9·157	3·157	3·332	3·319	3·249	4·257
	β_{02}	8·803	3·960	3·325	3·288	3·243	3·388
E	β_{20}	—	80	1·2	·56	5	2·8
	β_{02}	—	14	1·3	·98	5	2·5

The conditions are satisfied best for the data of Tables I and IV; for Table I certainly far better than were the conditions of the Logarithmic Surface. Unfortunately, the partial moment and marginal curves, not being expressed in finite form, cannot be directly discussed. The disagreement obtained in the case of Table IV will be sufficient, I think, to affect appreciably the goodness of fit of the surface.

5. *The Marginal Distributions.* Pearson's system of curves, the Type A series, Edgeworth's translated curves, and the Logarithmic Normal Curve were used in specifying these distributions. Edgeworth's curves do not appear as marginal totals in his translated surfaces, but being at the basis of the method of translation as applied to correlation, they could not be left out of consideration.

* The equations show that, strictly, the surface cannot describe distributions whose marginal totals have β's lying below the Type III line, i.e. in the Type I area.

The probability integral of the Type A series is

$$\int_{-\infty}^{t} z_x dx = N\left[\tfrac{1}{2}(1 \pm \alpha_t) - \frac{1}{\sqrt{6}}\sqrt{\beta_{10}}\,.\,\tau_3 - \frac{1}{\sqrt{24}}(\beta_{20}-3)\,\tau_4 - \frac{1}{\sqrt{120}}(\beta'_{30} - 10\sqrt{\beta_{10}})\,\tau_5 - \frac{1}{\sqrt{720}}(\beta_4 - 15\beta_2 + 30)\,\tau_6 - \ldots\right]$$

$$\equiv N\left[\tfrac{1}{2}(1 \pm \alpha_t) - a_3\tau_3 - a_4\tau_4 - a_5\tau_5 - a_6\tau_6 - \ldots\right] \quad \text{.................(72)},$$

where $t = x/\sigma_1$.

The equations of Edgeworth's curves and that of the Logarithmic Normal Curve are

$$\left.\begin{aligned} y &= \frac{N}{\sqrt{\pi}}\,.\,e^{-\xi^2} \\ x &= a(\xi + k\xi^2 + \lambda\xi^3) \end{aligned}\right\} \quad \text{..................................(73)},$$

and

$$y = \frac{N \log e}{\sqrt{2\pi}\,.\,s_1}\,.\,\frac{1}{x}\,.\,e^{-\frac{1}{2}\left(\frac{\log x - l_1}{s_1}\right)^2} \quad \text{........................(74)}.$$

In referring to a particular frequency group, I shall state only its central value.

Marriage Statistics. The β's of the margins are

$$\beta_{10} = 4{\cdot}057{,}531, \quad \beta_{01} = 3{\cdot}853{,}680,$$
$$\beta_{20} = 9{\cdot}290{,}441, \quad \beta_{02} = 8{\cdot}332{,}812.$$

Turning to Diagram (1) we notice that both pairs of β's fall outside the area demarcated for the application of Edgeworth's curves. They suggest further the application of the Logarithmic Normal rather than of Type Aa. But before passing on to fitting that curve and the Pearson curves, we shall consider the correction of the moments for abruptness.

The sub-frequencies necessary for applying these corrections were obtained from the original table in Knibbs' *Mathematical Theory of Population*, the sub-intervals being single years. Discarding in the distribution of age of bride the observations in the lowest age-group, I find for the uncorrected and corrected moments about the beginning of the age-group 15·5 the following values:

Uncorrected	Sheppard's Corrections applied	Abruptness Corrections applied
$\nu_1' = 3{\cdot}907{,}280$	$\mu_1' = 3{\cdot}907{,}280$	$\mu_1' = 3{\cdot}907{,}140$
$\nu_2' = 20{\cdot}635{,}594$	$\mu_2' = 20{\cdot}282{,}261$	$\mu_2' = 20{\cdot}282{,}190$
$\nu_3' = 142{\cdot}046{,}089$	$\mu_3' = 141{\cdot}069{,}269$	$\mu_3' = 141{\cdot}069{,}626$

The moments of the distribution of age of bridegroom about the beginning of age-group 16·5 are:

Uncorrected	Sheppard's Corrections applied	Abruptness Corrections applied
$\nu_1' = 4{\cdot}794{,}355$	$\mu_1' = 4{\cdot}794{,}355$	$\mu_1' = 4{\cdot}794{,}353$
$\nu_2' = 30{\cdot}042{,}820$	$\mu_2' = 29{\cdot}959{,}486$	$\mu_2' = 29{\cdot}959{,}558$
$\nu_3' = 247{\cdot}854{,}858$	$\mu_3' = 246{\cdot}656{,}269$	$\mu_3' = 246{\cdot}656{,}423$

The slight difference between the values of the corresponding moments in the last two columns establishes for these cases the sufficiency of Sheppard's corrections.

(*a*) *x-Margin.* (i) The β's suggest a Pearson Type VI curve. The curve was fitted by fixing its start and by determining its constants from the first three moments. As a first trial the start was fixed at the beginning of the age-group 15·5; but this gave a too high frequency in the first group. The start was then fixed at a distance of 3·7 grouping units from the mean, i.e. at age = 14·622 years. This led to

$$y = y_0 (x - 5{\cdot}26753)^{4{\cdot}34030} \,.\, x^{-13{\cdot}96486},$$

where $\log y_0 = 15{\cdot}584{,}963$, the β's of the curve being

$$\beta_{10} = 4{\cdot}058{,}346, \quad \beta_{20} = 11{\cdot}706{,}379.$$

(ii) The constants of the Logarithmic Normal Curve found from equations (12) and (13) are

$$\xi_1 = 3{\cdot}73516, \quad l_1 = {\cdot}50562, \quad s_1 = {\cdot}24068.$$

The curve starts at 14·516 years, and its β's are

$$\beta_{10} = 4{\cdot}057{,}531, \quad \beta_{20} = 10{\cdot}990.$$

The frequencies are shown in Table IX. Notwithstanding a relatively large difference between the β_2's, the curves are markedly similar, there being not much to choose between them. Both curves tail off too slowly at the higher age-groups and fit badly at the lower; but here the material is undoubtedly spurious. Taking the largeness of the numbers into consideration, I think the graduation is not unsatisfactory.

(*b*) *y-Margin.* (i) Instead of the Type I_1 indicated by the β's, I tried a Type III and Type I_2 by fixing arbitrarily the start of the former curve, and the start, range or mode of the latter. But not one of the many trials that were made produced reasonable results; the curves could not be made to give correspondence at the tails as well as at the maximum frequencies. Finally a Type III

$$x = x_0 \,.\, e^{-\gamma y} (1 + y/a)^p$$

was fitted with start at the centre of the age-group 19·5. The position of the mode* was fixed by choosing $a = 1{\cdot}76$; while in assigning to p the value 1·2, I paid more attention to the modal frequency of the distribution than to its variability. The other constants are

$$\gamma = {\cdot}68182, \quad x_0 = 70004{\cdot}6,$$

the β's of the curve being $\beta_{01} = 1{\cdot}818$, $\beta_{02} = 5{\cdot}727$.

(ii) The Logarithmic Normal Curve fitted from the first three moments has

$$\eta_1 = 4{\cdot}49910, \quad l_2 = {\cdot}58884, \quad s_2 = {\cdot}23629, \quad \beta_{02} = 10{\cdot}554,$$

its start being at 15·886 years.

* A smoothing cubic fitted to the five largest frequencies about the mode gave the position of the mode at 1·8 grouping units from the centre of age-group 19·5.

TABLE IX.

Marginal Distributions of Ages of Brides and Bridegrooms, Australian Marriages.

	Brides			Bridegrooms			
Age of Bride	Observed Frequency	Theor. Freq. Type VI	Theor. Freq Log. Normal	Age of Bridegroom	Observed Frequency	Theor. Freq. Type III	Theor. Freq Log. Normal
12·5	5	—	—	16·5	294	—	259
15·5	2,975	3,412	2,207	19·5	10,995	9,256	19,453
18·5	38,291	44,620	44,776	22·5	61,001	57,339	56,819
21·5	80,847	74,649	77,051	25·5	73,054	68,434	64,141
24·5	71,010	64,826	64,894	28·5	56,501	56,867	51,989
27·5	44,541	44,346	43,568	31·5	33,478	40,827	36,894
30·5	24,261	27,603	27,008	34·5	20,569	27,073	24,735
33·5	13,752	16,590	16,318	37·5	14,281	17,075	16,203
36·5	8,883	9,896	9,831	40·5	9,320	10,405	10,542
39·5	6,062	5,936	5,966	43·5	6,236	6,184	6,869
42·5	3,478	3,605	3,664	46·5	4,770	3,605	4,503
45·5	2,605	2,222	2,281	49·5	3,620	2,068	2,977
48·5	1,805	1,393	1,442	52·5	2,190	1,177	1,987
51·5	1,139	888	924	55·5	1,655	660	1,339
54·5	645	575	601	58·5	1,100	367	912
57·5	513	379	396	61·5	810	203	627
60·5	291	253	264	64·5	649	112	435
63·5	242	172	179	67·5	487	61	305
66·5	206	118	122	70·5	326	33	216
69·5	130	82	84	73·5	211	18	154
72·5	56	58	59	76·5	119	10	111
75·5	25	41	41	79·5	73		80
78·5	16	30	29	82·5	27	11	59
81·5	6	22	21	85·5	14		43
84·5	1	16	15	88·5	6	—	32
—	—	48	44	—	—	—	101
—	—			—	—	—	
Totals	301,785	301,780	301,785	—	301,785	301,785	301,785
χ^2	—	—	—	—	—	—	—
P	—	—	—	—	—	—	—

The frequencies are seen in Table IX. As could have been expected the Type III fails to give a description of the long tail; while the Logarithmic Normal, being better here than the Type III, is inferior to it at the mode. The divergence of the β's of the curves from those observed is noteworthy.

Birth Statistics. (*a*) *x-Margin.* The observed β's

$$\beta_{10} = \cdot 100{,}603, \quad \beta_2 = 2{\cdot}430{,}327$$

fall outside the compass of Edgeworth's method of translation; the distribution was used in Section C, 2 simply for illustrating the type of singularity to which the method of translation is subject.

(i) The equation of the Type I with start fixed at age 15·60*, and fitted from the first three moments, is

$$y = 75911{\cdot}6\left(1 + \frac{x}{6{\cdot}12055}\right)^{1{\cdot}81499}\left(1 - \frac{x}{12{\cdot}45834}\right)^{3{\cdot}69440}$$

TABLE X.

Marginal Distribution of Ages of Mothers, Australian Births.

Age of Mother	Observed Frequency	Theor. Freq. Type I	Theor. Freq. Type Aa
—	—	—	
—	—	—	−1766
—	—	—	
13·0	3	—	391
15·0	191	46	3,570
17·0	4,573	6,105	10,482
19·0	21,322	22,871	22,049
21·0	42,758	41,996	37,545
23·0	62,620	58,455	54,745
25·0	73,423	69,796	67,472
27·0	74,834	75,176	75,848
29·0	72,640	74,822	76,574
31·0	65,182	69,637	71,313
33·0	58,407	60,909	62,121
35·0	48,834	50,071	50,833
37·0	39,932	38,524	38,804
39·0	31,050	27,485	27,222
41·0	18,975	17,878	17,237
43·0	11,283	10,359	9,678
45·0	4,365	5,088	4,734
47·0	1,072	1,943	1,973
49·0	199	476	676
51·0	13	44	175
53·0	4	1	6
55·0	2	—	
Totals	631,682	631,682	631,682
χ^2	—	—	—
P	—	—	—

(ii) For the Type Aa probability integral we have

$$a_3 = {\cdot}129{,}488, \quad a_4 = -{\cdot}116{,}284.$$

The theoretical frequencies tabulated in relation to the observed in Table X show a superiority of the Type I over the Type Aa; that curve, though, does not fit very well itself.

* Again many experiments were made to find the best-fitting curve.

(*b*) *y-Margin.* (i) The Pearson curve determined from the observed β's

$$\beta_{01}=\cdot 524{,}655, \quad \beta_{02}=3\cdot 624{,}169$$

is
$$x=105{,}928\cdot 6\left(1+\frac{y}{4\cdot 80170}\right)^{3\cdot 64155}\left(1-\frac{y}{43\cdot 57116}\right)^{33\cdot 04388}$$

The start of the curve is at 16·089 years.

(ii) The Type A probability integral is

$$\int_{-\infty}^{t} z_y\,.\,dy=N\left[\tfrac{1}{2}(1\pm\alpha_t)-\cdot 295{,}707\tau_3-\cdot 127{,}408\tau_4-\cdot 060{,}222\tau_5-\cdot 234{,}704\tau_6-\ldots\right],$$

where $t=y/\sigma_2$. The higher β's were found from the observations; the series was fitted up to τ_4 only ($\equiv$ Type Aa).

(iii) The parameters in Edgeworth's curve have the following values:

$$\lambda=-\cdot 00717, \quad k=\cdot 17717, \quad a=3\cdot 51098.$$

The median is at 32·567 years.

(iv) For the Logarithmic Normal Curve

$$\eta_1=10\cdot 52763, \quad l_2=1\cdot 01046, \quad s_2=\cdot 10153.$$

The start of the curve is at 1·917 years; the "conditioned" $\beta_{02}=3\cdot 947$.

The corresponding frequencies are compared in Table XI. A "break" occurs in Edgeworth's curve at age 19·1, corresponding to a deviation of $\xi=-2\cdot 456$* from the mean of the normal curve. About one-thousandth part of the area under the normal curve is thus folded over. Beyond this "break" the curve fits very well but loses its advantage of course if the first two groups be taken into consideration; the Type I is then to be preferred. In the tetrachoric series the τ_5 and τ_6 terms are evidently not negligible; while the large difference between the observed β_{02} and that of the Logarithmic Normal Curve accounts for the failure of this curve.

Barometric Heights (*Whole Year*). The two marginal columns being practically similar, I examined only one of them: the *x-margin*. The observed β's

$$\beta_{10}=\cdot 203{,}214, \quad \beta_{20}=3\cdot 397{,}763$$

lie in the Type IV area very close to the Type V† line; they conform also approximately to the relation between the β's of the Logarithmic Normal Curve.

The theoretical curves with their constants are as follows:

(i) Pearson's Type IV:

$$y=y_0\left(1+\frac{x^2}{91\cdot 56833}\right)^{-86\cdot 41804}\times e^{176\cdot 85828\tan^{-1}\frac{x}{9\cdot 56918}},$$

where
$$\log y_0=\overline{57}\cdot 481{,}845.$$

(ii) Type Aa probability integral:

$$a_3=\cdot 184{,}036, \quad a_4=\cdot 081{,}193.$$

(iii) Edgeworth's translated curve:

$$\lambda=\cdot 01045, \quad k=\cdot 10559, \quad a=4\cdot 26484,$$

the median being at 29·803″. The curve presents no singularities in this case.

* There is another break at $\xi=+18\cdot 939$.

† The Type V, with the observed β_{10}, has its $\beta_{20}=3\cdot 385$ and gives $\chi^2=72\cdot 89$.

TABLE XI.

Marginal Distribution of Ages of Fathers, Australian Births.

Age of Father	Observed Frequency	Theor. Freq. Type I	Theor. Freq. Type Aa	Theor. Freq. Edgeworth	Theor. Freq. Log. Normal
—	—	—	-705	—	82
—	—	—		—	
16·5	181	131	3,154	?	1,689
19·5	7,936	9,175	13,464	8,918	11,342
22·5	40,789	39,776	35,168	41,244	36,613
25·5	79,964	76,959	66,936	76,912	71,852
28·5	99,328	100,662	97,806	98,682	99,451
31·5	102,303	104,277	112,097	103,614	107,052
34·5	90,670	92,260	102,082	92,246	95,787
37·5	73,609	72,643	75,335	72,981	74,643
40·5	52,930	52,183	47,500	52,510	52,376
43·5	35,507	34,744	28,799	34,904	33,913
46·5	21,817	21,669	19,101	21,675	20,639
49·5	12,781	12,748	13,527	12,682	11,968
52·5	6,717	7,109	8,920	7,035	6,684
55·5	3,587	3,775	4,991	3,719	3,626
58·5	1,821	1,908	2,294	1,882	1,922
61·5	911	921	862	914	1,001
64·5	489	425	266	427	515
67·5	183	187	68	193	262
70·5	85	79	14	84	132
73·5	38	32	3	35	67
76·5	25	12		15	33
79·5	9	4	—	6	17
82·5	2	3	—	4	8
—	—		—		8
—	—	—	—	—	
Totals	631,682	631,682	631,682	631,682	631,682
χ^2	—	480·4	—	299·4	—
P	—	·000	—	·000	—

(iv) Logarithmic Normal Curve:

$$\xi_1 = 20{\cdot}64737, \quad l_1 = 1{\cdot}31009, \quad s_1 = {\cdot}06442.$$

The curve starts at 31·846″ and its "conditioned" $\beta_{20} = 3{\cdot}363$.

The frequencies are seen in Table XII. It is obvious that the first three curves, with both their β's equal to those of the data, are superior to the Logarithmic Normal Curve—the difference between the observed and computed β_2 being only about ·6 of the standard error of the latter. The values of χ^2 are not directly comparable; the number of categories is less for the Type Aa than for the other representations. If this be borne in mind, then there is not much to choose between the first three sets of results.

TABLE XII.

x-Marginal Distribution of Barometric Heights (Whole Year).

Barometric Height	Observed Frequency	Theor. Freq. Type IV	Theor. Freq. Type Aa	Theor. Freq. Edgeworth	Theor. Freq. Log. Normal
—	—	2	·5	2·6	1·4
30·75	7	13	19·6	14·0	11·0
30·65	13	63	81·3	63·7	58·6
30·55	73	220	243·9	218·7	215·7
30·45	258	583	583·6	579·1	584·8
30·35	563	1219	1165·8	1214·1	1229·5
30·25	1148	2074	1976·6	2071·6	2087·2
30·15	1951	2948	2878·0	2952·8	2957·3
30·05	2951	3595	3623·7	3601·7	3592·0
29·95	3749	3836	3966·9	3840·8	3823·8
29·85	3921	3648	3798·1	3648·3	3633·8
29·75	3700	3140	3209·8	3136·7	3130·4
29·65	3176	2479	2434·0	2475·2	2476·6
29·55	2333	1817	1703·3	1813·4	1819·1
29·45	1752	1247	1143·8	1246·0	1252·2
29·35	1233	810	763·9	809·6	814·4
29·25	813	501	511·2	501·2	503·8
29·15	542	296	333·9	297·4	298·3
29·05	282	169	204·9	170·0	169·9
28·95	189	93	114·6	94·1	93·5
28·85	81	50	57·3	50·6	49·9
28·75	60	27	25·4	26·5	25·9
28·65	43	13	9·8	13·6	13·2
28·55	12	7	3·6	6·8	6·6
28·45	4	6	1·5	3·4	3·2
28·35	1			3·1	2·9
—	—	—	—		
Totals	28,855	28,855	28,855	28,855	28,855
χ^2	—	68·54	61·95	64·96	79·63
P	—	·000	·000	·000	·000

Barometric Heights. x-Margins of (a) Summer and (b) Winter Months.*

(a), (i) The β's satisfy approximately the condition of the Type III cur The resulting equation is

$$y = 2465{\cdot}72 \,.\, e^{-1{\cdot}843864x}\left(1 + \frac{x}{10{\cdot}21891}\right)^{18{\cdot}84804}$$

The start of the curve is at 30·866″.

(ii) The coefficients of the tetrachoric terms are

$$a_3 = {\cdot}183{,}299, \quad a_4 = {\cdot}056{,}014.$$

* Edgeworth's curve was not fitted to these distributions, for they are of more or less the sa degree of skewness as the distribution considered in the last paragraph.

(*b*), (i) The appropriate Pearson curve is

$$y = 1519{\cdot}38\left(1+\frac{x}{8{\cdot}86517}\right)^{3.02656}\left(1-\frac{x}{22{\cdot}08814}\right)^{9.03580}$$

with its start at 30·748″.

(ii) The Type Aa integral has

$$a_3 = {\cdot}153{,}949, \quad a_4 = -{\cdot}028{,}267.$$

The frequencies for both margins are shown in Table XIII. The goodness of fit, as measured by P, manifests the superiority of the Pearson curves over the Type Aa.

TABLE XIII.

x-Marginal Distributions of Barometric Heights (Summer and Winter Months).

	Summer Months			Winter Months		
Barometric Height in inches	Observed Frequency	Theor. Freq. Type III	Theor. Freq. Type Aa	Observed Frequency	Theor. Freq. Type I	Theor. Freq. Type Aa
—	—	—	—	—	—	
30·75	—	—	—	7		−11·1
30·65	—	—	—	13	11·6	48·7
30·55	—			73	81·3	129·7
30·45	10	10·5	11·20	248	250·8	270·2
30·35	81	79·0	102·6	482	508·9	478·3
30·25	338	329·9	344·3	810	811·6	743·9
30·15	830	862·2	825·4	1121	1103·0	1034·1
30·05	1576	1580·3	1513·1	1375	1334·8	1298·8
29·95	2256	2196·3	2182·3	1493	1475·6	1485·7
29·85	2427	2446·4	2516·2	1494	1514·7	1559·0
29·75	2279	2274·1	2353·8	1421	1458·7	1511·7
29·65	1812	1818·4	1827·2	1364	1327·0	1365·8
29·55	1236	1280·4	1226·1	1097	1145·1	1159·7
29·45	822	808·7	756·4	930	939·6	933·6
29·35	484	464·8	452·9	749	733·8	717·7
29·25	252	246·1	262·7	561	545·2	528·8
29·15	122	121·2	140·0	420	384·7	373·2
29·05	47	56·0	64·8	235	257·2	251·2
28·95	27	24·4	25·2	162	162·1	160·1
28·85	9	10·1	8·1	72	95·8	96·0
28·75	5	4·0	2·1	55	52·7	53·7
28·65	2	2·2	·6	41	26·6	27·9
28·55	—			12	12·2	13·5
28·45	—	—	—	4	4·9	6·0
28·35	—	—	—	1	2·1	3·8
—	—	—	—	—		
Totals	14,615	14,615	14,615	14,240	14,240	14,240
χ^2	—	7·73	33·76	—	34·49	89·64
P	—	·956	·006	—	·032	·000

Length and Breadth of Beans. (a) x-Margin.

(i) The observed β's

$$\beta_{10} = \cdot 829{,}136, \quad \beta_{20} = 4\cdot 862{,}944$$

led to the following Pearson Type IV curve

$$y = \cdot 395{,}121\left(1 + \frac{x^2}{17\cdot 30134}\right)^{-8\cdot 84886} \cdot e^{-18\cdot 38048 \tan^{-1}\frac{x}{4\cdot 15949}}.$$

(ii) An extensive trial with the Type A series was made in this case; successive approximations, including the Type Ab (see p. 114), were considered. These are denoted in Table XIV by the symbol Σ, the suffix indicating the order of the last tetrachoric term in the approximation. The higher β's involved in the coefficients of the tetrachoric terms have the observed values

$$\beta'_{30} = -12\cdot 574{,}125, \quad \beta_{40} = 53\cdot 221{,}083.$$

TABLE XIV.

Marginal Distribution of Length of Beans.

Length of Beans in mm.	Observed Frequency	Theor. Freq. Type IV	Theor. Freq. Type Aa $\equiv \Sigma_4$	Theor. Freq. Type A $\equiv \Sigma_5$	Theor. Freq. Type A $\equiv \Sigma_6$	Theor. Freq. Type Ab $\equiv \Sigma_{4,6}$	Theor. Freq. Edgeworth	Theor. Freq. Log. Normal
—	—	—					—	—
—	—		16·3	−16·2	2·0	30·2		
17·0	6	1·4	12·8	13·7	−35·3	−26·8	4·8	
16·5	55	28·5	25·6	116·6	22·3	−50·6	38·6	10·1
16·0	275	299·3	241·7	370·4	438·1	293·4	280·4	280·5
15·5	1129	1181·6	1012·7	926·2	1214·0	1244·9	1136·6	1255·2
15·0	2082	2132·6	2155·4	1833·0	1866·9	2182·7	2177·9	2163·8
14·5	2294	2229·6	2593·0	2506·4	2112·8	2275·6	2269·5	2179·6
14·0	1787	1638·9	1788·4	2082·6	1916·7	1654·6	1625·3	1598·5
13·5	929	968·9	713·4	921·3	1183·4	924·8	948·2	965·0
13·0	437	503·6	280·7	199·0	371·2	419·6	495·6	515·3
12·5	199	243·7	258·7	132·1	66·9	206·1	244·2	254·5
12·0	115	113·8	206·2	178·1	101·2	144·2	116·5	119·5
11·5	70	52·5	98·7	117·0	107·1	90·7	54·7	54·4
11·0	36	24·2	29·6	43·5	54·0	38·2	25·5	24·3
10·5	18	11·3	5·9	10·0	15·4	10·3	11·8	10·7
10·0	7	5·4					5·5	4·8
9·5	1	2·6	·9	1·7	3·3	2·1	2·6	2·1
—	—	1·9					2·3	1·7
—	—		—	—	—	—		
Totals	9440	9440	9440	9440	9440	9440	9440	9440
χ^2	—	50·3	325·9	608·9	425·3	85·6	46·04	89·0
P	—	·000	—	—	—	—	·000	—

Whence we get

$$\int_{-\infty}^{t} z_x . dx = N[\tfrac{1}{2}(1 \pm \alpha_t) + \cdot 371{,}738\tau_3 - \cdot 380{,}272\tau_4 + \cdot 316{,}623\tau_5 - \cdot 382{,}998\tau_6 + \ldots].$$

The coefficient of $\tau_3 = -\cdot 309{,}001$ in the Type Ab approximation.

(iii) Edgeworth's curve has

$$\lambda = \cdot 05836, \quad k = \cdot 20676, \quad a = 2\cdot 29416;$$

its median is at 14·523 mm.

(iv) The Logarithmic Normal has

$$\xi_1 = 6\cdot 10087, \quad l_1 = \cdot 76728, \quad s_1 = \cdot 12544;$$

it starts at 17·445 mm. and its $\beta_{20} = 4\cdot 509$.

The frequencies are compared in Table XIV. The practical non-convergency of the Type A expansion, which is suggested by the coefficients of the successive terms in the series, is brought out clearly by the results in columns 4, 5 and 6. The effect of the τ_5 and τ_6 terms is to reduce instead of to increase the goodness of fit. Moreover, there is a hump in the curves at x about 12 mm. which is contrary to the nature of the data. The Type Ab produces negative frequencies at the high values of the variable, but beyond this it fits better than any one of the other approximations. I fail, however, to find any *a priori* justification, in this case, for neglecting the τ_5 term. Edgeworth's curve gives the best representation of the data; it presents no singularities and fits better than the Type IV at the high values of the variable. The Logarithmic Normal is rather unsatisfactory, especially at its start.

(*b*) *y-Margin.* The observed β's

$$\beta_{01} = \cdot 194{,}333, \quad \beta_{02} = 3\cdot 654{,}374$$

led to the following curves:

(i) Pearson's Type IV

$$x = 266\cdot 445\left(1 + \frac{y^2}{28\cdot 19141}\right)^{-11\cdot 16896} . e^{-10\cdot 52409 \tan^{-1}\frac{y}{5\cdot 30956}}$$

(ii) Type Aa integral

$$a_3 = -\cdot 179{,}969, \quad a_4 = \cdot 133{,}574.$$

(iii) Edgeworth's curve

$$\lambda = \cdot 03079, \quad k = \cdot 10055, \quad a = 1\cdot 82812,$$

with its median at 7·998 mm. Again there are no singularities.

The frequencies are given in Table XV. The goodness of fit, as measured by P, shows a slight superiority of Edgeworth's curve over the Type IV, and a distinct superiority of these two curves over the Type Aa.

The results of this section point to the following main conclusions:

(i) Of the curves we dealt with, the Logarithmic Normal is easiest to apply; but because of the relation between its fourth and lower moments it is of small practical use. When β_1 and β_2 are not large, even a relatively small deviation from this relation affects appreciably the goodness of fit of the curve.

TABLE XV.

Marginal Distribution of Breadth of Beans.

Breadth of Beans	Observed Frequency	Theor. Freq. Type IV	Theor. Freq. Type Aa	Theor. Freq. Edgeworth
—	—	—	1·4	—
—	—	3·0		4·3
9·125	5		7·0	
8·875	48	48·6	50·4	49·4
8·625	400	393·6	374·8	385·7
8·375	1483	1515·4	1474·6	1513·6
8·125	2742	2732·5	2793·3	2749·7
7·875	2579	2530·5	2609·8	2528·2
7·625	1397	1413·3	1321·8	1403·0
7·375	530	557·9	516·3	557·4
7·125	170	177·2	213·5	179·7
6·875	72	49·9	65·0	51·1
6·625	10	13·3	11·0	13·4
6·375	4	4·8	1·1	4·5
—	—			
Totals	9440	9440	9440	9440
χ^2	—	14·38	18·18	13·50
P	—	·110	·033	·140

(ii) As a general frequency function the Type A expansion* with either of the approximations to it, the Type Aa or Type Ab, is inferior to the Pearson or Edgeworth curves. It is simple to apply, but unless the distributions be only moderately skew, this simplicity is not commensurate with the accuracy sacrificed.

(iii) Edgeworth's translated curves, although they also have definite limits of applicability, have a much wider range than the Type A series has. As set out in Section C, 2 the method is not a universal one for describing frequency distributions; it is discredited for all but a small portion of the Pearson Type I area. Within the set limits, however, they give as good a representation of the data as the Pearson curves do. The method has the disadvantage that the start of the curves cannot be fixed at will; and if a high degree of accuracy is desired, the determination of the fundamental constants as well as the solution of a number of cubic equations becomes rather lengthy; it has the advantage that the theoretical frequencies are directly obtainable.

(iv) The Pearson curves are the most useful and most general of those considered; they will describe almost every variety of distribution we have to deal with in practice; the start of the non-symmetrical curves, except in the case of

* See also the remarks on pp. 113—114.

Type IV, is adjustable to the nature of the data. The greatest drawback of the system lies at present in the amount of numerical work it entails.

F. *General Considerations.*

In Section C, I gave an account of the contributions that had been made to the solution of the problem of describing mathematically skew bivariate frequency distributions. After having further analysed, in Section D, some of the characteristics of two or three of the proposed frequency functions, I passed on to a detailed graphical analysis of six observed distributions with the object of studying the generality of the theoretical surfaces. These results will now be embodied in a short discussion of the more outstanding aspects of the problem.

As a general starting-point from which a system of skew surfaces can be developed, the idea of axes of independent probability does not appear of great value; the heteroscedasticity of sections parallel to the principal axes through the modes of the distributions, such as we have actually found to exist, proves the interdependency of the transformed variables. Of course, this method might be adequate in certain isolated cases, but as its application will generally have to be justified by a comparison with other representations, there seems to be no point in using it.

The method of linear transformation suggested by Steffensen includes the above as a special case. It, too, consists in replacing the correlation function by the simple product of two univariate functions, but the axes representing these new variables are not necessarily rectangular; the angle they enclose is determined from the moments about the original axes. It is under this group, I think, that Dr Rhodes' surface may be classed. Although these formulae have the advantage that they can be applied with relative ease, their breadth of application is limited by their not allowing for enough freedom in the variation of the array and marginal distributions. The small number of parameters they contain requires certain relations between at least the fourth and lower order momental constants to be satisfied. But such relations seem rarely to exist in practice; those pertaining to the Rhodes' surface are satisfied approximately by only one of the distributions considered. Apart from these limitations, the formulae suffer from the serious disadvantage that their array moment and marginal curves are not finite expressions and thus not of any direct practical use.

The supposition of homoclitic and homokurtic array-sections from which Narumi developed his system of surfaces is not supported by our numerical illustrations. The data of Tables I—V involve (entangled with an irregular variation) a regular variation in the shape of the arrays; for the arrays of Table VI no regularity, whether changing or constant, is discernible. I am inclined to think, in fact, that the conditions presupposed by Narumi will physically very seldom be realised if each variable has a more or less distinct upper limit and if neither of these limits is a function of the other variable concerned. Thus, if there be positive correlation the x (or, y)-variable attains a greater freedom of variation in the direction of its upper limit as y (or, x) decreases in value. The arrays tend to become more and more symmetrical and may ultimately even become skew in the

opposite direction if the variables have lower limits as well. This is what actually happens in the distributions with which we have been dealing. While the condition of similar parallel sections seems likely to be fulfilled sooner by the transformed variables represented by the principal axes through the mode or mean of the distribution, than by the observed variables, the enquiry we made as to this led to rather discouraging results; the method is moreover handicapped by the unwieldiness of the resulting array moment curves.

The direct extension of a system of frequency curves to the corresponding system of surfaces is, perhaps, the most natural way of approaching the skew-correlation problem. Several such attempts have been made involving the Pearson curves, Edgeworth's translated curves, the Type A series, and the logarithmically transformed Normal curve. Accordingly it was necessary to make a comparative study of these curves; the conclusions arrived at place the curves in the above order for general adequacy in representing observed data. Owing to the difficulty of mathematical analysis, it has up to the present not been possible to derive a general system of surfaces from the Pearson curves. Edgeworth extended his method of translation to two dimensions and discussed the two cases where the equations of translation involve the observed variables (i) separately: simple translation; and (ii) conjointly: composite translation. The relations between the moments which justify simple instead of composite translation are not even approximately fulfilled by our distributions; while composite translation is not feasible. It is further to be observed that here, too, the array moment curves and marginal curves cannot be directly discussed. In summarising the relative advantages of the Pearson curves and the method of translation, Edgeworth claimed as a proposition in favour of his system that "it is adapted to the representation of frequency *surfaces*"; the proposition is true, but in the light of the analysis we have made it carries hardly any weight.

The Type AaAa (or, 15-Constant) surface still remains the most general of the surfaces that have been propounded. Its value lies more in its having 15 available constants, however, than in the form of its equation. Its limitations are essentially the same as those of the Type A expansion which we discussed in Sections C, 2 and E, 5. The simplified form of the Type AA surface suggested by Jørgensen, where the generating surface is the simple product of two normal curves, is still less efficient as a skew bivariate frequency function. The differential terms alone are inadequate to give an account of the correlation. The Logarithmic surface, equation (30), is fully specified as soon as all the moments up to the third order are known; it has consequently only 10 available constants; the relations between its fourth and lower order moments severely restrict the generality of its application. For a discussion of the regression, scedastic, clitic, and kurtic curves of these two surfaces, we refer the reader to Section E, 2. From the scedastic curves onwards the fits obtained are rather unsatisfactory; but having only one surface with 15 constants and one with 10 at our disposal, we do not know what goodness of fit really can be obtained in fitting theoretical array moment curves to observational data. A consideration of other characteristics of the surfaces, however, leads

one to anticipate that better results could be obtained only with surfaces whose sections and marginal totals are more successful than the Type A and the Logarithmic Normal Curve have been found to be in representing observed univariate distributions.

It can still be said, in conclusion, that after more than thirty years the problem still remains the "most urgent task before mathematical statisticians." The solutions that have been put forward will be serviceable in certain special cases; but no satisfactory solution to the general problem has yet been reached.

I wish to acknowledge with gratitude my indebtedness to Professor Karl Pearson for proposing the subject of this study and for his continual kindly advice and criticism. I am also indebted to Miss Ida McLearn and Miss Joyce Townend for preparing the diagrams so well.

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THE DERIVATION OF CERTAIN HIGH ORDER SAMPLING PRODUCT MOMENTS FROM A NORMAL POPULATION.

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The recent publication, by R. A. Fisher*, of a paper on the derivation of moments and product moments of sampling distributions has not only brought to a focus the work that has been done in this field by a large number of workers, but has also set these different contributions in their true perspective, and shown them to be partial attempts to deal with the whole matrix of direct and product moments of the various symmetric functions customarily calculated from the sample, and used as presumptive values in defining the population of which the sample is a member. Dr Fisher has supplied all the formulae required up to the 10th degree, together with four others of special interest of the 12th degree. He gave in addition a rule rendering it unnecessary to deal in the above classification with moments involving powers of the first sample moment coefficient, i.e. the mean, a further rule (dealt with in greater detail elsewhere†) for finding, for normal populations, the variance of the rth symmetric function of the observations, as defined by him, and an extension to multivariate distributions. It might seem that these results were sufficient, and that still higher order results, if they could be derived, would be of no practical utility. But at the end of his paper, in a section dealing with measures of departure from normality, Dr Fisher considered a problem requiring new formulae of higher degree, in order to determine the second, fourth and sixth semi-invariants of an expression analogous to $\sqrt{\beta_1}$ of the sample, and similar expressions for the equivalent of $\beta_2 - 3$. These expressions are only approximate, being expansions in powers of $1/n$, and proceed as far as terms in $1/n^2$ in the first case and $1/n$ in the second. If it is desired to proceed to a higher degree of approximation in order to test the convergence of the series reached, particularly for the higher semi-invariants, and so to determine how large the sample must be before deductions can safely be drawn as to the normality or otherwise of the population from which the sample has been taken, then further formulae are required. It is the purpose of the present paper to supply the formulae which will enable the results to be pushed to a further stage in the approximation. Their application in this connection has been made by Dr E. S. Pearson in the paper following this one‡.

As a number of different papers have appeared on the subject in the last year or two, it may be well to begin by pointing out their essential differences. Thiele,

* R. A. Fisher, *Proc. Lond. Math. Soc.* (2), Vol. xxx. (1929), pp. 199—238.

† J. Wishart, *Proc. Roy. Soc. Edin.* Vol. xlix. (1929), pp. 78—90.

‡ See pp. 239—249.

in 1889*, after defining the semi-invariants, used symmetric functions of the observations of a sample which are the same functions of the sample moment coefficients as the population semi-invariants are of the population moment coefficients. He supplied an expression covering all the semi-invariants of the mean; the first three semi-invariants of his symmetric function of 2nd degree (i.e. of the variance); the first two semi-invariants of his 3rd and 4th degree expressions; and the first semi-invariant or mean value of his 5th and 6th degree expressions. The steps in the derivation of these formulae were only outlined, the results depending on the use of tables of symmetric functions of the roots of equations; such tables were given by Thiele at the end of his paper. Earlier tables of this kind had been given by Cayley and other writers†. Later, in 1903‡, Thiele added a fourth semi-invariant of the variance, and the mean values of his 7th and 8th degree expressions.

Later Tchouproff§ obtained certain general results for the moment coefficients of moment coefficients, and in particular gave formulae for the first four moment coefficients of the variance; these were also given later by Church‖, by a method which has been described as the method of "Student"¶; it is essentially the same as that employed by Thiele, but later workers were probably unaware that the tables of symmetric functions had already been published in works on pure mathematics. It is clear, as has been recently pointed out by Rider**, that the formulae of Tchouproff for the moment coefficients of the variance can be derived from the earlier results of Thiele by using the general formulae connecting moments and semi-invariants.

In an important recent paper, C. C. Craig developed Thiele's work quite extensively††. It should be remembered that there are three kinds of moment coefficients, or semi-invariants, arising in this work. These are (1) the moment coefficients, or other symmetric functions of the sample observations, which are regarded as estimates of (2), the moment coefficients or semi-invariants of the population of which the observations form a random sample. Finally the distribution in all possible random samples of the sample moment coefficients or other symmetric functions is specified by means of (3), its direct and product moment coefficients, or semi-invariants. Both Fisher and Craig postulate an infinite population having any law of distribution with finite moments. Craig supposes the ordinary moments, m_r, to be calculated from the sample, and he gives formulae for the semi-invariants of the multiple distribution of these moments, in terms of the population semi-invariants. His list is not exhaustive, for he considers only the semi-invariants for the simultaneous distributions of the second and third, and second and fourth,

* T. N. Thiele, *Forelæsninger over Almindelig Iagttagelseslære*, Copenhagen, 1889.

† See Salmon's *Higher Algebra*, where the functions are given up to the 10th order.

‡ T. N. Thiele, *Theory of Observations*, London, 1903, C. and E. Layton, pp. 45—46.

§ A. A. Tchouproff, *Biometrika*, Vol. XII. (1919), pp. 140—169 and 185—210.

‖ A. E. R. Church, *Biometrika*, Vol. XVII. (1925), pp. 79—83.

¶ Used by "Student" in calculating the moment coefficients of the mean and variance in samples from a normal population. *Biometrika*, Vol. VI. (1908), pp. 1—25.

** P. R. Rider, *Proc. National Academy of Sciences* (U.S.A.), Vol. XV. (1929), pp. 430—434.

†† C. C. Craig, *Metron*, Vol. VII. (1928), pp. 3—74.

moments. For the normal case, however, he does give three exact expressions ($S_{22}(\nu_2\nu_4)$, $S_{13}(\nu_2\nu_4)$ and $S_{04}(\nu_2\nu_4)$ in his notation) which were not tabulated by Fisher, although the latter, in connection with the tests for normality, gave the leading terms in the formulae corresponding to the last two of these formulae. Further results by Craig are general approximate formulae for the first four semi-invariants of $\sqrt{\beta_1}$, β_2-3 and σ, which are only worked out fully, however, for the normal case. A section of Fisher's paper also deals with this point and furnishes a higher degree of approximation for the semi-invariants of an expression equivalent to $\sqrt{\beta_1}$, adding the sixth (the fifth being zero), while for the equivalent of β_2-3 he goes as far as terms in $1/n^3$ as Craig does, but adds a fifth semi-invariant. I have been unable, however, in the case of β_2 to verify Craig's terms in $1/n^3$, which do not agree with Fisher's*. Important as Craig's contribution to the theory is, however, it should be pointed out that there is an essential difference in Fisher's method, a difference that makes for simplicity in the resulting formulae. Craig at one point remarks "it rather seems that the best hopes of effectively further simplifying the problem of sampling for statistical characteristics lie either in the discovery of a new kind of symmetric function of all the observations which may be used to characterise frequency functions and which will be more amenable than either moments or semi-invariants for use in sampling problems, or in, what may very well prove to be better and more feasible, the abandonment of the method of characterising frequency functions by symmetric functions of all the observations altogether." The line of Fisher's work had followed the first of these two suggestions. The symmetric functions of the observations which he supposes calculated, i.e. his k's, are such that the mean value of any k_r in all possible samples, is κ_r, the rth semi-invariant of the population from which the samples are taken. His k's are, therefore, more nearly allied to semi-invariants than to ordinary moments, and they are not the same functions of the sample moments as ordinarily defined as the semi-invariants are of the population moments. Thus, for example, if we denote by m_r the rth moment coefficient of the sample about its own mean, i.e.

$$m_r=\frac{1}{n}\overset{n}{\underset{1}{S}}(x-\bar{x})^r,\quad \bar{x}=\frac{1}{n}\overset{n}{\underset{1}{S}}(x),$$

then the first four of Fisher's symmetric functions are

$$k_1=m_1,\quad k_2=\frac{n}{n-1}m_2,$$

$$k_3=\frac{n^2}{(n-1)(n-2)}m_3,\quad k_4=\frac{n^2}{(n-1)(n-2)(n-3)}\{(n+1)m_4-3(n-1)m_2^2\}.$$

The employment of these functions of the ordinary moment coefficients leads to a great simplification, not only of the sampling results, but also of the methods by

* Craig's formula for c_1, the mean value of β_2, is certainly wrong in the terms in $1/n^3$, for the exact result is $3\dfrac{n-1}{n+1}$, which if expanded gives a term $-1/n^3$ in place of Craig's $-5/n^3$. It may be that Craig's degree of approximation hardly warranted his giving the terms in $1/n^3$ for the semi-invariants of β_2, although the term in $1/n^3$ in κ_2 (Craig's c_2) is correct.

which these are derived. For although Fisher shows by an example how to proceed by direct algebraic methods to determine the semi-invariants of the multiple distribution of powers and products of any number of the k's, expressing the results in terms of the population semi-invariants, he soon makes it clear that the intermediate steps may be left out and the final result written down, a term at a time, by methods of combinatorial analysis, following certain simple rules. The demonstration of the validity of the rules is admittedly a difficult piece of mathematics, but the rules themselves are easy to remember and far simpler to apply than the direct algebraic methods. To illustrate first the nature of the general problem, suppose that we are concerned with the derivation of the formula for $\kappa(3^p 2^q)$. First as to the meaning of this expression; k_2 and k_3 have already been defined in terms of the observations of the sample. If we write $\mu(3^p 2^q)$ for the mean value of $k_3^p k_2^q$ taken over all possible samples, then $\kappa(3^p 2^q)$ is the corresponding semi-invariant, the κ's and μ's being related by the identity in t_2 and t_3,

$$1+\mu(2)\,t_2+\mu(3)\,t_3+\mu(2^2)\frac{t_2^2}{2!}+\mu(23)\frac{t_2t_3}{1!\,1!}+\mu(3^2)\frac{t_3^2}{2!}+\dots$$

$$\equiv \exp\left\{\kappa(2)\,t_2+\kappa(3)\,t_3+\kappa(2^2)\frac{t_2^2}{2!}+\kappa(23)\frac{t_2t_3}{1!\,1!}+\kappa(3^2)\frac{t_3^2}{2!}+\dots\right\}.$$

$\kappa(3^p 2^q)$ may be expressed as the sum of terms each of order $3p+2q$, consisting of powers and products of the semi-invariants of the sampled population, $\kappa_2, \kappa_3, \dots, \kappa_{3p+2q}$, and the general rules of the combinatorial procedure for determining the coefficients have been given by Fisher*. A simple illustration is given at the end of the present paper.

In the case where the sampled population is normal we see at once the advantage of expressing the results in terms of semi-invariants, for all the κ's above κ_2 vanish, and thus we are left with only a single coefficient to evaluate—in the above example that of $(\kappa_2)^{\frac{1}{2}(3p+2q)}$†. Certain interesting generalisations also follow. Thus the semi-invariants of the distribution of powers and products of moment coefficients, or of k's, of the second order, may be solved by considering appropriate ring arrangements of rods‡, while general formulae for the variance of k_t and for the correlation between product moments of any order have also been determined for the normal case§.

We shall now consider in detail in the case of a normal population the derivation of the formula for $\kappa(3^2 2^3)$ which is one of the new results required for further development of the tests for normality. The result will consist of a single term in κ_2^6, since the expression evaluated is of the 12th degree, κ_2 being simply σ^2, the variance of the sampled population. In following out the rules we must therefore write down all the two-way partitions which have

5 *columns*, containing 3, 3, 2, 2 and 2 entries respectively (controlled by $\kappa(3^2 2^3)$),

6 *rows*, each containing 2 entries (controlled by κ_2^6).

* R. A. Fisher, *loc. cit.* pp. 219—228.

† The coefficient vanishes when p is odd.

‡ J. Wishart, *Proc. Lond. Math. Soc.* (2), Vol. XXIX. (1929), pp. 309—321.

§ J. Wishart, *Proc. Roy. Soc. Edin.* Vol. XLIX. (1929), pp. 78—90.

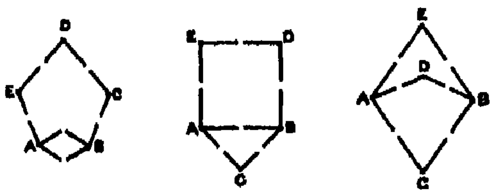
We find that there are three partitions that have other than zero coefficients, as follows:

$$
\begin{array}{ccccc|c}
1 & 1 & - & - & - & 2 \\
1 & 1 & - & - & - & 2 \\
1 & - & 1 & - & - & 2 \\
- & 1 & - & 1 & - & 2 \\
- & - & 1 & - & 1 & 2 \\
- & - & - & 1 & 1 & 2 \\
3 & 3 & 2 & 2 & 2 & 12
\end{array}
\qquad
\begin{array}{ccccc|c}
1 & 1 & - & - & - & 2 \\
1 & - & 1 & - & - & 2 \\
1 & - & - & - & 1 & 2 \\
- & 1 & 1 & - & - & 2 \\
- & 1 & - & 1 & - & 2 \\
- & - & - & 1 & 1 & 2 \\
3 & 3 & 2 & 2 & 2 & 12
\end{array}
\qquad
\begin{array}{ccccc|c}
1 & - & 1 & - & - & 2 \\
1 & - & - & 1 & - & 2 \\
1 & - & - & - & 1 & 2 \\
- & 1 & 1 & - & - & 2 \\
- & 1 & - & 1 & - & 2 \\
- & 1 & - & - & 1 & 2 \\
3 & 3 & 2 & 2 & 2 & 12
\end{array}
$$

(*a*) (*b*) (*c*)

No row can contain only a single entry, so that no 2 can occur, and the entries are therefore all units*. Now if the columns be regarded as corners having as many arms extending from them as there are units in the column, and if the rows be regarded as the junctions between the arms of different corners, we have the five corners shown in the diagram below to dispose in all possible ways so that there are six junctions

and two arms only at each junction. The partitions (*a*), (*b*) and (*c*) are then found to be equivalent to the symbolical figures



(a) (b) (c)

Figures (*a*) and (*b*) may be regarded as two-dimensional models, unless it is preferred to think of the arms connecting *A* to *B*, directly and through *C*, as in a plane perpendicular to the paper. Figure (*c*), on the other hand, is best regarded as a symmetrical figure in three dimensions.

In each case there are two parts to the desired result. One is the numerical coefficient, which is derived from the number of ways in which the particular diagram, or partition, may be set up, the arms of any corner having separate identities and being interchangeable, and the corners themselves being regarded as distinct. The other part is the coefficient in n, the size of the sample, and this is fixed from the nature of the pattern, i.e. the arrangement of entries and zeros, irrespective of what the entries actually are, and the number of rows and columns—alternatively, looking at the symbolical diagrams, the n-coefficient is fixed by the number of corners and the number of breaks between corners and by the design of the diagram. The two parts will be considered separately.

* For the conditions under which certain patterns have zero-coefficients see R. A. Fisher, *loc. cit.* pp. 220—221.

(1) *Numerical coefficient.* This is determined most simply from the symbolical diagram.

(*a*) Having placed the two 3-way corners (A and B) together, so that they can be doubly linked, there are six ways of disposing the three 2-way corners (C, D and E) so as to form a ring. An arm (B to C) may be selected out of the three at B in three ways, likewise for the arm A to E, and in the case of the three 2-arm corners there are two ways each of arranging the arms (C to B or D, etc.). Finally there are two ways of linking the double arms between A and B. The total numerical coefficient is therefore $6 . 3^2 . 2^3 . 2 = 864$.

(*b*) There are three ways of putting a 2-way corner (C, D or E) to one side of the line AB to form a triangle, and two ways of connecting up the others (D or E to B). These give a factor 6. The arms A to E, A to B and A to C may be stretched out in six ways, likewise for the three arms from B, while there are two ways of linking for each of the three 2-arm corners. The numerical coefficient is therefore $6^3 . 2^3 = 1728$.

(*c*) This is a symmetrical arrangement. There are six ways each of stretching out the arms at A and B, and two ways each of linking up to the arms from C, D and E. The numerical coefficient is then $6^2 . 2^3 = 288$.

As a check on the correctness of the total numerical coefficient we note that the total, $864 + 1728 + 288$, is equal to 288×10. 288 is the numerical coefficient of the term in κ_2^5 of $\kappa(3^2 2^2)$ (see R. A. Fisher, *loc. cit.* p. 213, formula (31)), and 10 is the degree of $\kappa(3^2 2^2)$. The coefficients can in fact be determined from the number of ways in which a new corner may be inserted into the pattern of lower degree. In the case of $\kappa(3^2 2^2)$ there are five junctions, and a new 2-way corner can therefore be inserted in five ways, and when its position has been decided, there are two ways of linking up the arms of the new corner to those at the broken junction. Hence the numerical coefficient for $\kappa(3^2 2^3)$, obtained by adding a 2 to $\kappa(3^2 2^2)$, is equal to the number of ways in which the pattern for $\kappa(3^2 2^2)$ can be arranged, namely 288, multiplied by 10. The diagrams on the top of p. 230 show the development from the simple result

$$\kappa(3^2) = \frac{6n}{(n-1)(n-2)} \kappa_2^3$$

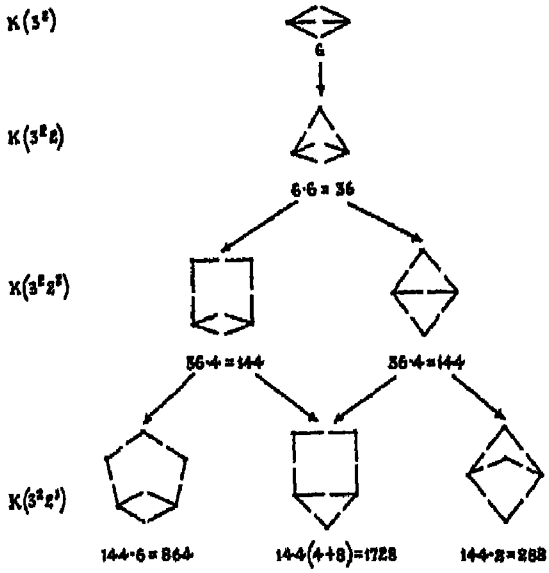
as far as the numerical coefficient is concerned.

The derivation of new formulae can evidently, as to the numerical coefficient, be pushed as far as desired. In fact the *total* numerical coefficient of the term in κ_2^{r+3} of $\kappa(3^2 2^r)$ is

$$2^r . \frac{(r+2)!}{2!} . 6.$$

(2) *The n-coefficient.* It is already known that the term in κ_2^{r+3} of $\kappa(3^2 2^r)$ is of the order $1/n^{r+1}$ so that, approximately,

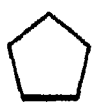
$$\kappa(3^2 2^r) = \frac{3 . 2^r . (r+2)!}{n^{r+1}} . \kappa_2^{r+3};$$



but if the exact coefficient is wanted, it is necessary to follow out the rules given by Fisher (pp. 221—222), and proved on pp. 226—230. It so happens that our three patterns, although essentially different in their structure, have the same n-coefficient, namely

$$\frac{n}{(n-1)^4(n-2)}.$$

This is due to the fact that they are all derived from the same pattern, namely that for $\kappa(3^2)$, by the insertion of fresh 2-way corners. By way of illustration of the general method one only of the above patterns will be evaluated, and this is more conveniently done from the symbolical diagram than from the 2-way partition. It should be noted that this work does not have to be repeated every time an example is worked: when the coefficient has been determined for any pattern it applies to all patterns of that kind, irrespective of the entries. R. A. Fisher has supplied some three pages of the more commonly occurring patterns (pp. 223—226), and it is only because our example is not covered by his list that it requires to be worked out*. We shall choose pattern (a) for illustration. The arms may now be regarded as being joined, thus:



and the reader is recommended to reconstruct this figure for himself by means of six matches, in order to follow more easily the reasoning.

* The reasoning which follows is given for the purpose of illustration, but it is unlikely that a pattern of such a complex nature will ever require to be worked out by the reader. An indication is given later of methods whereby the n-coefficients of such patterns can be derived from those of simpler patterns: e.g. in the present case the coefficient of the normal term of $\kappa(3^2)$ is all that is required to develop the normal term of such a semi-invariant as $\kappa(3^2 2^r)$.

The rods (matches) are now regarded as the rows of the 2-way partition, and we consider all the possible ways in which the rods can be separated into 1, 2, 3, ... 6 separate groups, or separates. With each of these there is associated a factor in n, according to the scheme of the following table, which also shows the number of such separates.

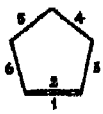
Separation into		*Number of ways*		*Factor*
(a) 1 separate		1		n
(b) 2 separates of	1 and 5	6	31	$n(n-1)$
	2 and 4	15		
	3 and 3	10		
(c) 3 separates of	1, 1 and 4	15	90	$n(n-1)(n-2)$
	1, 2 and 3	60		
	2, 2 and 2	15		
(d) 4 separates of	1, 1, 1 and 3	20	65	$n(n-1)(n-2)(n-3)$
	1, 1, 2 and 2	45		
(e) 5 separates of	1, 1, 1, 1 and 2	15		$n(n-1)(n-2)(n-3)(n-4)$
(f) 6 separates		1		$n(n-1)(n-2)(n-3)(n-4)(n-5)$

In each of these 203 separations we consider the corners separately. Each unbroken corner contributes a factor n^{-1}, a corner broken into two parts the factor $-\frac{1}{n(n-1)}$, into three parts the factor $\frac{2!}{n(n-1)(n-2)}$. The nature of the separations will be expressed by a quantity in brackets, such as $(1^p 2^q 3^r)$, which specifies p unbroken corners, q broken into two parts and r into three parts, so that for our example $p+q+r=5$ always, and such a group of separates would contribute a term

$$\left(\frac{1}{n}\right)^p \cdot \left(-\frac{1}{n(n-1)}\right)^q \cdot \left(\frac{2!}{n(n-1)(n-2)}\right)^r.$$

(a) Here all the corners are unbroken, and the coefficient is n/n^5.

(b) Let us number the rods as follows:



(i) Separation into two separates of 1 and 5, however it is done, leaves three corners unbroken, while the other two are broken into two parts. We therefore have a contribution of $6(1^3 2^2)$.

(ii) Separates of 2 and 4. The fifteen ways that this can be done may be divided into a number of sub-classes. Thus if we separate off 1 and 2 from the rest we obtain a term $(1^3 2^2)$ and the same result is obtained by separating off 3 and 4, 4 and 5, or 5 and 6. Total $4(1^3 2^2)$. Separating off 1 and 3 produces $(1^2 2^3)$ and the same is true of 1 and 6, 2 and 3, and 2 and 6. Total $4(1^2 2^3)$. Separating off 1 and 4 is the same as separating 1 and 5, 2 and 4, 2 and 5, 3 and 5, 3 and 6, and 4 and 6. Total $7(12^4)$. The separates of 2 and 4 therefore contribute $4(1^3 2^2)+4(1^2 2^3)+7(12^4)$.

(iii) Separates of 3 and 3. This can be done in ten ways, subdivided as follows:

1, 2, 3 and 1, 2, 6	$2(1^3 2^2)$,
1, 2, 4; 1, 2, 5; 1, 3, 6 and 1, 4, 5	$4(1\,2^4)$,
1, 3, 4 and 1, 5, 6	$2(1^2 2^3)$,
1, 3, 5 and 1, 4, 6	$2(2^5)$.

The total contribution from the 31 separations into 2 separates is therefore

$$12(1^3 2^2) + 6(1^2 2^3) + 11(12^4) + 2(2^5).$$

(*c*) The separations into three separates are a little more difficult to follow out*, but result in

$$1(1^3 3^2) + 10(1^2 2^3) + 12(1^2 2^2 3) + 22(12^4) + 12(12^3 3) + 6(12^2 3^2) + 13(2^5) + 12(2^4 3) + 2(2^3 3^2).$$

(*d*) The separations into four separates lead to

$$3(1^2 2\,3^2) + 5(12^4) + 12(12^3 3) + 9(12^2 3^2) + 9(2^5) + 16(2^4 3) + 11(2^3 3^2).$$

(*e*) For five separates we have

$$3(12^2 3^2) + 1(2^5) + 4(2^4 3) + 7(2^3 3^2).$$

(*f*) Here all the corners are broken completely and we have simply $(2^3 3^2)$.

The final n-coefficient is then made up as follows:

$$\frac{1}{n^4}\left[1 + \frac{12}{n-1} - \frac{6}{(n-1)^2} + \frac{11}{(n-1)^3} - \frac{2}{(n-1)^4} + \frac{4}{(n-1)(n-2)} - \frac{10(n-2)}{(n-1)^2} + \frac{24}{(n-1)^2}\right.$$
$$+ \frac{22(n-2)}{(n-1)^3} - \frac{24}{(n-1)^3} + \frac{24}{(n-1)^2(n-2)} - \frac{13(n-2)}{(n-1)^4} + \frac{24}{(n-1)^4} - \frac{8}{(n-1)^4(n-2)}$$
$$- \frac{12(n-3)}{(n-1)^2(n-2)} + \frac{5(n-2)(n-3)}{(n-1)^3} - \frac{24(n-3)}{(n-1)^3} + \frac{36(n-3)}{(n-1)^3(n-2)} - \frac{9(n-2)(n-3)}{(n-1)^4}$$
$$+ \frac{32(n-3)}{(n-1)^4} - \frac{44(n-3)}{(n-1)^4(n-2)} + \frac{12(n-3)(n-4)}{(n-1)^3(n-2)} - \frac{(n-2)(n-3)(n-4)}{(n-1)^4}$$
$$\left. + \frac{8(n-3)(n-4)}{(n-1)^4} - \frac{28(n-3)(n-4)}{(n-1)^4(n-2)} - \frac{4(n-3)(n-4)(n-5)}{(n-1)^4(n-2)}\right].$$

This reduces to

$$\frac{n}{(n-1)^4(n-2)}.$$

As already stated all three patterns for the contributory portions of $\kappa(3^2 2^3)$ have the same n-coefficient, and our result in full, therefore, is *for normality*

$$\kappa(3^2 2^3) = \frac{2880n}{(n-1)^4(n-2)}\kappa_2^6 \quad\text{..............................(1).}$$

* (*c*), (*d*) and (*e*) are best evaluated at the same time as (*b*), to avoid repetition of labour.

List of Higher Order Formulae.

The following results, which are of degree 12 and upwards, will enable expansions for the moments of $\sqrt{\beta_1}$ and β_2 from a normal population to be determined to a higher degree of approximation than has hitherto been reached.

$$\kappa(3^2 2^4) = \frac{34560n}{(n-1)^5(n-2)}\kappa_2^7 \quad \ldots\ldots(2), \qquad \kappa(3^4 2) = \frac{7776n^2(5n-12)}{(n-1)^4(n-2)^3}\kappa_2^7 \quad \ldots\ldots(3),$$

$$\kappa(3^4 2^2) = \frac{108864n^2(5n-12)}{(n-1)^5(n-2)^3}\kappa_2^8 \quad \ldots\ldots(4),$$

$$\kappa(3^4 2^3) = \frac{1741824n^2(5n-12)}{(n-1)^6(n-2)^3}\kappa_2^9 \quad \ldots\ldots(5),$$

$$\kappa(3^6) = \frac{466560n^3(22n^2-111n+142)}{(n-1)^5(n-2)^6}\kappa_2^9 \quad \ldots\ldots(6),$$

$$\kappa(3^6 2) = \frac{18}{n-1}\kappa_2\kappa(3^6) \quad \ldots\ldots(7), \qquad \kappa(3^6 2^2) = \frac{360}{(n-1)^2}\kappa_2^2\kappa(3^6) \quad \ldots\ldots(8),$$

$$\kappa(4^2 2^2) = \frac{1920n(n+1)}{(n-1)^3(n-2)(n-3)}\kappa_2^6 \quad \ldots\ldots(9),$$

$$\kappa(4^2 2^3) = \frac{23040n(n+1)}{(n-1)^4(n-2)(n-3)}\kappa_2^7 \quad \ldots\ldots(10),$$

$$\kappa(4^2 2^4) = \frac{322560n(n+1)}{(n-1)^5(n-2)(n-3)}\kappa_2^8 \quad \ldots\ldots(11),$$

$$\kappa(4^3 2) = \frac{20736n(n+1)(n^2-5n+2)}{(n-1)^3(n-2)^2(n-3)^2}\kappa_2^7 \quad \ldots\ldots(12),$$

$$\kappa(4^3 2^2) = \frac{290304n(n+1)(n^2-5n+2)}{(n-1)^4(n-2)^2(n-3)^2}\kappa_2^8 \quad \ldots\ldots(13),$$

$$\kappa(4^3 2^3) = \frac{4644864n(n+1)(n^2-5n+2)}{(n-1)^5(n-2)^2(n-3)^2}\kappa_2^9 \quad \ldots\ldots(14),$$

$$\kappa(4^4) = \frac{6912n(n+1)}{(n-1)^3(n-2)^3(n-3)^3}\{53n^4-428n^3+1025n^2-474n+180\}\kappa_2^8 \quad \ldots\ldots(15),$$

$$\kappa(4^4 2) = \frac{16}{n-1}\kappa_2\kappa(4^4) \quad \ldots\ldots(16), \qquad \kappa(4^4 2^2) = \frac{288}{(n-1)^2}\kappa_2^2\kappa(4^4) \quad \ldots\ldots(17),$$

$$\kappa(4^5) = \frac{364 \,.\, 12^5}{n^4}\kappa_2^{10}, \text{ approximately}^* \quad \ldots\ldots(18).$$

The derivation of the foregoing results has been rendered much simpler than it would otherwise have been by the discovery of a rule which applies whenever a fresh k_2 is introduced into the kappa expression to be evaluated. We may regard $\kappa(3^2)$, $\kappa(3^2 2), \ldots \kappa(3^2 2^r)$ for example as a train of formulae each derived from the preceding one by the adding of a k_2, or, looking at the symbolical diagram, by the insertion of

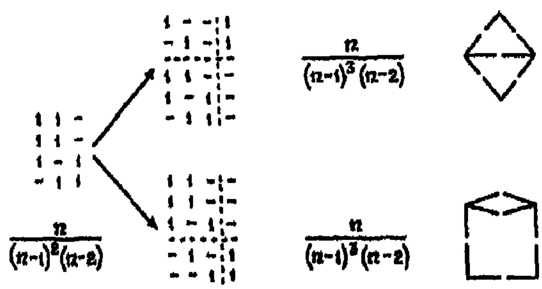
* There is a misprint in Dr Fisher's paper, p. 236, where $\kappa_5(x)$ (not κ_6 as printed) should read $71 \,.\, 144 \,.\, \sqrt{6} \,.\, n^{-\frac{3}{2}}$. Also on p. 233 (top) the three ways of evaluating the symbolical diagrams for $\kappa(3^4 2)$ should be 15552, 7776 and 15552 respectively.

a fresh 2-way corner. The effect this has on the numerical coefficient of the normal term in $\kappa(3^2 2^r)$ is to multiply by the degree of this expression, i.e. by $2r+6$, in order to produce the normal term of $\kappa(3^2 2^{r+1})$, as explained on p. 229. The effect on the n-coefficient is simply to divide by $n-1$ every time. For example, the normal term in $\kappa(3^2 2)$ is derived from that of $\kappa(3^2)$ as follows:

$$\begin{array}{cc} 1 & 1 \\ 1 & 1 \\ \hdashline 1 & 1 \end{array} \qquad \begin{array}{cc:c} 1 & 1 & - \\ 1 & 1 & - \\ \hdashline 1 & - & 1 \\ - & 1 & 1 \end{array}$$

$$\frac{n}{(n-1)(n-2)}, \qquad \frac{n}{(n-1)^2(n-2)}.$$

One of the rows (here the last) must be split to make two new rows, and the units of the new column placed in these rows. Now if in the working out of the n-coefficient of the pattern for the normal term of $\kappa(3^2 2)$ the separations are grouped into two classes, (1) those in which the last two rows occur together and therefore reproduce the separations of the normal term of $\kappa(3^2)$ together with the contribution $\frac{1}{n}$ from the new column, and (2) those in which the last two rows are separated, bringing in a contribution $-\frac{1}{n(n-1)}$ from the new column, it may be readily verified that the net effect on the coefficient of the normal term of $\kappa(3^2)$ is to divide by $n-1$. In the next stage two patterns are produced for $\kappa(3^2 2^2)$ corresponding with the symbolical diagrams on p. 230, according as one of the first two or one of the last two rows of the pattern for $\kappa(3^2 2)$ is split to form two new rows. Thus:



The n-coefficients are equal, both being derived from that of $\kappa(3^2 2)$ by dividing by $n-1$.

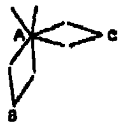
In the last stage of the process for the example illustrated, i.e. $\kappa(3^2 2^3)$, three new patterns are formed from the two patterns of $\kappa(3^2 2^2)$ as shown on p. 230, and their n-coefficients are all equal, and equal to $\frac{n}{(n-1)^4(n-2)}$. We are now able to write down the general formula

$$\kappa(p^q 2^r) = \frac{2^r (r+\frac{1}{2}pq-1)!}{(\frac{1}{2}pq-1)!\,(n-1)^r}\kappa_2{}^r \,.\, \kappa(p^q) \quad \text{.................(19)},$$

which is generally true for sampling from a normal population, for it holds when pq is even, while when pq is odd the whole expression vanishes, since $\kappa(p^q)$ has in general no term involving κ_2 only, when pq is odd. A special case of (19) occurs when $q=1$. We then find that, for the normal case,

$$\kappa(p2^r)=\frac{2^r(r+\frac{1}{2}p-1)!}{(\frac{1}{2}p-1)!(n-1)^r}\kappa_2{}^r \,.\, \kappa(p)=0 \quad\text{..................(20)},$$

when p is greater than 2. This follows from the general result that $\kappa(p)=\kappa_p$, while all κ's above κ_2 vanish for normality. It is obvious also from the impossibility of constructing a closed symbolical diagram to fit this case; for, to illustrate from $\kappa(p\,2^2)$, the only possible diagram is of the form



the number of arms extending from A being equal to p, and the conditions laid down by Fisher (*loc. cit.* pp. 220—221) are such that (1) no loose arms can exist and (2) a break at any one corner must not divide the figure into two separate pieces.

In the special case where $p=2$ we have

$$\kappa(2^{r+1})=\frac{2^r\,.\,r!}{(n-1)^r}\kappa_2{}^{r+1}\quad\text{............................(21)},$$

which is the $(r+1)$th semi-invariant of the distribution of k_2, in samples from a normal distribution. In terms of the more familiar m_2, the second moment coefficient of the sample, the result is

$$\frac{2^r\,.\,r!\,(n-1)}{n^{r+1}}\,.\,\sigma^{2(r+1)}\quad\text{..............................(22)},$$

a form which has already been published*, and which is derivable from "Student's" distribution of the variance†, σ being the standard deviation of the sampled normal population. For $p>2$ equation (20) shows that there can be no correlation in samples between k_p and any power of k_2. For comparison with this we have a more general result already reached, to the effect that no correlation can exist between k_t and k_u unless $t=u$, or, for bi-variate populations, between k_{tu} and k_{vw} unless $t+u=v+w$. This is another important property of the k functions which does not hold among moments. For correlation does exist between the sample moment coefficients of different orders, other than m_1, the mean, which is uncorrelated with any of the higher moments.

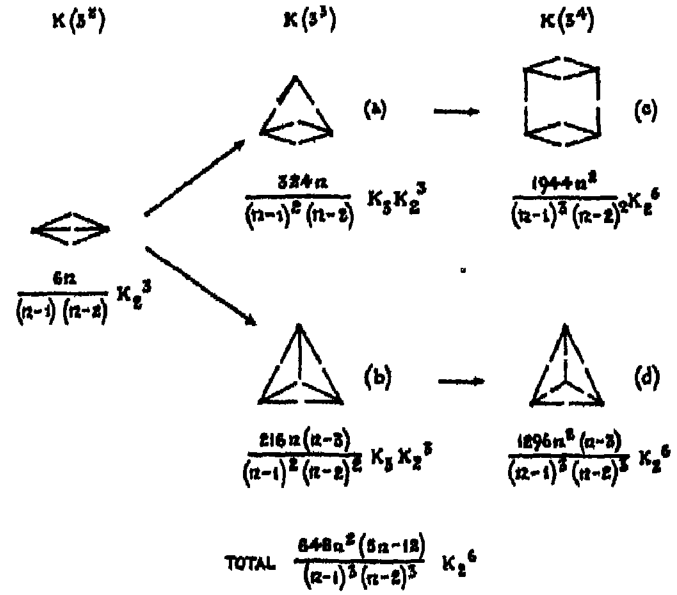
The tabulated formulae (1) to (18) are special cases of the general result (19) except for $\kappa(3^6)$, $\kappa(4^4)$ and $\kappa(4^5)$, and it is evident that any expression of the form $\kappa(p^q2^r)$ can be evaluated in full for the normal case as long as $\kappa(p^q)$ is known exactly. The numerical parts of $\kappa(3^6)$, $\kappa(4^4)$ and $\kappa(4^5)$ have already been worked out by R. A. Fisher‡, although he did not in his paper give the separate contribu-

* J. Wishart, *Proc. Lond. Math. Soc.* (2), Vol. xxix. (1929), p. 314.

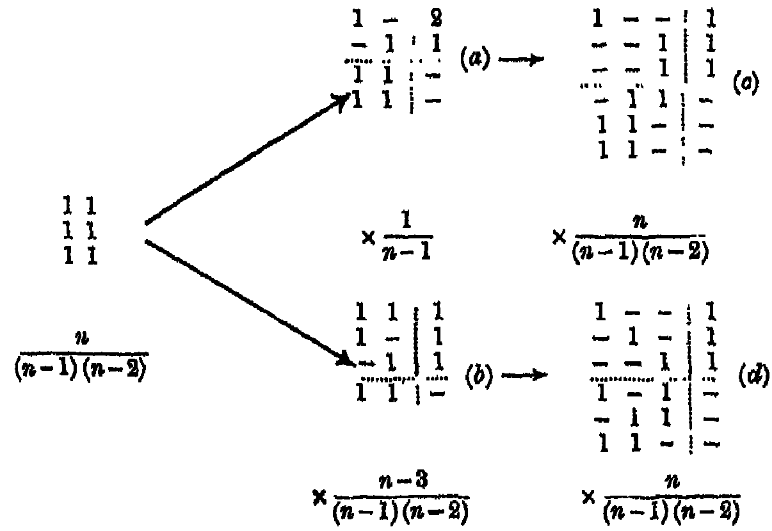
† "Student," *Biometrika*, Vol. vi. (1908), pp. 6—8.

‡ R. A. Fisher, *loc. cit.* pp. 233 and 236.

tions of the different patterns. An extension of the rule already described for the adding of a k_3 enables us to develop the n-coefficients of the patterns in $\kappa(3^6)$, e.g. from those of $\kappa(3^4)$, which are known*. By way of illustration we give the development of the two patterns for the normal term of $\kappa(3^4)$ from the single pattern for $\kappa(3^2)$.



Alternatively the development in terms of the 2-way partitions is as follows:



(*a*) The rule here is that already described, for the new k_3 added is in the form of a column having only two entries. The n-coefficient is therefore obtained from that of $\kappa(3^2)$ by dividing by $n-1$, while its numerical multiplier is multiplied by 54. (Three ways of breaking the old pattern—six ways of arranging the new 3-way corner at this break—and, since the resulting pattern is unsymmetrical, three ways of choosing which of the 3-way corners shall be unlike the others.)

* These rules will be described in a forthcoming paper by R. A. Fisher and the present author.

(b) Consideration of the separations shows that the n-coefficient of this pattern is of the form

$$\frac{n}{(n-1)(n-2)}A-\frac{1}{(n-1)(n-2)}B,$$

where A is the n-coefficient of the 2 row—2 column partition, i.e. $\frac{1}{n-1}$, and B is the n-coefficient of the pattern for the normal term of $\kappa(3^2)$, i.e. $\frac{n}{(n-1)(n-2)}$. The coefficient multiplier is therefore $\frac{n-3}{(n-1)(n-2)}$, while the numerical multiplier is 36. (Three ways of breaking old pattern—two junctions to either of which one of the three arms of the new corner may be connected, while the remaining two arms may be disposed in two ways at the break.)

(c) and (d) In both these cases there is only one junction, i.e. that where three arms meet, that can be broken if the normal term of $\kappa(3^4)$ is to be formed from the term in $\kappa_3\kappa_2^3$ of $\kappa(3^3)$. The new 3-way corner can then be disposed at this break in six ways, so that the numerical multiplier in (c) and (d) is 6, while the multiplier of the n-coefficients of the patterns for $\kappa(3^3)$ is in each case $\frac{n}{(n-1)(n-2)}$.

Patterns of $\kappa(3^6)$

Numerical multiplier

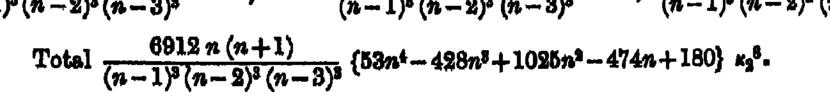
699840   2099520   4199040   2799360   466560

n-coefficient

$$\frac{n^3}{(n-1)^5(n-2)^3},\quad \frac{n^3}{(n-1)^5(n-2)^3},\quad \frac{n^3(n-3)}{(n-1)^5(n-2)^4},\quad \frac{n^3(n-3)^2}{(n-1)^5(n-2)^5},\quad \frac{n^3(n^2-6n+10)}{(n-1)^5(n-2)^5}.$$

$$\text{Total } \frac{466560\, n^3(22n^2-111n+142)}{(n-1)^5(n-2)^5}\kappa_2^6.$$

Patterns of $\kappa(4^4)$



Numerical multiplier

62208   248832   55296

n-coefficient

$$\frac{n(n+1)(n^4-8n^3+21n^2-14n+4)}{(n-1)^3(n-2)^3(n-3)^3},\quad \frac{n(n+1)(n^4-9n^3+23n^2-11n+4)}{(n-1)^3(n-2)^3(n-3)^3},\quad \frac{n^2(n+1)^2}{(n-1)^3(n-2)^2(n-3)^2}.$$

$$\text{Total } \frac{6912\, n(n+1)}{(n-1)^3(n-2)^3(n-3)^3}\{53n^4-428n^3+1025n^2-474n+180\}\,\kappa_2^8.$$

Patterns of $\kappa(4^5)$

Numerical multiplier $12^5.12$ $12^5.80$ $12^5.120$ $12^5.32$ $12^5.120$

Total $\frac{364.12^5}{n^4}\kappa_2^{10}$, approximately.

Procedure in the non-normal case.

In order to fix this simplified problem, which arises in the case of normality, in its place as a part of a wider scheme, it will perhaps be of value to conclude by indicating in a simple example the lines of procedure to be followed in the general case where the sampled population is not normal. Consider $\kappa(3^2)$, the second semi-invariant, or second moment coefficient about the mean, of k_3 in samples. The method of expressing this quantity in terms of the population semi-invariants may be illustrated as follows:

$$(1)\qquad a_1\,\kappa_6 \quad + \quad a_2\,\kappa_4\,\kappa_2 \quad + \quad a_3\,\kappa_3^2 \quad + \quad a_4\,\kappa_2^3$$

$$(2)\qquad \begin{array}{cc|c} 3 & 3 & 6 \\ \hline 3 & 3 & \end{array} \qquad \begin{array}{cc|c} 2 & 2 & 4 \\ 1 & 1 & 2 \\ \hline 3 & 3 & \end{array} \qquad \begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 2 & 3 \\ \hline 3 & 3 & \end{array} \qquad \begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \\ \hline 3 & 3 & \end{array}$$

(3) 

$$1 \qquad\qquad 9 \qquad\qquad 9 \qquad\qquad 6$$

$$(4)\qquad \frac{1}{n} \qquad\qquad \frac{1}{n-1} \qquad\qquad \frac{1}{n-1} \qquad\qquad \frac{n}{(n-1)(n-2)}$$

$$(5)\qquad \frac{1}{n}\kappa_6 \quad + \quad \frac{9}{n-1}\kappa_4\kappa_2 \quad + \quad \frac{9}{n-1}\kappa_3^2 + \frac{6n}{(n-1)(n-2)}\kappa_2^3.$$

(1) The result will be a sum of terms containing all the possible 6th order powers or products of the κ's ($\kappa_1 = 0$). There are four such terms.

(2) The 2-way partitions associated with each term are shown in this line. As we are dealing with $\kappa(3^2)$ we shall in each case have 2 columns with contents totalling 3. The number of rows and their contents vary but the rule is simple; for κ_6 one row containing 6; for $\kappa_4\kappa_2$ one row containing 4 and one containing 2; for κ_3^2 two rows each containing 3; for κ_2^3 three rows each containing 2. In this example there is only one possible partition associated with each coefficient (contrasted with the case of $\kappa(3^2 2^2)$ dealt with above) and all the cells are filled.

(3) The numerical coefficients are determined by considering in each case the number of ways of connecting up 2 junctions each having 3 arms. For a_1, a_2, a_3, and a_4 we must make connections of 6, of 4 and 2, of 3 and 3, and of 2, 2 and 2 respectively. It will be seen that the ways in which this can be done are 1, 9, 9 and 6 respectively.

(4) The n-coefficients depend upon the pattern of the four partitions in line (2) above*. These are given by Fisher (*loc. cit.* pp. 223—224), and are set out in line (4) above. The final result obtained by combining the expressions is shown in line (5). In the case where the sampled population is normal we are concerned only with securing the last term.

We have taken of course a very straightforward example in which only simple patterns and rod combinations are required, but the elegance of the method, once it has been grasped, can hardly fail to attract the worker even in far more complex problems.

* For example such partitions as $\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix}$, $\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$, $\begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix}$ etc. are all associated with the *same* "pattern," viz. $\begin{vmatrix} \times & \times \\ \times & \times \end{vmatrix}$.

A FURTHER DEVELOPMENT OF TESTS FOR NORMALITY.

By E. S. PEARSON, D.Sc.

(1) Many of the simpler methods of statistical analysis have been developed only for variables which are normally distributed. We have often *a priori* reasons based perhaps on parallel experience for believing that the material is so distributed, but in many cases it is important to obtain evidence on this point from the data, that is to say, it is necessary to apply some test for normality to the sample. The problem is of course two-sided; it is not enough to know that the sample *could* have come from a normal population; we must be clear that it is at the same time improbable that it has come from a population differing so much from the normal as to invalidate the use of "normal theory" tests in further handling of the material. When dealing with a single variable a knowledge of the sampling distribution of β_1 (or $\sqrt{\beta_1}$) and β_2 in terms of the population moments would go far towards the solution of the problem. It is true that there have long been available tables giving the standard errors of β_1 and β_2 and their inter-correlation in terms of the population β_1 and β_2*, but these are based upon the first order terms only in an expansion, and no precise information has been available regarding the size of sample for which the expressions may be considered as accurate, nor as to the shape of the sampling curves. Recent work of Dr C. C. Craig† and Dr R. A. Fisher‡ has, however, now made possible a considerable advance towards the solution of one side of the problem; that is to say, towards a knowledge of the sampling distributions of $\sqrt{\beta_1}$ and β_2 if the population be in fact normal. Reference has been made in the preceding paper to these two sets of results§; in order, however, to place the test on firmer ground and present it in form readily available for practical use, it seemed desirable to carry the expansions for the moments of $\sqrt{\beta_1}$ and β_2 to a higher order of approximation than was reached by these authors. The fresh expressions for the higher semi-invariants given by Dr Wishart have made this extension possible. It should be remembered that as in sampling from a normal population $\sqrt{\beta_1}$ and β_2 are uncorrelated, we have two separate tests. When dealing with other populations it becomes necessary to consider the co-variation of $\sqrt{\beta_1}$ and β_2.

* *Tables for Statisticians and Biometricians*, Part I. Cambridge University Press. Tables XXXVII—XXXIX. The values are of course based on the assumption that the population distribution may be represented approximately by one of the Pearson system of frequency curves.

† C. C. Craig, *Metron*, Vol. VII. 4 (1928), pp. 3—74.

‡ R. A. Fisher, *Proc. Lond. Math. Soc.* (2), Vol. XXX. (1929), pp. 199—238.

§ J. Wishart, pp. 224—238 of the present number of this Journal.

The present paper falls naturally into two parts:

(*a*) In which the results of Fisher and of Wishart have been used to obtain values of the first four moment coefficients of the sampling distributions of $\sqrt{\beta_1}$ and β_2 as far as the terms in n^{-3}.

(*b*) In which tables are given showing for different sizes of sample, n, the values of β_1 and β_2 corresponding to ·05 and ·01 probability points*; these being based on the assumption that Pearson Type VII and Type IV curves with the correct first four moment coefficients will adequately represent the true sampling distributions of the constants as far as these two levels of probability are concerned.

(2) *Moment Coefficients of the Sampling Distribution of* $\sqrt{\beta_1}$.

I shall use here the notation of Fisher and take†

$$x=\sqrt{\frac{(n-1)(n-2)}{6n}}\,\frac{k_3}{k_2^{\frac{3}{2}}}=\frac{n-1}{\sqrt{6(n-2)}}\sqrt{\beta_1}\quad\text{..................(1)},$$

$$=\sqrt{\frac{(n-1)(n-2)}{6n}}\,\frac{1}{\kappa_2^{\frac{3}{2}}}k_3\left\{1+\frac{k_2-\kappa_2}{\kappa_2}\right\}^{-\frac{3}{2}}\quad\text{..................(2)}.$$

Remembering that the population is symmetrical, it is clear that in sampling x will be symmetrically distributed about zero. Let us find the 2nd and 4th moment coefficients of x. We obtain from (2),

$$x^2=\frac{(n-1)(n-2)}{6n}\,\frac{1}{\kappa_2^3}\left\{k_3^2-\frac{3}{\kappa_2}k_3^2(k_2-\kappa_2)+\frac{6}{\kappa_2^2}k_3^2(k_2-\kappa_2)^2-\frac{10}{\kappa_2^3}k_3^2(k_2-\kappa_2)^3\right.$$
$$\left.+\frac{15}{\kappa_2^4}k_3^2(k_2-\kappa_2)^4-\frac{21}{\kappa_2^5}k_3^2(k_2-\kappa_2)^5+\frac{28}{\kappa_2^6}k_3^2(k_2-\kappa_2)^6-\dots\right\}\quad\text{.........(3)},$$

$$x^4=\frac{(n-1)^2(n-2)^2}{36n^2}\,\frac{1}{\kappa_2^6}\left\{k_3^4-\frac{6}{\kappa_2}k_3^4(k_2-\kappa_2)+\frac{21}{\kappa_2^2}k_3^4(k_2-\kappa_2)^2-\frac{56}{\kappa_2^3}k_3^4(k_2-\kappa_2)^3\right.$$
$$\left.+\frac{126}{\kappa_2^4}k_3^4(k_2-\kappa_2)^4-\frac{252}{\kappa_2^5}k_3^4(k_2-\kappa_2)^5+\frac{462}{\kappa_2^6}k_3^4(k_2-\kappa_2)^6-\dots\right\}\quad\text{......(4)}.$$

We must now take mean values of both sides of the equations, and shall need to evaluate terms of the form $\mu(k_3^2, (k_2-\kappa_2)^q)$ and $\mu(k_3^4, (k_2-\kappa_2)^q)$. Following the method of Fisher, these μ's must now be expressed in terms of the corresponding semi-invariants of k_2 and k_3. If u and v be two variables this process is, in general, carried out by means of the identity in t_1 and t_2,

$$1+\mu(u)t_1+\mu(v)t_2+\mu(u^2)\frac{t_1^2}{2!}+\mu(uv)\frac{t_1t_2}{1!\,1!}+\mu(v^2)\frac{t_2^2}{2!}+\dots$$
$$\equiv\exp\left\{\kappa(u)t_1+\kappa(v)t_2+\kappa(u^2)\frac{t_1^2}{2!}+\kappa(uv)\frac{t_1t_2}{1!\,1!}+\kappa(v^2)\frac{t_2^2}{2!}+\dots\right\}\quad\text{... (5)}.$$

* That is to say, the points at which ordinates of the sampling distribution cut off tail areas measuring 5 % and 1 % of the total area under the frequency curve.

† The relations between k_2, k_3 and k_4 and the sample moment coefficients have been given above on p. 226 by Wishart. x is really the ratio of m_3 (or k_3) to a sample estimate of its standard error.

In the present case $u = k_3$, $v = k_2 - \kappa_2$; it follows that

(a) $\mu(u) = 0 = \mu(v)$; hence $\kappa(u) = 0 = \kappa(v)$.

(b) The κ's of 2nd and higher order being independent of the origin chosen,

$\kappa(k_3^p, (k_2 - \kappa_2)^q) = \kappa(k_3^p, k_2^q) = (3^p 2^q)$ (for convenience in writing).

(c) Since the population is normal $\kappa(k_3, k_2^q) = (32^q) = 0$ (see Wishart's equation (20), p. 235 above).

Bearing these results in mind and retaining *within* the square brackets of (6) and (7) below terms up to the order of (n^{-4}) for $\mu_2(x)$ and (n^{-5}) for $\mu_4(x)$*, it will be found that

$$\mu(x^2) = \mu_2(x) = \frac{(n-1)(n-2)}{6n}\frac{1}{\kappa_2^3}\Big\{(3^2) - \frac{3}{\kappa_2}(3^2 2)$$

$$+ \frac{6}{\kappa_2^2}[(3^2 2^2) + (3^2)(2^2)] - \frac{10}{\kappa_2^3}[(3^2 2^3) + 3(3^2 2)(2^2) + (3^2)(2^3)]$$

$$+ \frac{15}{\kappa_2^4}[6(3^2 2^2)(2^2) + 4(3^2 2)(2^3) + (3^2)(2^4) + 3(3^2)(2^2)^2]$$

$$- \frac{21}{\kappa_2^5}[15(3^2 2)(2^2)^2 + 10(3^2)(2^3)(2^2)] + \frac{28}{\kappa_2^6}[15(3^2)(2^2)^3]\ldots\Big\} \quad \ldots\ldots\ldots\ldots(6),$$

$$\mu(x^4) = \mu_4(x) = \frac{(n-1)^2(n-2)^2}{36n^2}\frac{1}{\kappa_2^6}\Big\{(3^4) + 3(3^2)^2 - \frac{6}{\kappa_2}[(3^4 2) + 6(3^2 2)(3^2)]$$

$$+ \frac{21}{\kappa_2^2}[(3^4 2^2) + (3^4)(2^2) + 6(3^2 2^2)(3^2) + 6(3^2 2)^2 + 3(3^2)^2(2^2)]$$

$$- \frac{56}{\kappa_2^3}[3(3^4 2)(2^2) + (3^4)(2^3) + 6(3^2 2^3)(3^2) + 18(3^2 2^2)(3^2 2)$$
$$+ 18(3^2 2)(3^2)(2^2) + 3(3^2)^2(2^3)]$$

$$+ \frac{126}{\kappa_2^4}[3(3^4)(2^2)^2 + 36(3^2 2^2)(3^2)(2^2) + 36(3^2 2)^2(2^2) + 24(3^2 2)(3^2)(2^3)$$
$$+ 3(3^2)^2(2^4) + 9(3^2)^2(2^2)^2]$$

$$- \frac{252}{\kappa_2^5}[90(3^2 2)(3^2)(2^2)^2 + 30(3^2)^2(2^3)(2^2)] + \frac{462}{\kappa_2^6}[45(3^2)^2(2^2)^3]\ldots\Big\} \ldots(7).$$

We may now substitute into (6) and (7) the expressions for the semi-invariants of k_2 and k_3 tabled by Fisher and Wishart†. Since the population is normal we are concerned with the terms which contain powers of κ_2 only, the population variance; as is necessarily the case, since x is independent of scale the powers of κ_2 divide out and we are left with the following expressions in n, for the 2nd and 4th moment coefficients of x,

$$\mu_2(x) = 1 - \frac{6}{n-1} + \frac{28}{(n-1)^2} - \frac{120}{(n-1)^3} + \ldots$$

$$= 1 - \frac{6}{n} + \frac{22}{n^2} - \frac{70}{n^3} + \ldots \quad \ldots\ldots\ldots\ldots(8),$$

* The term $\kappa(3^p 2^q)$ is of order $n^{-(p+q-1)}$.

† Fisher, *loc. cit.* pp. 210—214; these results are general, for any population. Wishart, p. 233 above; these are for a normal population only.

$$\mu_4(x) = 3 - \frac{90}{n-1} + \frac{1680}{(n-1)^2} - \frac{25,200}{(n-1)^3} + \dots + \frac{5n-12}{n-2}\left(\frac{18}{n-1} - \frac{540}{(n-1)^2} + \frac{10,080}{(n-1)^3} - \dots\right)$$

$$= 3 - \frac{1056}{n^2} + \frac{24,132}{n^3} - \dots \quad \text{(9)}.$$

Whence, using the relation (1) between x and $\sqrt{\beta_1}$, we find*

$$\sigma_{\sqrt{\beta_1}} = \sqrt{\frac{6}{n}}\left(1 - \frac{3}{n} + \frac{6}{n^2} - \frac{15}{n^3} + \dots\right) \quad \text{(10)},$$

$$B_2(\sqrt{\beta_1}) = \frac{\mu_4(x)}{\{\mu_2(x)\}^2} = 3 + \frac{36}{n} - \frac{864}{n^2} + \frac{12,096}{n^3} - \dots \quad \text{(11)}.$$

(3) *Moment Coefficients of the Sampling Distribution of* β_2.

In this case Fisher has taken†

$$x = \sqrt{\frac{(n-1)(n-2)(n-3)}{24n(n+1)}}\,\frac{k_4}{k_2^2} = \sqrt{\frac{(n+1)(n-1)^3}{24n(n-2)(n-3)}}\left\{\beta_2 - \frac{3(n-1)}{n+1}\right\} \dots \text{(12)},$$

$$= \sqrt{\frac{(n-1)(n-2)(n-3)}{24n(n+1)}}\,\frac{1}{\kappa_2^2}\,k_4\left(1 + \frac{k_2-\kappa_2}{\kappa_2}\right)^{-2} \quad \text{(13)}.$$

Here the mean value of x is zero because in samples from a normal population k_4 and k_2 are completely uncorrelated and mean $(k_4) = 0$. It follows that

$$\text{Mean } \beta_2 = 3(n-1)/(n+1).$$

The 2nd, 3rd and 4th moment coefficients of x may be found from (13) by raising x to the appropriate power and expanding the right-hand side of the equation. As in the preceding section we obtain terms of the form $\mu(k_4^p, (k_2-\kappa_2)^q)$ ($p = 2$, 3 and 4), which can be expressed in terms of the semi-invariants $\kappa(k_4^p, k_2^q)$ or $(4^p 2^q)$ by means of the identity (5). It will be adequate to give only the final results in which the expressions within the square brackets have been carried as far as terms in n^{-4} for $\mu_2(x)$, and n^{-5} for $\mu_3(x)$ and $\mu_4(x)$.

$$\mu_2(x) = \frac{(n-1)(n-2)(n-3)}{24n(n+1)}\,\frac{1}{\kappa_2^4}\Big\{(4^2) - \frac{4}{\kappa_2}(4^2 2) + \frac{10}{\kappa_2^2}[(4^2 2^2) + (4^2)(2^2)]$$
$$- \frac{20}{\kappa_2^3}[(4^2 2^3) + 3(4^2 2)(2^2) + (4^2)(2^3)]$$
$$+ \frac{35}{\kappa_2^4}[6(4^2 2^2)(2^2) + 4(4^2 2)(2^3) + (4^2)(2^4) + 3(4^2)(2^2)^2]$$
$$- \frac{56}{\kappa_2^5}[15(4^2 2)(2^2)^2 + 10(4^2)(2^3)(2^2)] + \frac{84}{\kappa_2^6}[15(4^2)(2^2)^3]\dots\Big\} \quad \text{(14)},$$

$$\mu_3(x) = \left\{\frac{(n-1)(n-2)(n-3)}{24n(n+1)}\right\}^{\frac{3}{2}}\frac{1}{\kappa_2^6}\Big\{(4^3) - \frac{6}{\kappa_2}(4^3 2) + \frac{21}{\kappa_2^2}[(4^3 2^2) + (4^3)(2^2)]$$
$$- \frac{56}{\kappa_2^3}[(4^3 2^3) + 3(4^3 2)(2^2) + (4^3)(2^3)] + \frac{126}{\kappa_2^4}[6(4^3 2^2)(2^2) + 4(4^3 2)(2^3)$$
$$+ (4^3)(2^4) + 3(4^3)(2^2)^2]$$
$$- \frac{252}{\kappa_2^5}[15(4^3 2)(2^2)^2 + 10(4^3)(2^3)(2^2)] + \frac{462}{\kappa_2^6}[15(4^3)(2^2)^3]\dots\Big\} \quad \text{(15)},$$

* Mean $\sqrt{\beta_1}$ and $B_1(\sqrt{\beta_1})$ are both equal to zero.

† x is here the ratio of k_4 to a sample estimate of its standard error.

$$\mu_4(x) = \left\{\frac{(n-1)(n-2)(n-3)}{24n(n+1)}\right\}^2 \frac{1}{\kappa_2^6} \left\{(4^4) + 3(4^2)^2 - \frac{8}{\kappa_2}[(4^4 2) + 6(4^2 2)(4^2)]\right.$$

$$+ \frac{36}{\kappa_2^2}[(4^4 2^2) + (4^4)(2^2) + 6(4^2 2^2)(4^2) + 6(4^2 2)^2 + 3(4^2)^2(2^2)]$$

$$- \frac{120}{\kappa_2^3}[3(4^4 2)(2^2) + (4^4)(2^3) + 6(4^2 2^3)(4^2) + 18(4^2 2^2)(4^2 2)$$

$$+ 18(4^2 2)(4^2)(2^2) + 3(4^2)^2(2^3)]$$

$$+ \frac{330}{\kappa_2^4}[3(4^4)(2^2)^2 + 36(4^2 2^2)(4^2)(2^2) + 24(4^2 2)(4^2)(2^3) + 36(4^2 2)^2(2^2)$$

$$+ 3(4^2)^2(2^4) + 9(4^2)^2(2^2)^2]$$

$$\left. - \frac{792}{\kappa_2^5}[90(4^2 2)(4^2)(2^2)^2 + 30(4^2)^2(2^3)(2^2)] + \frac{1716}{\kappa_2^6}[45(4^2)^2(2^2)^3]\ldots\right\} \quad (16).$$

The values of the semi-invariants of k_4 and k_2 taken from the tables of Fisher and Wishart must now be substituted into (14), (15) and (16). If this is carried out, it is found after reduction that

$$\mu_2(x) = 1 - \frac{12}{n-1} + \frac{100}{(n-1)^2} - \frac{720}{(n-1)^3} + \ldots$$

$$= 1 - \frac{12}{n} + \frac{88}{n^2} - \frac{532}{n^3} + \ldots \quad \ldots\ldots(17),$$

$$\mu_3(x) = \frac{n^2 - 5n + 2}{\sqrt{6n(n+1)(n-1)(n-2)(n-3)}} \left\{36 - \frac{1080}{n-1} + \frac{20{,}160}{(n-1)^2} - \frac{302{,}400}{(n-1)^3} + \ldots\right\}$$

$$= 6\sqrt{\frac{6}{n}} \left\{1 - \frac{65}{2n} + \frac{4811}{8n^2} - \frac{136{,}605}{16n^3} + \ldots\right\} \quad \ldots\ldots(18),$$

$$\mu_4(x) = 3 - \frac{168}{n-1} + \frac{5544}{(n-1)^2} - \frac{141{,}120}{(n-1)^3} + \ldots$$

$$+ \frac{12(53n^4 - 428n^3 + 1025n^2 - 474n + 180)}{(n+1)n(n-1)(n-2)(n-3)} \left\{1 - \frac{56}{n-1} + \frac{1848}{(n-1)^2} - \ldots\right\}$$

$$= 3 + \frac{468}{n} - \frac{32{,}196}{n^2} + \frac{1{,}118{,}388}{n^3} - \ldots \quad \ldots\ldots(19).$$

Hence, using the relation (12) between x and β_2, it follows that

$$\text{Mean } \beta_2 = \frac{3(n-1)}{n+1} \quad \ldots\ldots(20),$$

$$\sigma_{\beta_2} = \sqrt{\frac{24}{n}}\left(1 - \frac{15}{2n} + \frac{271}{8n^2} - \frac{2319}{16n^3} + \ldots\right) \ldots\ldots(21),$$

$$B_1(\beta_2) = \frac{\{\mu_3(x)\}^2}{\{\mu_2(x)\}^3} = \frac{216}{n}\left(1 - \frac{29}{n} + \frac{519}{n^2} - \frac{7637}{n^3} + \ldots\right) \quad \ldots\ldots(22),$$

$$B_2(\beta_2) = \frac{\mu_4(x)}{\{\mu_2(x)\}^2} = 3 + \frac{540}{n} - \frac{20{,}196}{n^2} + \frac{470{,}412}{n^3} - \ldots \quad \ldots\ldots(23).$$

(4) *Approximations to the Probability Integrals of* $\sqrt{\beta_1}$ *and* β_2.

The first problem to consider is the degree of convergence of the series (10) and (11), and (21), (22) and (23). For this purpose the Tables I and II have been prepared in which the values entered for σ, B_1 and B_2 are based of course only on those terms of the expansions given above. It is necessary to assume that the adequacy of the convergence can be judged from the first four terms of each series. The expressions for the standard errors are clearly adequately represented by the series at $n=50$. Columns have been inserted showing the degree of approximation

TABLE I. *Moment Coefficients of* $\sqrt{\beta_1}$.

n	$\sqrt{6/n}$	Terms in $\sigma_{\sqrt{\beta_1}}$	$\sigma_{\sqrt{\beta_1}}$	Terms in $B_2(\sqrt{\beta_1})$	$B_2(\sqrt{\beta_1})$
50	·3464	1 − ·060 000 + ·002 400 − ·000 120	·3264	3 + ·720 000 − ·345 600 + ·096 768	3·4712
75	·2828	1 − ·040 000 + ·001 067 − ·000 036	·2718	3 + ·480 000 − ·153 600 + ·028 668	3·3551
100	·2449	1 − ·030 000 + ·000 600 − ·000 015	·2377	3 + ·360 000 − ·086 400 + ·012 096	3·2857
150	·2000	—	·1961	3 + ·240 000 − ·038 400 + ·003 580	3·2052
200	·1732	—	·1706	3 + ·180 000 − ·021 600 + ·001 512	3·1599
250	·1549	—	·1531	3 + ·144 000 − ·013 824 + ·000 774	3·1310
500	·1095	—	·1089	3 + ·072 000 − ·003 456 + ·000 097	3·0686
1000	·0775	—	·0772	3 + ·036 000 − ·000 864 + ·000 012	3·0351

TABLE II. *Moment Coefficients of* β_2.

n	$\sqrt{24/n}$	Terms in σ_{β_2}	σ_{β_2}	Terms in $B_1(\beta_2)$	$B_1(\beta_2)$
50	·6928	1 − ·150 000 + ·013 550 − ·001 159	·5975	1 − ·580 000 + ·207 600 − ·061 096	2·4473
75	·5657	1 − ·100 000 + ·006 022 − ·000 344	·5123	1 − ·386 667 + ·092 267 − ·018 100	1·9800
100	·4899	1 − ·075 000 + ·003 387 − ·000 145	·4547	1 − ·290 000 + ·051 900 − ·007 637	1·6292
150	·4000	1 − ·050 000 + ·001 506 − ·000 043	·3806	1 − ·193 333 + ·023 067 − ·002 261	1·1916
200	·3464	1 − ·037 500 + ·000 847 − ·000 018	·3337	1 − ·145 000 + ·012 975 − ·000 955	·9364
250	·3098	—	·3007	1 − ·116 000 + ·008 304 − ·000 489	·7705
500	·2191	—	·2158	1 − ·058 000 + ·002 076 − ·000 061	·4078
1000	·1549	—	·1538	1 − ·029 000 + ·000 519 − ·000 008	·2098

n	Terms in $B_2(\beta_2)$	$B_2(\beta_2)$
50	3 + 10·800 000 − 8·078 400 + 3·763 296	—
75	3 + 7·200 000 − 3·590 404 + 1·114 876	—
100	3 + 5·400 000 − 2·019 600 + ·470 412	6·8508
150	3 + 3·600 000 − ·897 591 + ·139 383	5·8418
200	3 + 2·700 000 − ·504 900 + ·058 801	5·2539
250	3 + 2·160 000 − ·323 136 + ·030 106	4·8670
500	3 + 1·080 000 − ·080 784 + ·003 763	4·0030
1000	3 + ·540 000 − ·020 196 + ·000 470	3·5203

of $\sqrt{6/n}$ to $\sigma_{\sqrt{\beta_1}}$ and of $\sqrt{24/n}$ to σ_{β_2}*. It may be said roughly that for most practical purposes a knowledge of B_1 and B_2 correct to the 2nd decimal place is sufficient. Thus at $n=100$ and perhaps at 75, the series for $B_2(\sqrt{\beta_1})$ and $B_1(\beta_2)$ may be considered as satisfactory. The convergence of the expression for $B_2(\beta_2)$ is a good deal slower. It would of course have been possible to develop the series to further terms by retaining semi-invariants of higher order, but even if for n less than 50 the resulting series were found to converge, it seems likely that the test in such cases would be of no great practical value. The test might enable us to say that in a sample of 20, let us suppose, values of $\beta_1 = \cdot 8$ and $\beta_2 = 4\cdot 8$ could well have occurred in random sampling from a normal population. But such a sample *might* have come from a population in which, let us say, $\beta_1 = 1\cdot 5$, $\beta_2 = 5\cdot 5$; hence the sample data alone would give us no confidence in assuming normality in the population. Such an assumption must be justified from outside evidence which the sample values, while they would not contradict, would hardly strengthen.

An exact solution of the problem must await a knowledge of the true sampling distributions of $\sqrt{\beta_1}$ and β_2. In the meantime an approximate solution of some practical value can be obtained. Consider first the distribution of $\sqrt{\beta_1}$. Table I shows that this is a symmetrical leptokurtic curve which tends fairly rapidly to the normal. Fisher has obtained an approximation to its probability integral by constructing a function of x which as far as terms in n^{-2} is normally distributed with unit standard deviation†. The relation between this function, ξ, and x is given as follows:

$$\xi = x\left(1+\frac{3}{n}+\frac{91}{4n^2}\right)-\frac{3}{2n}\left(1-\frac{111}{2n}\right)(x^3-3x)-\frac{33}{8n^2}(x^5-10x^3+15x) \quad \ldots(24),$$

the coefficients being determined so that $\kappa_2(x)$, $\kappa_4(x)$ and $\kappa_6(x)$ (or $\mu_2(x)$, $\mu_4(x)$ and $\mu_6(x)$) are correct as far as terms in n^{-2}. But the expression used for $\kappa_6(x)$, namely $15120/n^2$, containing only a single term is of doubtful value as an approximation to the 6th semi-invariant of x even at $n=100$. To proceed by this method to terms in n^{-3} would involve the calculation of $\kappa_8(x)$. It seems therefore possible that as good an approximation will be obtained by assuming that the distribution of $\sqrt{\beta_1}$ may be closely represented by a Type VII curve of form

$$y = y_0\left(1+(\sqrt{\beta_1})^2/a^2\right)^{-m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25),$$

whose constants are to be determined from the values of $\sigma_{\sqrt{\beta_1}}$, and $B_2(\sqrt{\beta_1})$ given in (10) and (11) above. Table I suggests that for $n \geqslant 50$ the expression (10) is completely adequate, while little error will be involved in using (11) for $n \geqslant 75$. For a curve of Type VII with the moments of (10) and (11) it can be shown that

$$\beta_4 = \frac{\mu_6}{\mu_2^3} = 15\left(1+\frac{36}{n}-\frac{288}{n^2}+\ldots\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(26).$$

* These approximations to the standard errors were first given by K. Pearson in 1901, *Phil. Trans.* Vol. 198 A, p. 278.

† *Loc. cit.* pp. 283—285.

On the other hand, using Fisher's value for $\kappa_6(x)$ quoted above, the true β_4 of the x-distribution as far as terms in n^{-2} is

$$\beta_4 = 15\left(1 + \frac{36}{n} + \frac{144}{n^2} + \dots\right) \quad \dots\dots\dots\dots\dots\dots\dots\dots(27).$$

At $n=100$ the error is about 4 °/₀ which will probably not affect the form of the curve seriously in the region of significant frequency. In Table III are compared at $n=50$ and 100 the chances P_1, P_2 and P_3 of $\sqrt{\beta_1}$ exceeding in sampling certain multiples of its standard error, $\sigma_{\sqrt{\beta_1}}$, found on three different hypotheses, namely:

(1) that $\sqrt{\beta_1}$ is normally distributed with $\sigma_{\sqrt{\beta_1}}$ given by (10),

(2) that $\sqrt{\beta_1}$ follows the Type VII curve,

(3) that the ξ of (24) is normally distributed with unit standard deviation, where $x=(n-1)\sqrt{\beta_1/6(n-2)}$.

TABLE III. *Approximations to Probability Integral of* $\sqrt{\beta_1}$.

Values of $\sqrt{\beta_1}/\sigma_{\sqrt{\beta_1}}$	$n=50$			$n=100$		
	P_1	P_2	P_3	P_1	P_2	P_3
1·2	·1151	·1094	·1159	·1151	·1113	·1125
1·6	·0548	·0534	·0532	·0548	·0539	·0535
2·0	·0228	·0241	·0205	·0228	·0237	·0227
2·4	·0082	·0102	·0067	·0082	·0096	·0087
2·8	·0026	·0042	·0020	·0026	·0037	·0032

The values required on hypothesis (2) have been found by interpolating in "Student's" Tables*. For $n=50$ a value of $B_2(\sqrt{\beta_1})=3{\cdot}45$ was used†. It will be seen that at $n=100$ the differences are of very little practical importance; at $n=50$, although they are larger, it seems impossible to say without further information whether P_2 or P_3 is the more accurate. There would be little error involved in assuming the distribution of $\sqrt{\beta_1}$ to be normal with $\sigma_{\sqrt{\beta_1}}=\sqrt{6/n}$ for $n \geqslant 100$. In the Table printed at the end of this paper giving the 5 °/₀ and 1 °/₀ points for different values of n, I have, however, assumed the distribution to be of Type VII with the moments given by (10) and (11). The deviations from the mean to the ordinates cutting off these tail areas were found with the help of "Student's" Tables and graphical interpolation.

The distribution of β_2 is less easy to deal with. Fisher has suggested the use of another normally distributed function, ξ, of the x of equation (12), but as the transformation depends upon the use of expressions for $\kappa_4(x)$ and $\kappa_5(x)$ containing each only the first term of an expansion in inverse powers of n, the degree of

* *Metron*, Vol. v. 3, 1925, pp. 113—120.

† As far as the term in n^{-3}, the value shown in Table I is 3·4712, and a rough guess at the effect of the term in n^{-4} was made.

accuracy of the method is very uncertain. If the values of $B_1(\beta_2)$ and $B_2(\beta_2)$ given in Table II be plotted it will be found that they fall on a curve in the Type IV area which converges on the Type V and Type III lines, slowly approaching the Normal Point as $n \to \infty$. I have therefore made the assumption that the distribution of β_2 can be approximately represented by a Pearson Type IV curve with the moment coefficients given by the expressions (20)—(23), that is to say, by an equation of form

$$y = y_0\left(1 + \frac{x^2}{a^2}\right)^{-p} e^{-\nu \tan^{-1}\frac{x}{a}} \quad \text{.........................(28).}$$

At $n = 100$ it will be seen from Table II that the four terms in the expansion for $B_2(\beta_2)$ are not sufficient to insure convergence even to the first decimal place, but for $B_1(\beta_2)$ they are so. Experience in curve fitting suggests, however, that this degree of uncertainty in B_2 when B_1 is known is not likely to have much influence on the deviation from the mean to the 5 °/$_\circ$ and 1 °/$_\circ$ probability points. That is to say, the chief danger of error present at $n = 100$ will not be due to uncertainty as to the B_2 of the Type IV curve, but to the use of a Type IV curve at all in place of the true curve. What this degree of approximation may be cannot at present be judged; Pearson curves based on theoretical values of the first four moment coefficients have been found in other problems to provide very satisfactory approximations to sampling distributions*, but these skew leptokurtic systems form a somewhat extreme case. It may, however, be said without hesitation that the results set out in Table IV below provide a test for normality of β_2 which will be far more accurate than has hitherto been available. $\sqrt{24/n}$ is a good approximation to the standard error of β_2 at $n = 50$, but even at $n = 1000$ the sampling distribution is not normal, viz. $B_1 = \cdot 21$, $B_2 = 3\cdot 52$.

The method of computation was as follows. The ordinates of Type IV curves with B_1 and B_2 as in (22) and (23) were calculated for the cases $n = 100$, 150, 250 and 1000. From these were obtained by quadrature the deviations from the mean in terms of the standard deviation to the ordinates cutting off 5°/$_\circ$ and 1°/$_\circ$ tail areas. As n increases these deviations tend to the normal curve values of $\pm 1\cdot 6449$ and $\pm 2\cdot 3263$ respectively. Also as n increases the deviations approach closely the corresponding deviations in the Type III curve which has the same value of B_1; these latter were found from the Tables of the Incomplete Gamma Function. With these results to form a guide, it was possible to obtain graphically with sufficient accuracy the deviations from the mean to the 5 °/$_\circ$ and 1 °/$_\circ$ points for all the other Type IV curves required†. These deviations with the appropriate means and standard deviations given by (20) and (21) have provided the limiting values of β_2 given in Table IV.

* E.g. when used with experimental data, in connection with the distributions of the mean and the variance. A. E. R. Church, *Biometrika*, Vol. XVIII. pp. 321—394. Or again when compared with true theoretical curves as for the sampling distributions of r_{11}. Pearson, Jeffery and Elderton, *Biometrika*, Vol. XXI. pp. 164—201.

† The error involved in this process should not be greater than a single unit in the last (2nd) decimal place of the values of β_2 tabled. This can hardly be *greater* than the error involved in the assumption that Type IV curves will represent the sampling distribution of β_2.

TABLE IV. 5 % *and* 1 % *Points for* $\sqrt{\beta_1}$, β_1 *and* β_2.

Size of Sample	$\sqrt{\beta_1}$ Lower and Upper Limits		β_1 Upper Limits		β_2 Lower Limits		β_2 Upper Limits	
	5 %	1 %	10 %	2 %	1 %	5 %	5 %	1 %
50	·533	·787	·285	·619	—	—	—	—
75	·445	·651	·198	·424	—	—	—	—
100	·389	·567	·152	·321	2·18	2·35	3·77	4·39
125	·350	·508	·123	·258	2·24	2·40	3·70	4·24
150	·321	·464	·103	·216	2·29	2·45	3·65	4·14
175	·298	·430	·089	·185	2·33	2·48	3·61	4·05
200	·280	·403	·078	·162	2·37	2·51	3·57	3·98
250	·251	·360	·063	·130	2·42	2·55	3·52	3·87
300	·230	·329	·053	·108	2·46	2·59	3·47	3·79
350	·213	·305	·045	·093	2·50	2·62	3·44	3·72
400	·200	·285	·040	·081	2·52	2·64	3·41	3·67
450	·188	·269	·035	·072	2·55	2·66	3·39	3·63
500	·179	·255	·032	·065	2·57	2·67	3·37	3·60
550	·171	·243	·029	·059	2·58	2·69	3·35	3·57
600	·163	·233	·027	·054	2·60	2·70	3·34	3·54
650	·157	·224	·025	·050	2·61	2·71	3·33	3·52
700	·151	·215	·023	·046	2·62	2·72	3·31	3·50
750	·146	·208	·021	·043	2·64	2·73	3·30	3·48
800	·142	·202	·020	·041	2·65	2·74	3·29	3·46
850	·138	·196	·019	·038	2·66	2·74	3·28	3·45
900	·134	·190	·018	·036	2·66	2·75	3·28	3·43
950	·130	·185	·017	·034	2·67	2·76	3·27	3·42
1000	·127	·180	·016	·032	2·68	2·76	3·26	3·41
1200	·116	·165	·013	·027	2·71	2·78	3·24	3·37
1400	·107	·152	·012	·023	2·72	2·80	3·22	3·34
1600	·100	·142	·010	·020	2·74	2·81	3·21	3·32
1800	·095	·134	·000	·018	2·76	2·82	3·20	3·30
2000	·090	·127	·008	·016	2·77	2·83	3·18	3·28
2500	·080	·114	·006	·013	2·79	2·85	3·16	3·25
3000	·073	·104	·005	·011	2·81	2·86	3·15	3·22
3500	·068	·096	·005	·009	2·82	2·87	3·14	3·21
4000	·064	·090	·004	·008	2·83	2·88	3·13	3·19
4500	·060	·085	·004	·007	2·84	2·88	3·12	3·18
5000	·057	·081	·003	·006	2·85	2·89	3·12	3·17

(5) *Illustration of Use of Table IV.*

In a sample of 500 the following values are found:

$$\sqrt{\beta_1} = -\cdot 2040; \quad \beta_1 = \cdot 0416; \quad \beta_2 = 3\cdot 7823.$$

Is it possible that the sampled population was normal?

The table shows that in 5 % of random samples from a normal population $\sqrt{\beta_1}$ may be expected to be less than $-\cdot 179$, and in 1 % less than $-\cdot 255$. The observed value falls in between these limits. In using β_1 positive and negative

values of $\sqrt{\beta_1}$ are clubbed together, and we see that in 10 % of random samples β_1 may be expected to be greater than ·032 and in 2 % greater than ·065. The observed value of course falls again between the limits. For β_2 we see that only 1 % of samples can be expected to give a β_2 greater than 3·60; the observed value of 3·7823 lies outside the limit. The test therefore provides a doubtful answer when applied to β_1 but a decisive one when applied to β_2, and we may conclude that it is practically certain that the sample has not been drawn randomly from a normal population.

(6) *Summary.*

(*a*) The work of Fisher and of Craig has made it comparatively simple to obtain expressions for the moment coefficients of the distributions of $\sqrt{\beta_1}$ and β_2 in samples of n from a normal population. These expressions are in the form of series in inverse powers of n. In order to see more clearly the degree of convergence of these series and to obtain more accurate values in smaller samples, it was necessary to extend the series beyond the point reached by those writers. This it has been possible to carry out with the aid of new results obtained by Wishart.

(*b*) The moment coefficients show that the distribution of $\sqrt{\beta_1}$ is a symmetrical leptokurtic curve which tends to the normal fairly rapidly as n increases. For rough purposes it may be taken as normal with a standard error of $\sqrt{6/n}$ for $n \geqslant 100$. The distribution of β_2 is an extremely skew curve at $n = 100$, and even when $n = 1000$ can hardly be considered as normal.

(*c*) A table has been given of the approximate 5 % and 1 % probability points for $\sqrt{\beta_1}$ and β_2, based on the assumption that the true distribution may be adequately represented by Pearson curves with the correct first four moment coefficients. This table starts at $n = 50$ for $\sqrt{\beta_1}$ and $n = 100$ for β_2.

(*d*) A complete test for normality must really be two-sided; it must help us not only to determine whether the population sampled could have been normal, but also to judge how far from normal the population might have been. A knowledge of the true sampling distribution of $\sqrt{\beta_1}$ and β_2, when the population is normal, would enable us to answer the first point however small the sample may be, but not the second point.

(*e*) It is to be hoped that the true sampling distributions will be found, not only to remove any doubt as to the accuracy of the test, but also for the light that will be thrown on the adequacy of these methods of approximation—information that will be of considerable value in handling similar problems in the future.

MISCELLANEA.

An unusual Frequency Distribution—The Term of Abortion.

By THOS. VIBERT PEARCE, F.R.C.S. England.

Abortion in women is becoming more prevalent, and gradually will assume economic and political importance. From being a purely medical problem it will gather biological and chemical interest, since the interlocking of the female reproductive hormones is slowly being laid bare.

Out of 300 women admitted to St Giles' Hospital who left the hospital following completed abortion, 283 patients were able to give enough information to allow of a fairly reliable estimate of the term of gestation prior to abortion. Term of abortion when used in this present connection does not mean the same thing as the time of development of the foetus. Even if the time of insemination is known—quite an unusual piece of information—the time of impregnation is quite unknown, and it is difficult to see how it can ever be ascertained. Impregnation "may be postponed for days or possibly three or four weeks." In the absence of careful measurement of the foetus and frequently in the absence of the foetus itself, this possible lapse of time between insemination and impregnation compels consideration of the term of abortion from the standpoint of the maternal partner in this pathological condition. Abortion is commonly regarded as a disease of the mother and not as a disease of the foetus, although there is no logical or objective support for that opinion. Incidentally it is quite possible there may be a type of abortion due to defect in the paternal germ plasm. At any rate the mother seems to be the more important sufferer, and the post-menstrual term in default of a better definition is used as the term of abortion. By plotting the frequencies of the post-menstrual term some light might be thrown on the likelihood of abortion occurring at the expected times of menstruation which are masked or abolished by impregnation.

The women could generally remember the date of their last menstruation, although for some obscure reason they found it quite difficult to forecast the date of the next. Frequently they were rather surprised at any attempt to find the exact inter-menstrual period. Some say it always occurs on the same day of the month, and has done so for months or years. They are then really claiming that the time of menstruation is sometimes governed by the calendar fixed by Pope Gregory—a pretension that is hardly convincing. Some women say that their menstrual cycle lasts exactly four weeks, but yet are ignorant of the day of the week on which it commences. The results of a sympathetic and veiled cross-examination really suggest that quite a large proportion of these women were ignorant of the length of the menstrual period beyond their estimate that it lasts "a month." Once a woman naïvely referred me to her husband. A rather young married woman told me to ask her mother, because she always menstruated concurrently with her. One patient was rather sorry for herself, for she unfortunately commenced menstruating on washing day, which was Monday—good evidence that she had a 28-day cycle. It was hoped that some estimate of the variability of the menstrual period might be got from the statements of these women. The statements hardly ever bore examination, and were abandoned as being hopelessly unreliable. These women who had aborted or were aborting did not seem more stupid than the generality of women. Certainly their period of amenorrhoea does seem long enough for them to have forgotten their proper menstrual periods.

Besides the commencement of the last menstruation, the other point of time that is fairly satisfactorily remembered is the date of the passage of the foetus, or, in default of which, the

dates of the maximum pain and bleeding which, if they happen to coincide, fix very well the date of abortion. The commencement of symptoms before abortion is difficult to date, and hence has not been used to calculate the term of abortion. It is very hard to disentangle post-impregnation menstruation from the symptoms of abortion. Term of abortion is therefore defined as the number of days between the commencement of the last menstrual period and the passage of the foetus.

For diagrammatic purposes the term of abortion has been plotted in nearest weeks. Division by 7 is very convenient and leads to no awkward half-divisions. The question of abortion at the expected menstrual times would be much better tackled by dividing the term in days by the number of days of the patient's own menstrual cycle. A frequency diagram of term of abortion along a scale of menstrual months could then be made. Such a diagram for these women at any rate would be unreliable. This is very disappointing, for it was rather hoped that by expressing the term of abortion in menstrual months it might be possible to get some evidence which would help to decide whether the abortion was spontaneous or artificial. Presumably spontaneous abortion would occur at an expected menstrual time, while induced abortion would occur after a menstrual period had been missed and the assault on the pregnancy would be renewed after the missing of the next expected menstruation.

The variability of the menstrual period in different women when compared with the optimum term for abortion will not cause the frequency curve of the term of abortion to give so little help as it might on this question of abortion at the expected menstrual times. The mean term of abortion is about 13·41 weeks or 3 calendar months, and the variability of menstrual period seldom exceeds 1 or 2 days on either side of 30 days. It is only in the later months that the error of 1 or 2 days would be multiplied to amount to a week. By reference to the table, the plateau at the 17th and 18th weeks does suggest that this kind of error has occurred there. The 17th and 18th weeks may include abortions that occurred at the 4th month of missed menstruation. On a scale of menstrual months, the frequencies for these weeks might be amalgamated.

Frequency Table of Term of Abortion.

Term (in weeks)	4	5	6	7	8	9	10	11	12	13	14	15	16
Afebrile	3	6	7	9	11	20	13	16	9	17	9	12	6
Febrile	0	1	3	4	3	9	9	5	9	11	7	7	4
Total	3	7	10	13	14	29	22	21	18	28	16	19	10

17	18	19	20	21	22	23	24	25	26	27	28	Total	Mean Term	Standard Devn.
7	8	7	2	1	4	2	2	2	4	4	1	182	13·08	5·57
6	6	1	2	1	6	2	2	1	0	2	0	101	14·00	5·03
13	14	8	4	2	10	4	4	3	4	6	1	283	13·41	5·41

To the writer the frequency diagram does suggest that abortion occurs especially at the menstrual times, even allowing for the human characteristic of rationalisation both in patients and the observer. Whether there is any statistical warrant for such opinion seems to be a difficult problem. Presumably it would be necessary to test the goodness of fit of a frequency curve which admits of a series of maxima at regular intervals occurring along a curve which itself mounts to an apex about its mid-point. Perhaps another way of treating the problem

would be one of dissection. The scale is one of 7 months. On the hypothesis of abortion being more frequent at the menstrual times, 7 summits to the curve could be postulated. The whole curve is then regarded as made up of the summation of 7 very pointed normal frequency curves. Such a method implies that abortion at any one month is a different "clinical entity" from abortion at any other. Such an argument could not be convincing, for women who have had multiple abortions do not miscarry at the same term every time. In fact the impression left is the reverse. A normal curve would not at all represent the terms of abortions of a patient who has had multiple abortions.

[A periodogram analysis of the above data shows, as far as weekly ranges will permit, a period of four and a half weeks. Ed.]

Socrates. And furthermore, the midwives, by means of drugs and incantations, are able to arouse the pangs of labour and if they wish, to make them milder, and to cause those to bear who have difficulty in bearing; and they cause miscarriages if they think them desirable.

Theaetetus. That is true.

Socrates. Well, have you noticed this also about them, that they are the most skilful of matchmakers, since they are very wise in knowing what union of man and woman will produce the best possible children?

Theaetetus. I do not know that at all.

Plato, *Theaetetus*
(Loeb Classical Library: H. N. Fowler).

TABLES OF THE PROBABILITY INTEGRALS OF SYMMETRICAL FREQUENCY CURVES IN THE CASE OF LOW POWERS SUCH AS ARISE IN THE THEORY OF SMALL SAMPLES.

BY KARL PEARSON, ASSISTED BY BRENDA STOESSIGER, M.Sc.

(i) THE symmetrical curves to be considered are those for which $\beta_1 = 0$ and β_2 takes any value from 1 to ∞. The curves are supposed completely determined by β_2 and their standard deviations.

Their differential equation will be

$$\frac{1}{y}\frac{dy}{dx'} = \frac{2mx'}{c_0 + x'^2},$$

leading to

$$y = y_0 (c_0 + x'^2)^m,$$

where

$$c_0 = \frac{2\beta_2}{\beta_2 - 3}\sigma^2, \text{ and } m = \frac{1}{2}\frac{9 - 5\beta_2}{\beta_2 - 3}.$$

We can throw them into the following forms:

(i) $\beta_2 = 1$ to $1{\cdot}8$ ($m_1 = 1$ to 0),

$$y = y_0 \frac{1}{\left(1 - \frac{x'^2}{a_1^2}\right)^{m_1}} \quad \text{......(i)},$$

where

$$a_1^2 = \frac{2\beta_2}{3 - \beta_2}\sigma^2, \text{ and } m_1 = \frac{1}{2}\frac{9 - 5\beta_2}{3 - \beta_2}.$$

This symmetrical curve passes from two equal lumps through U-curves to a rectangle.

(ii) $\beta_2 = 1{\cdot}8$ to 3 ($m_2 = 0$ to ∞),

$$y = y_0 \left(1 - \frac{x'^2}{a_2^2}\right)^{m_2} \quad \text{......(ii)},$$

where

$$a_2^2 = \frac{2\beta_2}{3 - \beta_2}\sigma^2, \text{ and } m_2 = \frac{5\beta_2 - 9}{2(3 - \beta_2)}.$$

This type of curve passes from a rectangle through limited range curves to the normal curve ($\beta_2 = 3$).

(iii) $\beta_2 = 3$ to ∞ ($m_3 = \infty$ to $\frac{5}{2}$),

$$y = y_0 \frac{1}{\left(1 + \frac{x'^2}{a_3^2}\right)^{m_3}} \quad \text{......(iii)},$$

where

$$a_3^2 = \frac{2\beta_2}{\beta_2 - 3}\sigma^2, \text{ and } m_3 = \frac{1}{2}\frac{5\beta_2 - 9}{\beta_2 - 3}.$$

The limit $\beta \to \infty$ occurs when $m_3 = \frac{1}{2}$ and $a_3^2 = 2\sigma^2$. This curve passes from the normal curve through all grades of leptokurtosis. The limits of range in (i) are from $-a_1$ to $+a_1$, in (ii) from $-a_2$ to $+a_2$, and in (iii) from $-\infty$ to $+\infty$.

We will now proceed to the probability integral of these three curves.

For (i) we have

$$ {}_1P_x = \tfrac{1}{2} + \frac{\int_0^{x'} y\,dx'}{2\int_0^{a_1} y\,dx'} = \tfrac{1}{2}\left\{1 + \frac{B_x(\frac{1}{2}, (1-m_1))}{B(\frac{1}{2}, (1-m_1))}\right\} $$

$$ = \tfrac{1}{2}\{1 + I_x(\tfrac{1}{2}, (1-m_1))\}, $$

where $B_x(p, q)$ is the incomplete and $B(p, q)$ the complete B-function, and $I_x(p, q)$ their ratio.

The required transformation is

$$ x = x'^2/a_1^2, \quad \text{or} \quad = \frac{x'^2}{\sigma^2}\,\frac{3-\beta_2}{2\beta_2}. $$

Now m_1 lies between 0 and 1, and accordingly to obtain the probability integral of the curve (i) we have only to add unity to the B-function ratio $I_x(\frac{1}{2}, (1-m_1))$ and divide by two.

Since m_1 only lies between 0 and 1, this involves the tabulation of $I_x(\frac{1}{2}, (1-m_1))$ for small ranges of m_1; but this has not yet been completed, and we cannot at present provide a table of the probability integral of the symmetrical curve (i).

Meanwhile, and until the required table be completed, a good method to determine $I_x(\frac{1}{2}, (1-m_1))$ is to use the formula provided by Soper* for the integral

$$ \int_0^{x'} x'^{\,p-1}(1-x')^{q-1}dx', $$

when p and q are small.

We shall not further consider the probability integral of the curve (i).

For (ii) we have to make the same transformation,

$$ x = \left(\frac{x'}{a_2}\right)^2 = \frac{x'^2}{\sigma^2}\,\frac{3-\beta_2}{2\beta_2}, $$

and have
$$ {}_2P_x = \tfrac{1}{2}\{1 + I_x(\tfrac{1}{2}, (1+m_2))\}. $$

Table I gives the value of

$$ \tfrac{1}{2}\{1 + I_x(\tfrac{1}{2}, \tfrac{1}{2}(n-1))\}, $$

and accordingly we must take $n = 2m_2 + 3$; it runs from $m_2 = -\frac{1}{2}$ to $m_2 = 14$.

When $m_2 = 14$, $\beta_2 = 2{\cdot}818{,}182$, and we have not yet reached closely enough the normal curve ($\beta_2 = 3$) to use its probability integral as anything but a rough approximation.

* *Tracts for Computers*, No. VII. pp. 21—22, Cambridge University Press. See also *Tables for Statisticians and Biometricians*, Part II.

For (iii) the requisite transformation is

$$\frac{x'^2}{a_3^2}=\frac{x}{1-x}, \text{ or } x=\frac{x'^2}{x'^2+a_3^2},$$

and we have $${}_3P_{x'}=\tfrac{1}{2}\{1+I_x(\tfrac{1}{2},(m_3-\tfrac{1}{2}))\};$$

our table will accordingly give ${}_3P_{x'}$ from $m_3=15\cdot5$ to $m_3=2\cdot5$, or from $\beta_2=3\cdot230{,}769$ to $\beta_2=\infty$. The former value of β_2 is still too far from $\beta_2=3$ to allow anything but a rough approximation to be obtained from the normal curve.

If we choose our curve to be $y=\dfrac{y_0}{(1+x'^2)^{n/2}}$

as is frequently done, then $n=2m_3$, and

$$a=\sqrt{\frac{2\beta_2}{\beta_2-3}}\,\sigma=\sqrt{2m_3-3}\,\sigma=\sqrt{n-3}\,\sigma,$$

or $$\sigma=\frac{1}{\sqrt{n-3}} \text{ if we take } a=1.$$

Accordingly at the end of Table I we have placed the probability integral of the normal curve with a standard deviation $\dfrac{1}{\sqrt{n-3}}$, where $n=31$, for comparison with that of the curve

$$y=\frac{y_0}{(1+x'^2)^{15\cdot5}}.$$

The result confirms the inference drawn from the value of β_2, i.e. that the normal curve will only give a rough approximation to the exact probability integral at $n=31$. At the top of the table we may be in error in two to three units in the third place of decimals*.

(ii) We will now describe the two tables here provided.

Table I gives the value of

$$\tfrac{1}{2}\{1+I_x(\tfrac{1}{2},\tfrac{1}{2}(n-1))\},$$

where the argument x increases by ·01.

We need to know the relations between the m's and n's, and x and x'.

Curve (i). m_1 lies between 0 and 1, and the only values available in our table are for $n=2$ and 3, or $m_1=0\cdot5$ and 0, while x is determined by $x=\dfrac{x'^2}{a_1^2}$.

Curve (ii). m_2 ranges from 0 to ∞, but the table only supplies values from 0 to 14, since $m_2=\tfrac{1}{2}(n-3)$. x is found from $x=\dfrac{x'^2}{a_2^2}$.

Curve (iii). m_3 ranges from 2·5 to ∞, or our table will supply the probability integrals of this curve from 2·5 to 15·5. The x is to be found from $x=\dfrac{x'^2}{x'^2+a_3^2}$.

* Actually the unpublished tables of the B-function will carry us up to $n=101$, $m_3=50\cdot5$, a value which gives a much closer approximation to a normal curve.

When the curve is written in the form

$$y = \frac{y_0}{(1+z^2)^{\frac{1}{2}n}},$$

our table will supply the probability integrals for $n=5$ to 31. If we choose to neglect the infinity of the fourth moment we can proceed to $n=2$.

In the last form of this curve $x = \frac{z^2}{1+z^2}$, or $z^2 = x/(1-x)$. This value of z^2 is given to five decimal places in the second column of each sheet of the table. This enables the user to ascertain rapidly whereabouts he is in the x-variate for a given value of z or z^2.

(iii) We need two kinds of interpolation into Table I: (*a*) we need to interpolate between the tabulated values of n, and (*b*) we need to interpolate between the tabulated values of x. Both these interpolations give rise to difficulties, which require some consideration.

(*a*) After $n=8$, interpolations for n lying between tabled values are successful, if we use δ^2 and occasionally δ^4. Neither Table I (nor the supplementary Table II) will give satisfactory brief interpolations for n less than 8. It may even be doubted, if the argument n were tabled by 0·1 instead of 1·0, whether satisfactory brief interpolation could be achieved. Although the graphs of the function for constant x give very simple smooth curves, after many trials no short interpolation process has been yet discovered. Luckily the chief use of the present tables is their application to small samples, and in such cases n is a whole number. For interpolation by the forward difference formulae, see the Appendix to this paper.

(*b*) With regard to direct interpolation for x, this is feasible for $x=·11$ onward throughout the table using δ^2, or occasionally if greater accuracy be required δ^2 and δ^4. But from $x=·00$ to ·10, ordinary interpolation formulae cannot be applied, owing to the infinite differential coefficients appearing with the factor $x^{-\frac{1}{2}}$ in the integral. Accordingly an auxiliary table—Table II—has been formed which gives the function

$$\mathscr{P}_x(n) = \frac{P_x(n) - 0·5}{\sqrt{x}},$$

and provides its δ^2*. This will suffice to ascertain $\mathscr{P}_x(n)$ for any value of x from ·00 to ·10, and therefore

$$P_x(n) = \mathscr{P}_x(n)\sqrt{x} + 0·5.$$

The user of Table II must therefore find the square root of the argument† with which he enters it, as the multiplier for $\mathscr{P}_x(n)$.

* Determined from the nine-figure B-function Table. For $\delta^2\mathscr{P}_x(n)$ we used the formula

$$\delta_0^2 = 4\delta_1^2 - 6\delta_2^2 + 4\delta_3^2 - \delta_4^2.$$

† x will not generally exceed four decimals, so that any table of square roots will provide what is required.

(iv) Illustrations of the use of the Tables.

Illustration (i). The frequency curve for the distribution of the correlation coefficient r in samples of size p taken from a parent population in which the correlation is zero is given by the curve

$$y = y_0 (1 - r^2)^{\frac{1}{2}(p-4)},$$

where the mean, $\bar{r}$, $= 0$, and since $\alpha_2 = 1$, $\sigma = \dfrac{1}{\sqrt{p-1}}$. What is the chance that in a sample of 20,

(*a*) r will lie outside twice its standard deviation?

(*b*) r will lie outside the limits $\pm \cdot 50$?

The above curve is our Type (ii), and therefore $m_2 = \frac{1}{2}(p-4) = 8$ for this special case. Now $m_2 = \frac{1}{2}(n-3) = 8$, and accordingly $n = 19$. The proper transformation is $r^2 = x$. We have $\sigma = \dfrac{1}{\sqrt{19}} = \cdot 229{,}4157$.

If $r = 2\sigma = \cdot 458{,}8314$, then $x = r^2 = \cdot 210526$. If $r = \cdot 50$, $x = \cdot 25$.

We have accordingly to find from Table I, for $n = 19$, the value of the function tabled for $x = \cdot 210526$ and $x = \cdot 25$.

The latter comes without interpolation at once as $\frac{1}{2}(1 + \alpha_2) = \cdot 987{,}6152$, or $\frac{1}{2}\alpha_2 = \cdot 487{,}6152$, hence doubling, we find the chance is $\cdot 975{,}2304$, or the odds are about 975 to 25, or 39 to 1, that in taking a sample of 20 individuals from a normal population two characters of zero correlation will not show a correlation in the sample exceeding numerically $\pm \cdot 50$.

In the first we have to interpolate between the values for x of $\cdot 21$ and $\cdot 22$, i.e.

$$u_0 = \cdot 978{,}9245, \qquad u_1 = \cdot 981{,}5217,$$
$$\delta^2 u_0 = -3461, \qquad \delta^2 u_1 = -3059.$$

Fourth differences are here unnecessary.

$$\theta = \cdot 0526, \quad \phi = \cdot 9474, \quad \tfrac{1}{8}\theta\phi = \cdot 0083{,}0554,$$
$$u_\theta = \cdot 9790{,}6111 + 0000{,}0827 = \cdot 979{,}0694.$$

The chance therefore of r falling within the range $\pm 2\sigma$ is $\cdot 958{,}1388$. Had we assumed the distribution of r to be a normal curve, the chance of r falling within the range $\pm 2\sigma$ would be $\cdot 954{,}4998$.

Illustration (ii). In a sample of 12, the correlation coefficient is found to be $\cdot 3$. What is the chance that in the original population there was no correlation?

In this case $$m_2 = \tfrac{1}{2}(p-4) = 4 = \tfrac{1}{2}(n-3),$$

and $$n = 11, \quad x = r^2 = \cdot 09.$$

Our table under $n = 11$ gives for $x = \cdot 09$ the value $\cdot 828{,}2807$. The chance accordingly, of r exceeding $\pm \cdot 30$, if the correlation were zero, would be

$$2(1 - \cdot 828{,}2807),$$

or if the population sampled had no correlation between the variants considered, a correlation of numerical intensity $\cdot 30$ or more would occur in 343 out of 1000

samples, i.e. in more than one sample in three. We cannot therefore assert that the correlation found in the sample marks a significant correlation in the parent population.

Even if the observed correlation in the sample were ·50, there would still be 98 samples in 1000 with a correlation of $\pm$·50 or more if the parent population had no correlation. Indeed correlation coefficients found from very small samples are of small service in indicating significant correlation in the parent population unless the correlation in the sample be very high. For example, if the correlation in the sample of 12 were ·80, samples from an uncorrelated population would only give rise to such a value once in 500 trials.

Illustration (iii). What is the chance in a sample of 31 that the regression coefficient will not differ from that of the parent population, supposed normal, by more than twice its standard deviation?

If ρ be the correlation, Σ_1, Σ_2 the standard deviations in the parent population and R_1 the regression coefficient in the sample, the distribution of R_1 is given by

$$y = \frac{y_0}{\left\{(n-3)\,\sigma_{R_1}^2 + \left(R_1 - \rho\,\frac{\Sigma_1}{\Sigma_2}\right)^2\right\}^{\frac{1}{2}n}}$$

$$= \frac{y_0'}{\left\{1 + \frac{x'^2}{n-3}\right\}^{\frac{1}{2}n}},$$

where $\bar{R}_1 = \text{Mean } R_1 = \rho\,\frac{\Sigma_1}{\Sigma_2}$, the value of the regression in the parent population,

$$\sigma_{R_1}^2 = \frac{1}{n-3}\,\frac{\Sigma_1^2}{\Sigma_2^2}(1-\rho^2),$$

and $$x' = \frac{\text{Deviation of } R_1 \text{ from } \bar{R}_1}{\text{Standard Deviation of } R_1},$$

n being the size of the sample.

The requisite transformation is

$$x'^2/(n-3) = x/(1-x) \quad \text{or} \quad \frac{x'^2}{x'^2 + a_3^2} = x.$$

Thus if $x' = 2$, we have

$$x = \frac{4}{4+n-3} = \frac{4}{n+1} = \text{in our case } \tfrac{1}{8} = \cdot 125.$$

We have accordingly to compute

$${}_2P_{x'} = \tfrac{1}{2}\left\{1 + I_{\cdot 125}\left(\tfrac{1}{2}, \tfrac{1}{2}(n-1)\right)\right\}.$$

The value will be found in the column for $n = 31$, or $\frac{1}{2}(n-1) = 15$, between the values of ·12 and ·13 of x. We have

$$u_0 = \cdot 973{,}9461, \quad \delta^2 u_0 = -10529, \quad \delta^4 u_0 = -517,$$

$$u_1 = \cdot 978{,}6801, \quad \delta^2 u_1 = -\;\;8559, \quad \delta^4 u_1 = -396.$$

We are therefore at a part of the table where it is requisite to use δ^4's as well as δ^2's, if we desire an accurate value of $P_{x'}$. Now

$$\theta = \cdot 5, \quad \phi = \cdot 5, \quad \tfrac{1}{6}\theta\phi = \cdot 041{,}6667,$$

and
$$u_\theta = \tfrac{1}{2}(\cdot 973{,}9461 + \cdot 978{,}6801) + 041{,}6667 \times 1\cdot 5\,(\cdot 001{,}9088)$$
$$- \cdot 041{,}6667 \times \cdot 1125 \times 2\cdot 5\,(\cdot 000{,}0913)$$
$$= \cdot 976{,}3131 + \cdot 000{,}1193 - \cdot 000{,}0011$$
$$= \cdot 976{,}4313.$$

Hence ·952,8626 is the chance that the regression coefficient will lie within $\pm$ twice its standard deviation from the true value in the parent population.

Illustration (iv). In a long series of observations on Fathers and Sons the correlation coefficient for span was found to be ·454, and the standard deviations were 3″·14 and 3″·11 respectively. The regression R_1 of Son on Father for span = ·44976. The standard deviation of R_1 in samples is

$$\sigma_{R_1}{}^2 = \frac{1}{n-3}\frac{(3''\cdot 11)^2}{(3''\cdot 14)^2}(1 - (\cdot 4497)^2),$$

or
$$\sigma_{R_1} = \frac{1}{\sqrt{n-3}} \times \cdot 884{,}666.$$

Hence, if we take this to be a reasonable normal parent population for span, let us ask whether a sample of 19 pairs of Father and Son giving a correlation of ·390 and standard deviations for span: Fathers 3″·19 and Sons 2″·98, may be supposed reasonably to have been drawn from this parent population.

Now R_1 for the sample = ·36432 and $\sigma_{R_1} = \cdot 221{,}1665$.

Thus
$$x' = \frac{\cdot 36432 - \cdot 44966}{\cdot 221{,}1665} = - \cdot 385{,}863.$$

Accordingly
$$x'^2 = \cdot 1488{,}9025,$$

and
$$x = \frac{\cdot 1488{,}9025}{\cdot 1488{,}9025 + 16} = \cdot 0921{,}9844.$$

This clearly lies within the first part of Table I where the differences are unsatisfactory. We therefore use the auxiliary Table II. For $n = 19$, we have

$$u_0 = 1\cdot 335{,}4038, \quad \delta^2 u_0 = 12631,$$
$$u_1 = 1\cdot 305{,}4459, \quad \delta^2 u_1 = 12083.$$

δ^4's will be unnecessary.

$$\theta = \cdot 219{,}844, \quad \phi = \cdot 780{,}156, \quad \tfrac{1}{6}\theta\phi = \cdot 028{,}5854,$$
$$u_\theta = 1\cdot 328{,}8177 - \cdot 000{,}1064$$
$$= 1\cdot 328{,}7113 = \mathscr{P}_x(19).$$

But
$$P_x(19) = \cdot 5 + \mathscr{P}_x(19)\sqrt{x},$$
$$P_x(19) = \cdot 5 + \cdot 303{,}6420 \times 1\cdot 328{,}7113 = \cdot 903{,}452,$$

or the chance if the above sample were really drawn from the above parent population that its regression coefficient would differ as much as or more than it does do from the regression in the parent population = ·193,096.

We see therefore that in about 19 in 100 samples the deviation of the regression would be greater than that observed.

Let us, however, look at this problem in another way, which will illustrate a further application of our present table.

Illustration (v). In the sample of the previous illustration the first product moment coefficient $= p_{11} = \cdot 390 \times 2\cdot 98 \times 3\cdot 19 = 3\cdot 707{,}4180$. What is the chance that a sample of 19 with this p_{11} could have been extracted at random from a parent population with no correlation, but with standard deviations $3''\cdot 14$ and $3''\cdot 11$?

We compute v:
$$v = n\frac{p_{11}}{3\cdot 14 \times 3\cdot 11} = \frac{19\,(3\cdot 707{,}4180)}{9\cdot 7654}$$
$$= 7\cdot 213{,}3187,$$

then the problem reduces to determining the chance that values of v will differ from zero by an amount as great as or greater than this. The distribution of v is given by

$$y_v = \frac{N}{\sqrt{\pi(n^2-1)}}\,\frac{\Gamma(\frac{1}{2}(n+4))}{\Gamma(\frac{1}{2}(n+3))}\,\frac{1}{\left(1+\dfrac{v^2}{n^2-1}\right)^{\frac{1}{2}(n+4)}},$$

where
$$v = n\frac{p_{11}}{\sigma_1\sigma_2},$$

and σ_1, σ_2 are the standard deviations in the parent population. The curve obviously falls under our Type (iii) above.

We write
$$y = y_0\frac{1}{\left(1+\dfrac{v^2}{n^2-1}\right)^{\frac{1}{2}(n+4)}}$$
$$= \tilde{\sigma}_0\left(1+\frac{v^2}{360}\right)^{-11\cdot 5}$$

We have accordingly to take $m_3 = 11\cdot 5$, and $a_3{}^2 = 360$, which gives*

$$n = 23,$$
$$x = \frac{52\cdot 031{,}967}{360 + 52\cdot 031{,}967} = \cdot 1262{,}8138.$$

Hence from column for $n = 23$ of Table I we find

$$u_0 = \cdot 951{,}3679,\quad \delta^2 u_0 = -11583,\quad \delta^4 u_0 = -407,$$
$$u_1 = \cdot 958{,}2584,\quad \delta^2 u_1 = -\ \ 9804,\quad \delta^4 u_1 = -310,$$
$$\theta = \cdot 628{,}138,\qquad \phi = \cdot 371{,}862,\quad \tfrac{1}{8}\theta\phi = \cdot 038{,}9301,$$
$$u_\theta = \cdot 955{,}6961 + \cdot 000{,}1240 - \cdot 000{,}0008$$
$$= \cdot 955{,}8193.$$

Thus in 884 out of 10,000 samples a v and therefore a p_{11} *numerically* as large or larger than the observed product moment coefficient could have arisen from a parent population without correlation. The odds are therefore only about 116 to

* This n is that of the Tables, and not the n above which is the size of the sample, the former $n =$ the latter $n+4$.

10 that p_{11} did not arise from a population without correlation. It would occur about once in 11 trials. We cannot assert significance in the observed

$$p_{11} = 3{\cdot}707{,}4180.$$

It is well to investigate the significance of the correlation observed.

The correlation is ·390 and the size of the sample 19. The distribution curve will then be

$$y = y_0 (1 - r^2)^{7\cdot5},$$

and

$$m_2 = 7{\cdot}5 = \tfrac{1}{2}(n-3), \text{ or } n = 18,$$

$$x = r^2 = {\cdot}1521.$$

Turning to our Table I:

$$u_0 = {\cdot}949{,}3160, \quad \delta^2 u_0 = -7531,$$

$$u_1 = {\cdot}955{,}1406, \quad \delta^2 u_1 = -6616,$$

$$\theta = {\cdot}21, \quad \phi = {\cdot}79, \quad \tfrac{1}{6}\theta\phi = {\cdot}02765,$$

and the use of δ^4 is unnecessary. Accordingly

$$u_\theta = {\cdot}950{,}5392 + {\cdot}000{,}0594$$
$$= {\cdot}950{,}5986.$$

The chance is therefore $1 - 2\,({\cdot}950{,}5986 - {\cdot}5) = {\cdot}098{,}8028$ that a sample of 19 from a population of zero correlation would show a correlation *numerically* greater than ·390. Thus such a correlation will occur in samples of this size about once in 10 trials.

It will be clear from the results in this illustration:

(*a*) That the introduction of the observed standard deviations into the sample (i.e. using $p_{11} = r\sigma_1\sigma_2$ instead of r) lessens the probability of the parent population being one of zero correlation.

(*b*) That very little of definite value can be learnt as to correlation from small samples, i.e. in the above illustrations the sample might have been easily obtained from a parent population of correlation = ·00 or ·45 *.

Illustration (vi). In the long series of observations referred to in Illustration (iv) the mean spans of Fathers and of Sons were 68″·67 and 69″·94 respectively. Hence the regression line of Son's span on Father's span is

$$\tilde{y} = 39''{\cdot}06 + 0''{\cdot}44966x.$$

If $\tilde{y}_x$ be the value of $\tilde{y}$ found in a particular sample from the regression line of that sample, the standard deviation of $\tilde{y}_x$'s for numerous samples is

$$\sigma^2_{\tilde{y}_x} = \frac{\Sigma_2^2(1-\rho^2)}{n-3}\left\{1 - \frac{2}{n} + \left(\frac{x - m_1}{\Sigma_1}\right)^2\right\}$$

$$= \frac{(3{\cdot}11)^2(1 - ({\cdot}454)^2)}{n-3}\left\{1 - \frac{2}{n} + \frac{(x-m_1)^2}{(3{\cdot}14)^2}\right\}$$

$$= \frac{7{\cdot}678{,}5254}{n-3}\left\{1 - \frac{2}{n} + \frac{(x - 68''{\cdot}67)^2}{9{\cdot}8596}\right\}.$$

* Inferences like these in character may easily be drawn by looking at Table I for $n = 19$ and examining the entries above $+{\cdot}39$ and below $-{\cdot}39$ in the column with $\rho = 0$, ·4 and ·5.

Now suppose we fix our attention on Fathers with spans between 66″ and 67″, i.e. put $x = 66''\cdot5$, and suppose samples taken of size 19. Then

$$\sigma^2_{\bar{y}_x} = \cdot4799,0784 \{1 - \cdot1052,6316 + \cdot4775,9544\}$$
$$= \cdot6585,9302,$$

and $$\sigma_{\bar{y}_x} = \cdot811,5374.$$

For $x = 66''\cdot5$, we have $\qquad \bar{y} = 68''\cdot96$

from the regression line.

Now we will suppose the regression line for the sample of 19 (!) has been found and gives for the mean span of Sons of Fathers of 66″ to 67″ span the value $\bar{y}_x = 68''\cdot26$. The parent population gives $68''\cdot96$. Is this a reasonable difference?

The distribution of $\bar{y}_x - \bar{y}$ will be given by the curve

$$y = \frac{y_0}{\left\{1 + \frac{1}{n-3}\left(\frac{\bar{y}_x - \bar{y}}{\sigma_{\bar{y}_x}}\right)^2\right\}^{\frac{1}{2}n}},$$

and we have $$x' = \frac{\bar{y}_x - \bar{y}}{\sigma_{\bar{y}_x}} = \frac{\cdot70}{\cdot811,5374},$$

or $$x'^2 = \cdot74401.$$

Thus $$x = \frac{x'^2}{x'^2 + a_2^2} = \frac{\cdot74401}{16\cdot74401} = \cdot04443.$$

We have accordingly to interpolate from our tables for $x = \cdot04443$ in the column $n = 19$. This for accuracy must be done by aid of Table II.

We have $\quad u_0 = 1\cdot505,3176, \quad \delta^2 u_0 = 15731, \quad \delta^4 u_0 = 27,$

$\qquad\qquad u_1 = 1\cdot468,4491, \quad \delta^2 u_1 = 15060, \quad \delta^4 u_1 = 27.$

Clearly we need not use δ^4's.

$$\theta = \cdot443, \quad \phi = \cdot557, \quad \tfrac{1}{6}\theta\phi = \cdot041,1252,$$
$$u_\theta = 1\cdot488,98485^+ - \cdot000,19010 = 1\cdot488,7948.$$

Thus $$\mathcal{P}_x = 1\cdot488,7948,$$

and $$P_x = \cdot5 + \sqrt{\cdot04443} \times 1\cdot488,7948$$
$$= \cdot813,8145.$$

Hence assuming the sample to lie within the range $\pm 0''\cdot7$ from the value $68''\cdot96$ for Sons of Fathers having spans of 66″ to 67″ in the sampled population, the chance of a deviation *numerically* as large as or larger than this $= 2(1 - P_x) = \cdot372,3710$, or we might expect 37·2 % of samples of 19 to give a worse disagreement with the value in the sampled population.

N.B. The reader will note that we are *not* comparing the mean of actual isolated individuals in the sample with Fathers having spans between 66″ and 67″, but we are comparing the mean of the Sons of this array of Fathers *found from the regression line of the sample* with the value of the same mean as given by the parent population.

We can use our tables as applied to the third type of curve to test whether a sample of which we know the mean and standard deviation comes from a parent population of which we know the mean.

Let the size of the sample be n, the mean and standard deviation of the sample be m and s, and the mean of the parent population be M. Then, if

$$x' = (m - M)/s,$$

the distribution of x' in samples of size n is given by*

$$y = \frac{y_0}{(1 + x'^2)^{\frac{1}{2}n}},$$

provided the parent population be normally distributed. E. S. Pearson has shown the extent to which this result may still be applied in a certain range of non-normal distributions†.

It is difficult to imagine a practical case in which we know M so accurately that its probable error relative to that of m is negligible, and yet do not know Σ the standard deviation of the parent population with corresponding accuracy. If we know both M and Σ we have two *independent* variables m and s to compare with them, and the writer of the present paper personally much prefers in all such cases the double test to the single test which involves both characters.

Illustration (vii). Among samples of 10 from a normal population of mean variate zero and standard deviation 10, a sample occurred with mean 7·0 and standard deviation 14·64‡. What is the probability of such a sample occurring *at a single draw* as judged by the present test?

$$x' = \frac{7 \cdot 0}{14 \cdot 64} = \cdot 4781, \text{ and } x'^2 = \cdot 2286.$$

The distribution curve of x' is

$$y = y_0 \frac{1}{(1 + x'^2)^{\frac{1}{2}n}},$$

and the proper transformation $\quad x = \dfrac{x'^2}{1 + x'^2} = \cdot 1861.$

Turning to Table I under $n = 10$ and $x = \cdot 1861$, we have

$$u_0 = \cdot 903{,}2890, \quad \delta^2 u_0 = -4832,$$

$$u_1 = \cdot 909{,}9040, \quad \delta^2 u_1 = -4443,$$

$$\theta = \cdot 61, \quad \phi = \cdot 39, \quad \tfrac{1}{8}\theta\phi = \cdot 03965,$$

$$u_\theta = \cdot 907{,}3241{,}5 + \cdot 000{,}0549{,}9$$

$$= \cdot 907{,}3791.$$

* This is the case really proved by "Student," *Biometrika*, Vol. v. pp. 7—8; however, the actual examples he gives do not belong to this case, but indicate that he foresaw a wider application of it.

† *Biometrika*, Vol. xxi. pp. 259 *et seq.*

‡ Such a sample was one of a set of 700 samples actually drawn from a normal population.

Thus the chance that a value of x' should occur as large as or larger than this is ·185,2418, taking positive and negative excesses in x' together. The odds are only about 4·5 to 1 against such occurrence.

Now let us consider the two characters m and s which have been combined in "Student's" test separately.

The means in the samples are distributed normally with standard deviation of $\Sigma/\sqrt{n}$ = in our case 3·1623, or the ratio of the observed deviation in the sample mean to the standard deviation of sample means is 2·2136.

From Table II of Part I of these *Tables for Statisticians*:

$$u_0 = \cdot 986{,}4474, \quad \delta^2 u_0 = -77,$$

$$u_1 = \cdot 986{,}7906, \quad \delta^2 u_1 = -75,$$

$$\theta = \cdot 36, \quad \phi = \cdot 64, \quad \tfrac{1}{8}\theta\phi = \cdot 0384.$$

Accordingly
$$u_\theta = \cdot 986{,}5709{,}5 + \cdot 000{,}0008{,}8$$
$$= \cdot 986{,}5718.$$

Thus the chance of a mean as great as or greater than this occurring = ·013,4282, or taking both positive and negative excesses = ·026,8564. Thus the odds against such a mean occurring in a single sample are of the order 36 to 1, while those as judged by "Student's" test are about 4·5 to 1.

Now turn to the standard deviation, which is 14·64 against the 10 of the parent population.

If we judged roughly, assuming the distribution of standard deviations to be approximately normal with a standard deviation $\dfrac{\Sigma}{\sqrt{2n}} = 2{\cdot}2361$ about a mean of $\Sigma = 10$, the deviation $14{\cdot}64 - 10 = 4{\cdot}64$ would be 2·075 times the standard deviations, or deviations as great as or greater than this would only occur about 38 times in 1000 trials, or the odds are of the order 25 to 1 against such an occurrence.

For a more accurate appreciation of the odds, we must note that the curve of distribution of s in samples from a normal population is

$$y = y_0 x'^{n-2} e^{-\frac{1}{2}x'^2},$$

where $x' = s/(\Sigma/\sqrt{2n})$ in our present notation. But this curve has not yet had its probability integral tabled for various values of n and x'.

If, however, we write $z = \frac{1}{2}x'^2$, the probability integral becomes

$$P(z, n) = \frac{\int_0^z z^{\frac{n-3}{2}} e^{-z} dz}{\int_0^\infty z^{\frac{n-3}{2}} e^{-z} dz}$$

= Probability Integral of a Type III curve as tabled in the *Tables of the Incomplete Γ-Function**.

* Published by H.M. Stationery Office, 1922.

The integral there given is

$$I(u, p) = \frac{\int_0^{u\sqrt{p+1}} z^p e^{-z} dz}{\int_0^{\infty} z^p e^{-z} dz}.$$

In our case $$p = \tfrac{1}{2}(n-3) = 3{\cdot}5,$$

$$u = \frac{1}{\sqrt{4{\cdot}5}} z = \frac{1}{4\sqrt{4{\cdot}5}} \left(\frac{s}{\Sigma/\sqrt{2n}}\right)^2 = 5{\cdot}0516.$$

For our interpolation in excess of mean we have from the above tables, under $p = 3{\cdot}5$:

Argument	Entry	δ^2	δ^4
5·0	·988,2683	− 2497	Negligible
5·1	·989,8982	− 2185	"

$$\theta = {\cdot}516, \quad \phi = {\cdot}484, \quad \tfrac{1}{6}\theta\phi = {\cdot}041{,}624.$$

$$\text{Required value} = {\cdot}989{,}1069{,}1 + {\cdot}000{,}0292{,}1$$

$$= {\cdot}989{,}1361,$$

or the chance of values of s as great as or greater than $14{\cdot}64 = {\cdot}010{,}8639$.

If on the side of defect we take as our limit $14{\cdot}64 - 10 = 4{\cdot}64$, we find $u = {\cdot}5074$. Our tables give us:

Argument	Entry	δ^2	δ^4
0·50	·010,5995	+ 37648	− 2442
0·60	·020,3677	+ 43857	− 2855

$$\theta = {\cdot}074, \quad \phi = {\cdot}926, \quad \tfrac{1}{6}\theta\phi = {\cdot}011{,}4207, \quad \tfrac{1}{20}(1+\theta)(1+\phi) = {\cdot}103{,}4202.$$

$$\text{Required value} = {\cdot}011{,}3223{,}5 - {\cdot}000{,}1366{,}1 - {\cdot}000{,}0015{,}4$$

$$= {\cdot}011{,}1842.$$

Accordingly the probability that s will differ from the population value by as much as or more than $4{\cdot}64$

$$= {\cdot}010{,}8639 + {\cdot}011{,}1842$$

$$= {\cdot}022{,}0481,$$

or the odds are about 44 to 1 against such a deviation from the population standard deviation occurring. Now it would appear that these two sets of odds—36 to 1 against such an excess in the mean and 44 to 1 against such an excess in the standard deviation—especially when we remember that by our hypothesis as to the parent population these two results are independent—are entirely screened when we apply "Student's" test, with its odds of only 4·5 to 1. The fact is that when the two characters, on the ratio of which "Student's" test is based, deviate in the same direction, this test may be very misleading, when we use it as an indication of the rarity of a particular sample; it is the measurement

of the rarity of a particular ratio connected with the sample, but may be dangerous if interpreted as a measure of the rarity of the sample itself*.

That "Student" himself has not laid too great emphasis on his test is, I think, clear, but the emphasis used by others must lead us to be cautious in its application.

While "Student's" analysis follows the lines indicated above of the probability of his ratio in the case of a sample drawn from a normal parent population, he uses it in the examples he gives for a somewhat different purpose, where its application needs some consideration.

Let u and v be two variates, each of which follows the normal law, then their difference $u-v$ will also follow a normal curve with mean $\bar{u}-\bar{v}$ and standard deviation $\sqrt{\sigma_1^2+\sigma_2^2-2r\sigma_1\sigma_2}$, which latter is the standard deviation of the difference, σ_{u-v}, if r be the correlation coefficient of u and v.

Accordingly if we take samples from these populations with means m_u, m_v and standard deviations s_u, s_v and correlation r, then

$$m_u - m_v \text{ and } s_{u-v} = \sqrt{s_u^2 + s_v^2 - 2rs_u s_v}$$

will follow the two curves used by "Student" to obtain his ratio distribution, and if we write

$$x' = \frac{m_u - m_v - (\bar{u}-\bar{v})}{s_{u-v}} \quad\quad\quad (e),$$

then x' will follow the law of distribution in samples of n given by

$$y = \frac{y_0}{(1+x'^2)^{\frac{1}{2}n}}.$$

"Student" tacitly takes $\bar{u}=\bar{v}$, or he assumes the mean difference of the population from which he is sampling to be zero. He is therefore measuring the probability of the ratio x' on the assumption that u and v are taken at random† from the same parent population. If the ratio x' gives a very small chance of occurrence, he very properly assumes that on his hypothesis u and v are not drawn from the same parent population.

But with "Student" u and v are not *independent* samples of necessity as in J. Neyman and E. S. Pearson's test for two samples (see below).

Take the following series of values from "Student's" original paper:

* A cephalic index among Englishmen of 80·0 is not uncommon, but if we say it has arisen from a skull length of 210 mm. and a skull breadth of 168 mm. we recognise that we are dealing with a very exceptional case on two counts. That is the non-rarity of a ratio is not sufficient to justify us in considering the individual whom it characterises as of common occurrence.

† Actually, however, this is not so, in for example his Illustration I; his two populations are linked by a high correlation due to individual reaction to soporifics. If he gets a high u, he will get a high v and if he gets a low u he will have a low v.

Additional Hours of Sleep gained by the use of hyoscyamine hydrobromide.

Patient	1 (Dextro-)	2 (Laevo-)	Difference (2 − 1)
1	+0·7	+1·9	+1·2
2	−1·6	+0·8	+2·4
3	−0·2	+1·1	+1·3
4	−1·2	+0·1	+1·3
5	−0·1	−0·1	0·0
6	+3·4	+4·4	+1·0
7	+3·7	+5·5	+1·8
8	+0·8	+1·6	+0·8
9	+0·0	+4·6	+4·6
10	+2·0	+3·4	+1·4
Mean	+0·75	+2·33	+1·58
S.D.	1·70	1·90	1·17

Now it is clear that $s_{u-v}=1{\cdot}17$ is much less than $\sqrt{(1{\cdot}70)^2+(1{\cdot}90)^2}$, which it should be, were u and v independent. Actually worked out on these ten cases the correlation is over ·79. Is it likely even on ten cases that the correlation would exceed numerically ·79, if it were really zero in the parent population?

The curve of distribution is (see p. 257)

$$y=y_0(1-r^2)^3.$$

We have therefore to enter our table with $x=r^2=\cdot 6241$, and as $\frac{1}{2}(n-3)=3$ with $n=9$, we find that the chance of such a correlation coefficient from a population of zero correlation lying outside the limits $\pm\cdot 79$ is between ·006 and ·007. There is small doubt therefore that u and v in the sampled population are correlated, probably highly correlated, as the influence of any sleeping draught whatever is a characteristic of the individual. "Student," in applying his test to the difference, has noted this fact as accounting for the low value of the probable error of the difference.

But what, I think, it is desirable to emphasise is that this correlation may exist in most of the examples to which "Student" applies his test, either owing to the influence of the same individual, or of the same year, etc. Accordingly the denominator of "Student's" ratio will be subject to large variation owing to the presence of this correlation in s_{u-v}, the correlation itself being subject to large variation in small samples of such sizes as 10, the numerator m_u-m_v being not subject to this influence of the correlation to the same extent.

Now "Student" takes $x'=1{\cdot}58/1{\cdot}17=1{\cdot}35$, $x'^2=1{\cdot}8225$, and accordingly the transformed $x=x'^2/(1+x'^2)=\cdot 6457$, while $n=10$.

Entering our Table I with $n=10$, we have:

x		δ^2	δ^4
·64	·998,4448	−215	Negligible
·65	·998,6380	−196	"

$$\theta=\cdot 57,\quad \phi=\cdot 43,\quad \tfrac{1}{4}\theta\phi=\cdot 04085.$$

$$\text{Required value}=\cdot 998{,}5549+\cdot 000{,}0025$$
$$=\cdot 998{,}5574.$$

"Student" gives the value ·9985, quite in keeping.

The chance therefore that x' will not lie between the limits

$$\pm 1{\cdot}35 = 2 \times {\cdot}00144 = {\cdot}0029,$$

or the odds are 9971 to 29 against it or 344 to 1 against it.

Now let us suppose the 10 patients who had dextro-hyoscyamine hydrobromide were not identical with those who had the laevo- form. Then there is no doubt about the application of formula (ϵ). If we suppose them to be independent samples of the same population $r = 0$, and $\bar{u} = \bar{v}$. In this case $m_u - m_v = 2{\cdot}33 - 0{\cdot}75 = 1{\cdot}58$, and

$$s_{u-v} = \sqrt{s_u^2 + s_v^2} = \sqrt{(1{\cdot}70)^2 + (1{\cdot}90)^2}$$

$$= 2{\cdot}5495.$$

Thus $$x' = 1{\cdot}58/2{\cdot}5495 = {\cdot}6197, \quad x'^2 = {\cdot}3840,$$

and $$x = \frac{x'^2}{1 + x'^2} = {\cdot}2775, \text{ and } n = 10.$$

We have from Table I:

x		δ^2	δ^4
27	·949,3108	−2475	Negligible
28	·952,9130	−2318	"

$$\theta = {\cdot}75, \quad \phi = {\cdot}25, \quad \tfrac{1}{8}\theta\phi = {\cdot}0336.$$

$$\text{Required value} = {\cdot}952{,}0124 + {\cdot}000{,}0224$$

$$= {\cdot}952{,}0348,$$

or the odds are about 9·4 to 1 that x' does not lie in the range $\pm\,{\cdot}6197$.

A further test has been provided by J. Neyman and E. S. Pearson* to determine whether two samples, of which the means are m_1, m_2 and the standard deviations s_1 and s_2, have been drawn from the same normal population.

They take $$x' = \frac{m_u - m_v}{\sqrt{n_1 s_1^2 + n_2 s_2^2}} \sqrt{\frac{n_1 n_2}{n_1 + n_2}},$$

and find its distribution curve to be

$$y = \frac{y_0}{(1 + x'^2)^{\frac{1}{2}(n_1 + n_2 - 1)}}.$$

In the above case of "Student" $n_1 = n_2 = 10$, and $s_1 = 1{\cdot}70$, $s_2 = 1{\cdot}90$, $m_1 = 0{\cdot}75$, $m_2 = 2{\cdot}33$.

Accordingly $$x' = \frac{1{\cdot}58}{2{\cdot}5495\sqrt{2}} = {\cdot}4382, \text{ and } x'^2 = {\cdot}1920,$$

while $$x = \frac{x'^2}{1 + x'^2} = {\cdot}1611,$$

* *Biometrika*, Vol. xx^A, pp. 175 *et seq.*

and we must look up the column for $n=19$ in Table I. We have:

x		δ^2	
·16	·959,7231	− 6554	δ^4 may be neglected
·17	·964,5820	− 5745	for present purposes.

$$\theta = ·11, \quad \phi = ·89, \quad \tfrac{1}{8}\theta\phi = ·016,3167.$$

$$\text{Required result} = ·960,2576 + ·000,0306$$
$$= ·960,2882.$$

The chance accordingly of x' exceeding the limits $\pm ·1611$ is ·0794, or the odds against this are about 11·6 to 1.

This is roughly in keeping with the previous determination. Or, we conclude that there is some, but far from overwhelming, evidence that a population treated with the laevo- form of the soporific would have longer hours of sleep than another sample of the same population treated with the dextro- form. On the other hand, if we can trust the application of formula (ϵ) to the case where the samples are not independent samples, then the odds are 344 to 1 that the *same* individual gets longer hours of sleep from the laevo- than from the dextro- form.

The difference lies and can lie only in the correlation in the individual between hours of sleep due to the two forms. What real trust, however, can be put upon a correlation due to 10 pairs? We need, further, some more definite demonstration of how (ϵ) applies to this case with $\bar{u}-\bar{v}=0$, which seems to involve the assumption that u and v are drawn *at random from the same population.*

It may be of interest in regard to problems of this sort to exhibit a further example of the use of the Neyman-Pearson test as given by them on p. 206 of their paper cited above.

Illustration. A piece of work is carried out by one set of 30 workmen according to Method I, and by a *second* set of 40 workmen according to Method II. The two sets of workmen are supposed of like ability. The resulting frequencies were:

Time in seconds	50	51	52	53	54	55	56	57	58	59	60	Totals
Method I	1	3	5	4	7	5	3	1	1	—	—	$30=n_1$
Method II	—	1	2	5	8	9	6	3	3	1	2	$40=n_2$

Here for I, $\quad m_u = 53·700$ secs., $\quad s_1 = 1·882$ secs.

" $\quad$ II, $\quad m_v = 55·175$ secs., $\quad s_2 = 2·072$ secs.

Now according to the test we take

$$x' = \frac{m_u - m_v}{\sqrt{n_1 s_1^2 + n_2 s_2^2}} \sqrt{\frac{n_1 n_2}{n_1 + n_2}} = -·3663,$$

$$x'^2 = ·1342 \text{ and } x = \frac{·1342}{1·1342} = ·11832,$$

$$n = n_1 + n_2 - 1 = 69.$$

This lies outside our Table I for n, but the probability integral is

$$\tfrac{1}{2}(1 + I_x(\tfrac{1}{2}, 34)),$$

and found from the Tables of the Incomplete B-function* = ·998,2199, which agrees with the value ·9982 given by Neyman and Pearson. Thus the odds are about 277 to 1 that if the two samples were from the same population x' would lie outside the limits $\pm$ ·3662.

It is clear that such a problem cannot be solved directly by "Student's" ratio as originally given, unless we have the two samples of equal size. In any real case this would be likely to occur, for to produce equal ability in the two samples, the same men would probably be used for both methods. But if this were so, correlation would almost certainly come in and the Neyman-Pearson test would be inapplicable. Hence it becomes all the more important to be certain that "Student's" test can be safely used, when the two populations are correlated member for member.

It appears to me that in applying his test "Student" has really to face two problems, which cannot be solved by a single investigation in the manner he proposes:

(i) If we take two *wholly independent* sets of individuals, and administer the laevo- form of the soporific to one and the dextro- to the other, is there a probability, and what value has it, that the two means differ, and so can we determine which is the more efficient?

(ii) If we administer both soporifics to the *same* set of individuals, i.e. allowing for the individual reactions to the two forms of the drug, will the data indicate that the one is more effective than the other?

Now (i) can be answered by "Student's" test, because he can suppose the samples drawn from the same population, and thus see how improbable the results are. Or, Neyman and Pearson's test may be used, if the samples are of different sizes.

But (i) must be answered before (ii). If (i) show there to be no substantial difference in the hours of sleep of the two sets, then $\bar{u}$ may be put $= \bar{v}$ in (ii). But if the answer to (i) is that $\bar{u}$ and $\bar{v}$ in all probability differ, then it does not seem valid to put $\bar{u} = \bar{v}$ in (ii). It is clear that if $\bar{u}$ be not equal to $\bar{v}$, then a very different value and a much smaller value will be obtained for x' than that given by "Student." The problem thus raised appears to repeat itself in others of "Student's" illustrations, and my object is to press for caution in the application of his test, and indeed in other tests similar to it.

* For most practical purposes, it is adequate to use here the normal curve with standard deviation $1/\sqrt{n-8}$, the standard deviation of the x' curve. Now $x' = -·3662$ and $1/\sqrt{n-8} = ·1231$, therefore $x'/\sigma_{x'} = 2·975$, and the corresponding probability ·99858, which for most practical inferences is as serviceable as the correct value ·99822 obtained from the B-function tables.

APPENDIX.

On Interpolation into Table I for small values of $q = \frac{1}{2}(n-1)$.

Interpolation for $q = \frac{1}{2}(n-1)$ is bound to be laborious, even if it be straightforward. In interpolating for q into Table I, it will be found best, particularly in the earlier part of the table, to use a forward difference formula, e.g.

$$u_0(\theta) = u_0 + \theta\Delta u_0 - \frac{\theta(1-\theta)}{2!}\Delta^2 u_0 + \frac{\theta(1-\theta)(2-\theta)}{3!}\Delta^3 u_0$$
$$- \frac{\theta(1-\theta)(2-\theta)(3-\theta)}{4!}\Delta^4 u_0 + \ldots.$$

If we use the tabled value $P_x(\frac{1}{2}, q)$, we may have to find, even at $x = \cdot 25$, eight or nine differences to get the correct result to seven decimal places. But if we reduce the $P_x(\frac{1}{2}, q)$ to $B_x(\frac{1}{2}, q)$ by the relation $B_x(\frac{1}{2}, q) = \{2P_x(\frac{1}{2}, q) - 1\} \times B(\frac{1}{2}, q)$ four or five differences will suffice for 7-figure accuracy when $x = \cdot 25$. For $x = \cdot 50$, the seventh difference is required for the $B_x(\frac{1}{2}, q)$'s. The $I_x(\frac{1}{2}, q)$'s would need far more. In order that Δq may not exceed ·25, we can use when it appears desirable a negative interpolation.

The value of $B(\frac{1}{2}, q)$ is given at the top of each column to assist the reader in reducing the tabled entries to $B_x(\frac{1}{2}, q)$. From the interpolated value of the latter we find $P_x(\frac{1}{2}, q)$ by determining from a table of the complete Γ-functions the complete B-function corresponding to the interpolated value.

Illustrations.

(i) Find the value of $P_{\cdot 25}(\frac{1}{2}, 3{\cdot}25)$.

(*a*) Let us work first with $I_x(\frac{1}{2}, q) = 2P_x(\frac{1}{2}, q) - 1$:

q	$I_{\cdot 25}(\frac{1}{2}, q)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7	Δ
3	·792,9688,								
3·5	·829,5293,	·036,5605,							
4	·858,8867,	·029,3574,	−·007,2031,						
4·5	·882,6932,	·023,8065,	−·005,5509,	·001,6522,					
5	·902,1454,	·019,4522,	−·004,3543,	·001,1966,	−·000,4556,				
5·5	·918,1358,	·015,9904,	−·003,4618,	·000,8925,	−·000,3041,	·000,1515,			
6	·931,3450,	·013,2092,	−·002,7812,	·000,6806,	−·000,2119,	·000,0922,	−·000,0593,		
6·5	·942,3012,	·010,9562,	−·002,2530,	·000,5282,	−·000,1524,	·000,0595,	−·000,0327,	·000,0266,	
7	·951,4197,	·009,1185,	−·001,8377,	·000,4153,	−·000,1129,	·000,0395,	−·000,0200,	·000,0127,	−·000,0139.

Here we must go as far as Δ^8.

$$I_{\cdot 25}(\tfrac{1}{2}, 3{\cdot}25) = {\cdot}792{,}9688 + \tfrac{1}{2}({\cdot}036{,}5605) - \tfrac{1}{8}(-{\cdot}007{,}2031) + \tfrac{1}{16}({\cdot}001{,}6522)$$
$$- \tfrac{5}{128}(-{\cdot}000{,}4556) + \tfrac{7}{256}({\cdot}000{,}1515) - \tfrac{21}{1024}(-{\cdot}000{,}0593)$$
$$+ \tfrac{33}{2048}({\cdot}000{,}0266) - \tfrac{429}{32768}(-{\cdot}000{,}0139)$$
$$= {\cdot}792{,}9688 + {\cdot}018{,}2803 + {\cdot}000{,}9004 + {\cdot}000{,}1033 + {\cdot}000{,}0178 + {\cdot}000{,}0042$$
$$+ {\cdot}000{,}0012 + {\cdot}000{,}0004 + {\cdot}000{,}0002$$
$$= {\cdot}792{,}9688 + {\cdot}019{\cdot}3078 = {\cdot}812{,}2766,$$

and $P_{\cdot 25}(\frac{1}{2}, 3{\cdot}25) = {\cdot}906{,}1383$, which is accurate to the last figure.

The process is somewhat lengthy and can be shortened by using $B_x(\frac{1}{2}, q)$.

(*b*) Starting from the $I_x(\frac{1}{2}, q)$'s, multiply them by their respective $B(\frac{1}{2}, q)$'s and we obtain the following series:

q	$B_x(\frac{1}{2}, q)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
3	·845,8334,					
3·5	·814,3885,	−·031,4449,				
4	·785,2678,	−·029,1207,	·002,3242,			
4·5	·758,2593,	−·027,0085,	·002,1122,	−·000,2120,		
5	·733,1721,	−·025,0872,	·001,9213,	−·000,1909,	+·000·0211,	
5·5	·709,8349,	−·023,3372,	·001,7500,	−·000,1713,	+·000,0196,	−·000,0015.

The differencing here is briefer and more effective.

$$B_{\cdot 25}(\tfrac{1}{2}, 3{\cdot}25) = {\cdot}845{,}8334 - {\cdot}015{,}7224(5) - {\cdot}000{,}2905(3) - {\cdot}000{,}0132(5)$$
$$- {\cdot}000{,}0008(2) - {\cdot}000{,}0000(4)$$
$$= {\cdot}845{,}8334 - {\cdot}016{,}0271 = {\cdot}829{,}8063.$$

But $B(\frac{1}{2}, 3{\cdot}25) = 1{\cdot}021{,}58087$, hence

$$I_{\cdot 25}(\tfrac{1}{2}, 3{\cdot}25) = B_{\cdot 25}(\tfrac{1}{2}, 3{\cdot}25)/B(\tfrac{1}{2}, 3{\cdot}25) = {\cdot}812{,}27668$$

and $$P_{\cdot 25}(\tfrac{1}{2}, 3{\cdot}25) = \tfrac{1}{2}\{1 + I_{\cdot 25}(\tfrac{1}{2}, 3{\cdot}25)\} = {\cdot}906{,}1383,$$

the correct value, as before.

(ii) Find the value of $P_{\cdot 50}(\frac{1}{2}, 3{\cdot}25)$.

Here, even using the B_x's, we must go as far as Δ^7 to be accurate to the seventh figure. Our scheme is as follows:

		Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7
$B_{\cdot 50}(\frac{1}{2}, 3\ )$	1·013,5197,							
$B_{\cdot 50}(\frac{1}{2}, 3{\cdot}5)$	·949,2072,	−·064,3125,						
$B_{\cdot 50}(\frac{1}{2}, 4\ )$	·893,9850,	−·055,2222,	·009,0903,					
$B_{\cdot 50}(\frac{1}{2}, 4{\cdot}5)$	·846,1813,	−·047,8037,	·007,4185,	−·001,6718,				
$B_{\cdot 50}(\frac{1}{2}, 5\ )$	·804,4742,	−·041,7071,	·006,0966,	−·001,3219,	·000,3499,			
$B_{\cdot 50}(\frac{1}{2}, 5{\cdot}5)$	·767,8131,	−·036,6611,	·005,0460,	−·001,0506,	·000,2713,	−·000,0786,		
$B_{\cdot 50}(\frac{1}{2}, 6\ )$	·735,3579,	−·032,4552,	·004,2059,	−·000,8401,	·000,2105,	−·000,0608,	·000,0178,	
$B_{\cdot 50}(\frac{1}{2}, 6{\cdot}5)$	·706,4329,	−·028,9250,	·003,5302,	−·000,6757,	·000,1644,	−·000,0461,	·000,0147,	−·000,0031.

Substituting these results in the forward difference formula, we have

$$B_{\cdot 50}(\tfrac{1}{2}, 3{\cdot}25) = 1{\cdot}013{,}5197 - \tfrac{1}{2}({\cdot}064{,}3125) - \tfrac{1}{8}({\cdot}009{,}0903) - \tfrac{1}{16}({\cdot}001{,}6718)$$
$$- \tfrac{5}{128}({\cdot}000{,}3499) - \tfrac{7}{256}({\cdot}000{,}0786) - \tfrac{21}{1024}({\cdot}000{\cdot}0178)$$
$$- \tfrac{33}{2048}({\cdot}000{,}0031)$$
$$= 1{\cdot}013{,}5197 - {\cdot}032{,}1562(5) - {\cdot}001{,}1362(9) - {\cdot}000{,}1044(8)$$
$$- {\cdot}000{,}0136(7) - {\cdot}000{,}0021(5) - {\cdot}000{,}0003(7)$$
$$- {\cdot}000{,}0000(5)$$
$$= {\cdot}980{,}1064, \text{ and again } B(\tfrac{1}{2}, 3{\cdot}25) = 1{\cdot}021{,}58087.$$

Hence $$I_{\cdot 50}(\tfrac{1}{2}, 3{\cdot}25) = {\cdot}959{,}4016(7),$$

and thus $$P_{\cdot 50}(\tfrac{1}{2}, 3{\cdot}25) = {\cdot}979{,}7008(3),$$

which is the correct value to seven figures.

(iii) Find the value of $P_{\cdot 10}(\frac{1}{2}, 3\cdot 25)$.

Now $P_{\cdot 10}(\frac{1}{2}, 3\cdot 25)$ is easy to find; we have the following series of differences for $B_{\cdot 10}(\frac{1}{2}, q)$:

q	$B_{\cdot 10}(\frac{1}{2}, q)$	Δ	Δ^2	Δ^3	Δ^4
3	·591,5567,				
3·5	·582,0941,	− ·009,4626,			
4	·572,9144,	− ·009,1797,	+ ·000,2829,		
4·5	·564,0073,	− ·008,9071,	+ ·000,2726,	− ·000,0103,	
5	·555,3632,	− ·008,6441,	+ ·000,2630,	− ·000,0096,	+ ·000,0007.

Hence
$$B_{\cdot 10}(\tfrac{1}{2}, 3\cdot 25) = \cdot 591{,}5567 - \cdot 004{,}73130 - \cdot 000{,}03536 - \cdot 000{,}00064 - \cdot 000{,}00003$$
$$= \cdot 591{,}5567 - \cdot 004{,}7073$$
$$= \cdot 586{,}7894.$$

And
$$I_{\cdot 10}(\tfrac{1}{2}, 3\cdot 25) = B_{\cdot 10}(\tfrac{1}{2}, 3\cdot 25)/B(\tfrac{1}{2}, 3\cdot 25)$$
$$= \cdot 574{,}3935.$$

Thus
$$P_{\cdot 10}(\tfrac{1}{2}, 3\cdot 25) = \cdot 787{,}19675,$$
which is exact.

Beyond $x = \cdot 75$ the forward difference method will still apply, but the number of differences required is excessive, if we start with $q = 2$ or 3. For $q = 4$ the eighth difference suffices for 7-figure accuracy; for $q = 5$ the fifth difference will suffice for like accuracy, and so on; this supposes working with $B_x(\frac{1}{2}, q)$ instead of $I_x(\frac{1}{2}, q)$. Thus by the time we get to $q = 8$, there is no trouble. For many statistical purposes four or five figure accuracy is adequate, and accordingly there is less trouble with forward difference work.

For such an extreme case as $I_{\cdot 90}(\frac{1}{2}, 3\cdot 25)$ the limiting difference that the present table provides is the twelfth. Even if we use this and the forward difference formula we shall be out slightly more than unity in the fifth decimal place. If we proceed also to the twelfth difference, using $B_{\cdot 90}(\frac{1}{2}, 3\cdot 25)$, we shall be out less than unity in the sixth decimal place, and assuming the thirteenth difference to be about half the twelfth (as it must be here) we can obtain a value differing from the true value by less than five units in the seventh decimal place. The labour, if straightforward, is considerable, and some will prefer to obtain the result by expansion methods rather than by using the present table of $I_x(\frac{1}{2}, q)$ or $B_x(\frac{1}{2}, q)$ when x approaches unity and q is fractional and small.

TABLE I. *Values of $P_x(n)$.*

$n=$ / $\frac{1}{2}(n-1)=$		2 / 0·5	3 / 1·0	4 / 1·5	5 / 2·0	6 / 2·5	7 / 3·0	8 / 3·5
$B(\frac{1}{2}, \frac{1}{2}(n-1))=$		3·1415,9265+	2·0000,0000	1·5707,9633	1·3333,3333	1·1780,9725−	1·0666,6667	·9817,4770
x	z^2							
·00	·00000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000
·01	·01010	·531,8843	·550,0000	·563,5557	·574,7500•	·584,4589	·593,1269	·601,0142
·02	·02041	·545,1672	·570,7107	·589,7306	·605,3589	·618,8454	·630,8254	·641,6713
·03	·03093	·555,4123	·586,6025+	·609,7110	·628,6048	·644,8257	·659,1614	·672,0739
·04	·04167	·564,0942	·600,0000	·626,4700	·648,0000	·666,3904	·682,5600	·697,0494
·05	·05263	·571,7831	·611,8034	·641,1572	·664,9100	·685,0941	·702,7485−	·718,4861
·06	·06383	·578,7712	·622,4745−	·654,3656	·680,0375−	·701,7381	·720,6194	·737,3622
·07	·07527	·585,2317	·632,2876	·666,4475+	·693,8013	·716,8013	·736,7071	·754,2646
·08	·08696	·591,2774	·641,4214	·677,6328	·706,4752	·730,5973	·751,3623	·769,5793
·09	·09890	·596,9867	·650,0000	·688,0812	·718,2500•	·743,3452	·764,8306	·783,5773
·10	·11111	·602,4164	·658,1139	·697,9093	·729,2651	·755,2051	·777,2922	·796,4581
·11	·12360	·607,6095+	·665,8312	·707,2054	·739,6261	·766,2990	·788,8842	·808,3736
·12	·13636	·612,5995−	·673,2051	·716,0379	·749,4153	·776,7218	·799,7141	·819,4433
·13	·14943	·617,4127	·680,2776	·724,4614	·758,6983	·786,5497	·809,8678	·829,7631
·14	·16279	·622,0709	·687,0829	·732,5203	·767,5285+	·795,8446	·819,4159	·839,4117
·15	·17647	·626,5917	·693,6492	·740,2510	·775,9501	·804,6580	·828,4168	·848,4547
·16	·19048	·630,9899	·700,0000	·747,6842	·784,0000	·813,0330	·836,9200	·856,9474
·17	·20482	·635,2781	·706,1553	·754,8458	·791,7097	·821,0065+	·844,9674	·864,9373
·18	·21951	·639,4672	·712,1320	·761,7578	·799,1062	·828,6101	·852,5953	·872,4651
·19	·23457	·643,5663	·717,9449	·768,4395+	·806,2127	·835,8711	·859,8353	·879,5668
·20	·25000•	·647,5836	·723,6068	·774,9076	·813,0495+	·842,8137	·866,7151	·886,2736
·21	·26582	·651,5263	·729,1288	·781,1765+	·819,6347	·849,4585+	·873,2594	·892,6135−
·22	·28205+	·655,4006	·734,5208	·787,2593	·825,9839	·855,8258	·879,4898	·898,6113
·23	·29870	·659,2121	·739,7916	·793,1673	·832,1113	·861,9309	·885,4260	·904,2893
·24	·31579	·662,9660	·744,9490	·798,9108	·838,0296	·867,7894	·891,0855+	·909,6677
·25	·33333	·666,6667	·750,0000	·804,4989	·843,7500•	·873,4150+	·896,4844	·914,7647
·26	·35135+	·670,3183	·754,9510	·809,9399	·849,2828	·878,8199	·901,6370	·919,5969
·27	·36986	·673,9247	·759,8076	·815,2414	·854,6374	·884,0155+	·906,5567	·924,1796
·28	·38889	·677,4892	·764,5751	·820,4100	·859,8222	·889,0120	·911,2556	·928,5267
·29	·40845+	·681,0150+	·769,2582	·825,4520	·864,8449	·893,8188	·915,7448	·932,6512
·30	·42857	·684,5051	·773,8613	·830,3730	·869,7127	·898,4447	·920,0347	·936,5648
·31	·44928	·687,9620	·778,3882	·835,1782	·874,4322	·902,8976	·924,1349	·940,2787
·32	·47059	·691,3883	·782,8427	·839,8723	·879,0092	·907,1850+	·928,0542	·943,8032
·33	·49254	·694,7865−	·787,2281	·844,4598	·883,4496	·911,3139	·931,8008	·947,1477
·34	·51515+	·698,1585+	·791,5476	·848,9447	·887,7583	·915,2906	·935,3826	·950,3213
·35	·53846	·701,5067	·795,8040	·853,3308	·891,9403	·919.1213	·938,8067	·953,3323
·36	·56250•	·704,8328	·800,0000	·857,6215+	·896,0000	·922,8114	·942,0800	·956,1886
·37	·58730	·708,1387	·804,1381	·861,8201	·899,9416	·926,3663	·945,2088	·958,8976
·38	·61290	·711,4263	·808,2207	·865,9296	·903,7691	·929,7909	·948,1991	·961,4662
·39	·63934	·714,6971	·812,2499	·869,9528	·907,4861	·933,0900	·951,0567	·963,9010
·40	·66667	·717,9529	·816,2278	·873,8923	·911,0961	·936,2680	·953,7868	·966,2084
·41	·69492	·721,1951	·820,1562	·877,7505+	·914,6023	·939,3290	·956,3947	·968,3940
·42	·72414	·724,4253	·824,0370	·881,5298	·918,0078	·942,2769	·958,8850+	·970,4636
·43	·75439	·727,6449	·827,8719	·885,2324	·921,3154	·945,1156	·961,2625+	·972,4224
·44	·78571	·730,8553	·831,6625−	·888,8601	·924,5280	·947,8486	·963,5315−	·974,2755−
·45	·81818	·734,0579	·835,4102	·892,4150+	·927,6480	·950,4793	·965,6961	·976,0276
·46	·85185+	·737,2540	·839,1165−	·895,8988	·930,6780	·953,0110	·967,7603	·977,6834
·47	·88679	·740,4450−	·842,7827	·899,3132	·933,6202	·955,4466	·969,7280	·979,2472
·48	·92308	·743,6321	·846,4102	·902,6597	·936,4768	·957,7892	·971,6028	·980,7231
·49	·96078	·746,8167	·850,0000	·905,9398	·939,2500•	·960,0417	·973,3881	·982,1152
·50	1·00000	·750,0000	·853,5534	·909,1549	·941,9417	·962,2061	·975,0874	·983,4272

TABLE I (*continued*).

$n=$ $\frac{1}{2}(n-1)=$		2 0·5	3 1·0	4 1·5	5 2·0	6 2·5	7 3·0	8 3·5
$B(\frac{1}{2}, \frac{1}{2}(n-1))=$		3·1415,9265+	2·0000,0000	1·5707,9633	1·3333,3333	1·1780,9725−	1·0666,6667	·9817,4770
x	z^2							
·50	1·0000	·750,000	·853,5534	·909,1549	·941,9417	·962,2061	·975,0874	·983,4272
·51	1·04081	·753,1833	·857,0714	·912,3064	·944,5539	·964,2866	·976,7037	·984,6629
·52	1·08333	·756,3679	·860,5551	·915,3955−	·947,0884	·966,2843	·978,2403	·985,8256
·53	1·12766	·759,5550+	·864,0055−	·918,4232	·949,5468	·968,2019	·979,7001	·986,9187
·54	1·17391	·762,7460	·867,4235−	·921,3908	·951,9309	·970,0419	·981,0859	·987,9455+
·55	1·22222	·765,9421	·870,8099	·924,2993	·954,2422	·971,8065−	·982,4005+	·988,9090
·56	1·27273	·769,1447	·874,1657	·927,1496	·956,4822	·973,4977	·983,6466	·289,8122
·57	1·32558	·772,3551	·877,4917	·929,9426	·958,6524	·975,1177	·984,8268	·990,6580
·58	1·38095+	·775,5747	·880,7887	·932,6793	·960,7543	·976,6685+	·985,9434	·991,4489
·59	1·43902	·778,8049	·884,0573	·935,3603	·962,7890	·978,1521	·986,9990	·992,1878
·60	1·50000	·782,0471	·887,2983	·937,9865−	·964,7580	·979,5703	·987,9959	·992,8771
·61	1·56410	·785,3029	·890,5125−	·940,5585−	·966,6624	·980,9250−	·988,9363	·993,5193
·62	1·63158	·788,5737	·893,7004	·943,0770	·968,5035−	·982,2179	·989,8223	·994,1167
·63	1·70270	·791,8613	·896,8627	·945,5427	·970,2823	·983,4507	·990,6562	·994,6715+
·64	1·77778	·795,1672	·900,0000	·947,9560	·972,0000	·984,6253	·991,4400	·995,1860
·65	1·85714	·798,4933	·903,1129	·950,3175−	·973,6576	·985,7431	·992,1756	·995,6623
·66	1·94118	·801,8415−	·906,2019	·952,6276	·975,2562	·986,8058	·992,8651	·996,1023
·67	2·03030	·805,2135+	·909,2676	·954,8869	·976,7968	·987,8150+	·993,5103	·996,5081
·68	2·12500	·808,6117	·912,3106	·957,0956	·978,2803	·988,7722	·994,1130	·996,8814
·69	2·22581	·812,0380	·915,3312	·959,2542	·979,7075+	·989,6789	·994,6750+	·997,2242
·70	2·33333	·815,4949	·918,3300	·961,3629	·981,0795+	·990,5364	·995,1982	·997,5381
·71	2·44828	·818,9850−	·921,3075−	·963,4219	·982,3971	·991,3464	·995,6841	·997,8249
·72	2·57143	·822,5108	·924,2641	·965,4316	·983,6610	·992,1101	·996,1344	·998,0861
·73	2·70370	·826,0753	·927,2002	·967,3920	·984,8722	·992,8290	·996,5508	·998,3234
·74	2·84615+	·829,6817	·930,1163	·969,3033	·986,0314	·993,5044	·996,9348	·998,5382
·75	3·00000	·833,3333	·933,0127	·971,1656	·987,1393	·994,1376	·997,2880	·998,7320
·76	3·16667	·837,0340	·935,8899	·972,9788	·988,1967	·994,7300	·997,6119	·998,9062
·77	3·34783	·840,7879	·938,7482	·974,7430	·989,2043	·995,2828	·997,9079	·999,0621
·78	3·54545+	·844,5994	·941,5880	·976,4581	·990,1627	·995,7974	·998,1776	·999,2011
·79	3·76190	·848,4737	·944,4097	·978,1240	·991,0727	·996,2750+	·998,4222	·999,3244
·80	4·00000	·852,4164	·947,2136	·979,7403	·991,9350−	·996,7169	998,6432	·999,4331
·81	4·26316	·856,4337	·950,0000	·981,3070	·992,7500	·997,1242	·998,8419	·999,5285−
·82	4·55556	·860,5328	·952,7693	·982,8235−	·993,5185−	·997,4984	·999,0196	·999,6116
·83	4·88235+	·864,7219	·955,5217	·984,2895+	·994,2410	·997,8405+	·999,1777	·999,6834
·84	5·25000	·869,0101	·958,2576	·985,7045−	·994,9182	·998,1518	·999,3174	·999,7451
·85	5·66667	·873,4083	·960,9772	·987,0677	·995,5505+	·998,4336	·999,4400	·999,7976
·86	6·14286	·877,9291	·963,6809	·988,3785+	·996,1386	·998,6871	·999,5466	·999,8417
·87	6·69231	·882,5873	·966,3690	·989,6360	·996,6829	·998,9135+	·999,6385+	·999,8784
·88	7·33333	·887,4005+	·969,0416	·990,8390	·997,1841	·999,1141	·999,7169	·999,9085+
·89	8·09091	·892,3905−	·971,6991	·991,9864	·997,6425+	·999,2901	·999,7828	·999,9328
·90	9·00000	·897,5836	·974,3416	·993,0766	·998,0587	·999,4428	·999,8375+	·999,9521
·91	10·11111	·903,0133	·976,9696	·994,1078	·998,4332	·999,5735−	·999,8820	·999,9670
·92	11·50000	·908,7226	·979,5832	·995,0779	·998,7665−	·999,6835−	·999,9175−	·999,9782
·93	13·28571	·914,7683	·982,1825+	·995,9841	·999,0589	·999,7742	·999,9449	·999,9864
·94	15·66667	·921,2288	·984,7680	·996,8232	·999,3110	·999,8470	·999,9655−	·999,9921
·95	19·00000	·928,2169	·987,3397	·997,5909	·999,5232	·999,9034	·999,9801	·999,9959
·96	24·00000	·935,9058	·989,8979	·998,2815+	·999,6959	·999,9449	·999,9898	·999,9981
·97	32·33333	·944,5877	·992,4429	·998,8873	·999,8295+	·999,9732	·999,9957	·999,9993
·98	49·00000	·954,8328	·994,9747	·999,3961	·999,9245−	·999,9903	·999,9987	·999,9998
·99	99·00000	·968,1157	·997,4937	·999,7872	·999,9812	·999,9983	·999,9998	1·000,0000
1·00	∞	1·000,0000	1·000,0000	1·000,0000	1·000,0000	1·000,0000	1·000,0000	

TABLE I (*continued*).

$n=$		9	10	11	12	13	14	15
$\frac{1}{2}(n-1)=$		4·0	4·5	5·0	5·5	6·0	6·5	7·0
$B(\frac{1}{2}, \frac{1}{2}(n-1))=$		·9142,8571	·8590,2924	·8126,9841	·7731,2632	·7388,1674	·7086,9912	·6819,8468
x	z^2							
·00	·00000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000
·01	·01010	·608,2878	·615,0625+	·621,4209	·627,4250+	·633,1226	·638,5513	·643,7418
·02	·02041	·651,6230	·660,8451	·669,4569	·677,5476	·685,1864	·692,4281	·699,3168
·03	·03093	·683,8613	·694,7289	·704,8254	·714,2626	·723,1270	·731,4877	·739,4002
·04	·04167	·710,2080	·722,2773	·733,4323	·743,8051	·753,4981	·762,5930	·771,1560
·05	·05263	·732,7039	·745,6768	·757,6044	·768,6378	·778,8943	·788,4677	·797,4342
·06	·06383	·752,4086	·766,0651	·778,5552	·790,0479	·800,6752	·810,5424	·819,7353
·07	·07527	·769,9591	·784,1281	·797,0179	·808,8153	·819,6662	·829,6872	·838,9738
·08	·08696	·785,7758	·800,3193	·813,4786	·825,4578	·836,4166	·846,4828	·855,7609
·09	·09890	·800,1543	·814,9584	·828,2807	·840,3423	·851,3163	·861,3410	·870,5318
·10	·11111	·813,3125+	·828,2818	·841,6785+	·853,7408	·864,6550+	·874,5708	·883,6106
·11	·12360	·825,4173	·840,4705+	·853,8675−	·865,8628	·876,6560	·886,4074	·895,2477
·12	·13630	·836,5998	·851,6675−	·865,0019	·876,8740	·887,4963	·897,0391	·905,6418
·13	·14943	·846,9657	·861,9880	·875,2065+	·886,9085+	·897,3191	·906,6184	·914,9538
·14	·16279	·856,6019	·871,5270	·884,5844	·896,0772	·906,2428	·915,2711	·923,3169
·15	·17647	·865,5809	·880,3638	·893,2216	·904,4729	·914,3668	·923,1026	·930,8424
·16	·19048	·873,9640	·888,5658	·901,1913	·912,1742	·921,7752	·930,2024	·937,6248
·17	·20482	·881,8039	·896,1908^{5}	·908,5564	·919,2491	·928,5406	·936,6475+	·943,7452
·18	·21951	·889,1461	·903,2890	·915,3714	·925,7561	·934,7256	·942,5044	·949,2736
·19	·23457	·896,0306	·909,9040	·921,6840	·931,7469	·940,3853	·947,8311	·954,2710
·20	·25000*	·902,4922	·916,0747	·927,5362	·937,2666	·945,5678	·952,6788	·958,7911
·21	·26582	·908,5623	·921,8353	·932,9655−	·942,3554	·950,3161	·957,0926	·962,8809
·22	·28205+	·914,2687	·927,2165−	·938,0052	·947,0494	·954,6683	·961,1127	·966,5824
·23	·29870	·919,6362	·932,2459	·942,6854	·951,3806	·958,6584	·964,7748^{5}	·969,9328
·24	·31579	·924,6876	·936,9484	·947,0330	·955,3780	·962,3173	·968,1112	·972,9653
·25	·33333	·929,4434	·941,3466	·951,0727	·959,0679	·965,6725−	·971,1506	·975,7099
·26	·35135+	·933,9221	·945,4611	·954,8267	·962,4741	·968,7491	·973,9191	·978,1932
·27	·36986	·938,1410	·949,3108	·958,3154	·965,6182	·971,5700	·976,4404	·980,4395+
·28	·38889	·942,1156	·952,9130	·961,5575−	·968,5201	·974,1558	·978,7358	·982,4706
·29	·40845+	·945,8606	·956,2834	·964,5700	·971,1980	·976,5253	·980,8247	·984,3063
·30	·42857	·949,3892	·959,4369	·967,3689	·973,6684	·978,6960	·982,7249	·985,9643
·31	·44928	·952,7140	·962,3870	·969,9686	·975,9467	·980,6837	·984,4524	·987,4610
·32	·47059	·955,8463	·965,1463	·972,3826	·978,0471	·982,5028	·986,0221	·988,8111
·33	·49254	·958,7969	·967,7266	·974,6234	·979,9824	·984,1668	·987,4473	·990,0280
·34	·51515+	·961,5760	·970,1386	·976,7026	·981,7648	·985,6879	·988,7405+	·991,1239
·35	·53846	·964,1927	·972,3927	·978,6310	·983,4054	·987,0774	·989,9129	·992,1100
·36	·56250*	·966,6560	·974,4984	·980,4186	·984,9146	·988,3458	·990,9750−	·992,9964
·37	·58730	·968,9741	·976,4644	·982,0747	·986,3019	·989,5027	·991,9361	·993,7924
·38	·61290	·971,1546	·978,2993	·983,6080	·987,5762	·990,5570	·992,8050+	·994,5063
·39	·63934	·973,2051	·980,0108	·985,0268	·988,7459	·991,5169	·993,5898	·995,1459
·40	·66667	·975,1322	·981,6063	·986,3385+	·989,8185−	·992,3900	·994,2979	·995,7182
·41	·69492	·976,9426	·983,0926	·987,5505−	·990,8012	·993,1833	·994,9358	·996,2296
·42	·72414	·978,6424	·984,4764	·988,6692	·991,7008	·993,9033	·995,5100	·996,6860
·43	·75439	·980,2374	·985,7637	·989,7011	·992,5233	·994,5560	·996,0260	·997,0927
·44	·78571	·981,7331	·986,9604	·990,6519	·993,2746	·995,1470	·996,4891	·997,4545−
·45	·81818	·983,1348	·988,0718	·991,5272	·993,9601	·995,6814	·996,9042	·997,7758
·46	·85185+	·984,4474	·989,1032	·992,3321	·994,5847	·996,1640	·997,2756	·998,0608
·47	·88679	·985,6757	·990,0594	·993,0714	·995,1532	·996,5992	·997,6075−	·998,3131
·48	·92308	·986,8241	·990,9451	·993,7497	·995,6699	·996,9909	·997,9035−	·998,5359
·49	·96078	·987,8968	·991,7645	·994,3713	·996,1389	·997,3431	·998,1670	·998,7325−
·50	1·00000	·988,8980	·992,5218	·994,9402	·996,5638	·997,6592	·998,4011	·998,9054

TABLE I (*continued*).

	$n=$	9	10	11	12	13	14	15
	$\frac{1}{2}(n-1)=$	4·0	4·5	5·0	5·5	6·0	6·5	7·0
	$B(\frac{1}{2}, \frac{1}{2}(n-1))=$	·9142,8571	·8590,2924	·8126,9841	·7731,2632	·7388,1674	·7086,9912	·6819,8468
x	x^2							
·50	1·00000	·988,8980	·992,5218	·994,9402	·996,5638	·997,6592	·998,4011	·998,9054
·51	1·04081	·989,8316	·993,2209	·995,4601	·996,9484	·997,9423	·998,6088	·999,0573
·52	1·08333	·990,7011	·993,8654	·995,9346	·997,2958	·998,1955$^+$	·998,7927	·999,1903
·53	1·12766	·991,5101	·994,4589	·996,3670	·997,6091	·998,4215$^-$	·998,9550$^+$	·999,3066
·54	1·17391	·992,2620	·995,0047	·996,7604	·997,8911	·998,6227	·999,0981	·999,4080
·55	1·22222	·992,9599	·995,5057	·997,1177	·998,1444	·998,8016	·999,2239	·999,4962
·56	1·27273	·993,6069	·995,9651	·997,4416	·998,3716	·998,9602	·999,3342	·999,5727
·57	1·32558	·994,2059	·996,3856	·997,7348	·998,5749	·999,1005$^-$	·999,4307	·999,6388
·58	1·38095$^+$	·994,7597	·996,7699	·997,9996	·998,7564	·999,2243	·999,5148	·999,6958
·59	1·43902	·995,2708	·997,1204	·998,2383	·998,9180	·999,3333	·999,5880	·999,7448
·60	1·50000	·995,7419	·997,4395$^-$	·998,4530	·999,0616	·999,4289	·999,6515$^+$	·999,7868
·61	1·56410	·996,1753	·997,7294	·998,6456	·999,1889	·999,5127	·999,7064	·999,8226
·62	1·63158	·996,5733	·997,9923	·998,8180	·999,3014	·999,5857	·999,7536	·999,8531
·63	1·70270	·996,9382	·998,2301	·998,9720	·999,4005$^+$	·999,6492	·999,7942	·999,8789
·64	1·77778	·997,2720	·998,4448	·999,1091	·999,4876	·999,7043	·999,8280	·999,9007
·65	1·85714	·997,5767	·998,6380	·999,2308	·999,5638	·999,7518	·999,8584	·999,9190
·66	1·94118	·997,8543	·998,8116	·999,3386	·999,6304	·999,7927	·999,8835$^-$	·999,9343
·67	2·03030	·998,1065$^-$	·998,9669	·999,4336	·999,6882	·999,8278	·999,9046	·999,9470
·68	2·12500	·998,3350$^+$	·999,1057	·999,5172	·999,7383	·999,8577	·999,9224	·999,9576
·69	2·22581	·998,5416	·999,2291	·999,5904	·999,7815$^+$	·999,8831	·999,9372	·999,9662
·70	2·33333	·998,7278	·999,3385$^+$	·999,6543	·999,8186	·999,9045$^+$	·999,9496	·999,9733
·71	2·44828	·998,8951	·999,4352	·999,7099	·999,8503	·999,9225$^+$	·999,9598	·999,9791
·72	2·57143	·999,0449	·999,5203	·999,7579	·999,8773	·999,9376	·999,9682	·999,9837
·73	2·70370	·999,1785$^-$	·999,5949	·999,7993	·999,9001	·999,9501	·999,9750$^+$	·999,9875$^-$
·74	2·84615$^+$	·999,2972	·999,6600	·999,8347	·999,9193	·999,9604^5	·999,9806	·999,9904
·75	3·00000	·999,4023	·999,7165$^+$	·999,8649	·999,9353	·999,9689	·999,9850$^+$	·999,9928
·76	3·16667	·999,4949	·999,7653	·999,8904	·999,9486	·999,9758	·999,9886	·999,9946
·77	3·34783	·999,5761	·999,8072	·999,9119	·999,9595$^+$	·999,9814	·999,9914	·999,9960
·78	3·54545$^+$	·999,6469	·999,8430	·999,9298	·999,9685$^-$	·999,9858	·999,9936	·999,9971
·79	3·76190	·999,7083	·999,8733	·999,9446	·999,9757	·999,9893	·999,9953	·999,9979
·80	4·00000	·999,7612	·999,8988	·999,9568	·999,9815$^+$	·999,9921	·999,9966	·999,9985$^+$
·81	4·26316	·999,8064	·999,9200	·999,9668	·999,9861	·999,9942	·999,9976	·999,9990
·82	4·55556	·999,8448	·999,9376	·999,9748	·999,9898	·999,9958	·999,9983	·999,9993
·83	4·88235$^+$	·999,8771	·999,9520	·999,9811	·999,9926	·999,9971	·999,9988	·999,9995$^+$
·84	5·25000	·999,9040	·999,9636	·999,9861	·999,9947	·999,9980	·999,9992	·999,9997
·85	5·66667	·999,9262	·999,9729	·999,9900	·999,9963	·999,9986	·999,9995$^-$	·999,9998
·86	6·14286	·999,9443	·999,9802	·999,9930	·999,9975$^-$	·999,9991	·999,9997	·999,9999
·87	6·69231	·999,9587	·999,9859	·999,9952	·999,9983	·999,9994	·999,9998	·999,9999
·88	7·33333	·999,9702	·999,9902	·999,9968	·999,9989	·999,9996	·999,9999	1·000,0000
·89	8·09091	·999,9790	·999,9934	·999,9979	·999,9993	·999,9998	·999,9999	
·90	9·00000	·999,9857	·999,9957	·999,9987	·999,9996	·999,9999	1·000,0000	
·91	10·11111	·999,9907	·999,9974	·999,9992	·999,9998	·999,9999		
·92	11·50000	·999,9942	·999,9984^5	·999,9996	·999,9999	1·000,0000		
·93	13·28571	·999,9966	·999,9992	·999,9998	·999,9999			
·94	15·66667	·999,9982	·999,9996	·999,9999	1·000,0000			
·95	19·00000	·999,9991	·999,9998	1·000,0000				
·96	24·00000	·999,9996	·999,9999					
·97	32·33333	·999,9999	1·000,0000					
·98	49·00000	1·000,0000						
·99	99·00000							
1·00	∞							

TABLE I (*continued*).

$n=$ $\frac{1}{2}(n-1)=$		16 7·5	17 8·0	18 8·5	19 9·0	20 9·5	21 10·0	22 10·5
$B(\frac{1}{2}, \frac{1}{2}(n-1))=$		·6580,7776	·6365,1904	·6169,4790	·5990,7674	·5826,7301⁵	·5675,4639	·5535,3936
x	t^2							
·00	·00000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000
·01	·01010	·648,7191	·653,5039	·658,1140	·662,5644	·666,8679	·671,0359	·675,0782
·02	·02041	·705,8891	·712,1754	·718,2015+	·723,9893	·729,5579	·734,9237	·740,1014
·03	·03093	·746,9108	·754,0578	·760,8738	·767,3869	·773,6213	·779,5979	·785,3355−
·04	·04167	·779,2420	·786,8968	·794,1595−	·801,0635+	·807,6379	·813,9080	·819,8961
·05	·05263	·805,8570	·813,7891	·821,2756	·828,3552	·835,0616	·841,4242	·847,4690
·06	·06383	·828,3252	·836,3720	·843,9267	·851,0331	·857,7294	·864,0489	·870,0211
·07	·07527	·847,6050−	·855,6473	·863,1575+	·870,1845+	·876,7706	·882,9530	·888,7645−
·08	·08696	·864,3378	·872,2865+	·879,6693	·886,5399	·892,9446	·898,9244	·904,5151
·09	·09890	·878,9811	·886,7689	·893,9628	·900,6211	·906,7943	·912,5264	·917,8563
·10	·11111	·891,8758	·899,4520	·906,4120	·912,8183	·918,7250+	·924,1795+	·929,2235−
·11	·12360	·903,2856	·910,6124	·917,3057⁵	·923,4323	·929,0497	·934,2081⁵	·938,9517
·12	·13636	·913,4195−	·920,4692	·926,8732	·932,7019	·938,0160	·942,8686	·947,3056
·13	·14943	·922,4470	·929,2002	·935,2998	·940,8199	·945,8240	·950,3674	·954,4981
·14	·16279	·930,5081	·936,9517	·942,7383	·947,9448	·952,6376	·956,8737	·960,7029
·15	·17647	·937,7198	·943,8463	·949,3160	·954,2088	·958,5931	·962,5276	·966,0635+
·16	·19048	·944,1812	·949,9875−	·955,1406	·959,7231	·963,8050−	·967,4466	·970,7000
·17	·20482	·949,9774	·955,4636	·960,3036	·964,5820	·968,3702	·971,7298	·974,7131
·18	·21951	·955,1815+	·960,3508	·964,8837	·968,8664	·972,3716	·975,4613	·978,1884
·19	·23457	·959,8572	·964,7151	·968,9489	·972,6460	·975,8800	·978,7132	·981,1987
·20	·25000⁴	·964,0602	·968,6140	·972,5583	·975,9812	·978,9569	·981,5476	·983,8063
·21	·26582	·967,8395−	·972,0980	·975,7635−	·978,9245−	·981,6552	·984,0178	·986,0647
·22	·28205+	·971,2383	·975,2116	·978,6098	·981,5217	·984,0213	·986,1701	·988,0201
·23	·29870	·974,2951	·977,9939	·981,1370	·983,8130	·986,0953	·988,0448	·989,7123
·24	·31579	·977,0440	·980,4798	·983,3803	·985,8338	·987,9127	·989,6769	·991,1760
·25	·33333	·979,5155+	·982,7002	·985,3710	·987,6152	·989,5042	·991,0967	·992,4410
·26	·35135+	·981,7370	·984,6826	·987,1365+	·989,1847	·990,8971	·992,3311	·993,5335+
·27	·36986	·983,7329	·986,4518	·988,7015−	·990,5665−	·992,1152	·993,4033	·994,4761
·28	·38889	·985,5252	·988,0297	·990,0877	·991,7821	·993,1795+	·994,3337	·995,2884
·29	·40845+	·987,1339	·989,4361	·991,3148	·992,8507	·994,1086	·995,1403	·995,4878
·30	·42857	·988,5767	·990,6867	·992,3999	·993,7891	·994,9187	·995,8387	·996,5891
·31	·44928	·989,8698	·991,8033	·993,3587	·994,6123	·995,6244	·996,4428	·997,1054
·32	·47059	·991,0279	·992,7943	·994,2049	·995,3336	·996,2383	·996,9644	·997,5481
·33	·49254	·992,0641	·993,6745+	·994,9511	·995,9650−	·996,7716	·997,4143	·997,9272
·34	·51515+	·992,9903	·994,4555−	·995,6082	·996,5169	·997,2344	·997,8018	·998,2512
·35	·53846	·993,8174	·995,1476	·996,1862	·996,9986	·997,6353	·998,1349	·998,5276
·36	·56250⁴	·994,5552	·995,7602⁵	·996,6938	·997,4185+	·997,9820	·998,4209	·998,7631
·37	·58730	·995,2126	·996,3019	·997,1392	·997,7840	·998,2815+	·998,6659	·998,9633
·38	·61290	·995,7975+	·996,7800	·997,5292	·998,1016	·998,5397	·998,8754	·999,1382
·39	·63934	·996,3174	·997,2015+	·997,8703	·998,3771	·998,7618	·999,0543	·999,2770
·40	·66667	·996,7787	·997,5726	·998,1680	·998,6156	·998,9526	·999,2067	·999,3985+
·41	·69492	·997,1876	·997,8986	·998,4276	·998,8218	·999,1161	·999,3362	·999,5010
·42	·72414	·997,5494	·998,1847	·998,6534	·998,9997	·999,2560	·999,4461	·999,5871
·43	·75439	·997,8689	·998,4353	·998,8494	·999,1528	·999,3754	·999,5390	·999,6594
·44	·78571	·998,1508	·998,6544	·999,0193	·999,2843	·999,4770	·999,6175−	·999,7199
·45	·81818	·998,3989	·998,8455−	·999,1662	·999,3970	·999,5634	·999,6885+	·999,7704
·46	·85185+	·998,6170	·999,0119	·999,2920	·999,4934	·999,6366	·999,7390	·999,8123
·47	·88679	·998,8082	·999,1565+	·999,4021	·999,5756	·999,6984	·999,7854	·999,8472
·48	·92308	·998,9756	·999,2819	·999,4959	·999,6456	·999,7505+	·999,8242	·999,8760
·49	·96078	·999,1217	·999,3904	·999,5762	·999,7050−	·999,7944	·999,8565−	·999,8998
·50	1·00000	·999,2491	·999,4840	·999,6448	·999,7552	·999,8311	·999,8833	·999,9193

TABLE I (*continued*).

$n=$ / $\frac{1}{2}(n-1)=$		16 / 7·5	17 / 8·0	18 / 8·5	19 / 9·0	20 / 9·5	21 / 10·0	22 / 10·5
$B(\frac{1}{2}, \frac{1}{2}(n-1))=$		·6580,7776	·6365,1904	·6169,4790	·5990,7674	·5826,7301^5	·5675,4639	·5535,3936
x	z^2							
·50	1·00000	·999,2491	·999,4840	·999,6448	·999,7552	·999,8311	·999,8833	·999,9193
·51	1·04081	·999,3599	·999,5645+	·999,7033	·999,7976	·999,8617	·999,9054	·999,9353
·52	1·08333	·999,4559	·999,6337	·999,7530	·999,8332	·999,8872	·999,9237	·999,9483
·53	1·12766	·999,5390	·999,6929	·999,7951	·999,8631	·999,9084	·999,9387	·999,9589
·54	1·17391	·999,6106	·999,7434	·999,8307	·999,8881	·999,9259	·999,9509	·999,9675−
·55	1·22222	·999,6723	·999,7864	·999,8606	·999,9089	·999,9404	·999,9609	·999,9744
·56	1·27273	·999,7252	·999,8229	·999,8857	·999,9261	·999,9522	·999,9690	·999,9799
·57	1·32558	·999,7704	·999,8537	·999,9067	·999,9404	·999,9619	·999,9756	·999,9843
·58	1·38095+	·999,8089	·999,8797	·999,9242	·999,9521	·999,9697	·999,9808	·999,9879
·59	1·43902	·999,8416	·999,9015+	·999,9387	·999,9617	·999,9761	·999,9851	·999,9906
·60	1·50000	·999,8693	·999,9197	·999,9506	·999,9696	·999,9812	·999,9884	·999,9928
·61	1·56410	·999,8927	·999,9349	·999,9605−	·999,9759	·999,9853	·999,9911	·999,9945+
·62	1·63158	·999,9123	·999,9475−	·999,9685+	·999,9811	·999,9886	·999,9932	·999,9959
·63	1·70270	·999,9286	·999,9579	·999,9751	·999,9852	·999,9912	·999,9948	·999,9969
·64	1·77778	·999,9423	·999,9664	·999,9804	·999,9885+	·999,9933	·999,9961	·999,9977
·65	1·85714	·999,9536	·999,9733	·999,9847	·999,9912	·999,9949	999,9971	·999,9983
·66	1·94118	·999,9629	·999,9790	·999,9881	·999,9932	·999,9962	·999,9978	·999,9988
·67	2·03030	·999,9705+	·999,9836	·999,9908	·999,9949	·999,9971	·999,9984	·999,9991
·68	2·12500	·999,9767	·999,9872	·999,9930	·999,9961	·999,9979	·999,9988	·999,9993
·69	2·22581	·999,9818	·999,9902	·999,9947	·999,9971	·999,9984	·999,9991	·999,9995
·70	2·33333	·999,9858	·999,9925−	·999,9960	·999,9979	·999,9989	·999,9994	·999,9997
·71	2·44828	·999,9891	·999,9943	·999,9970	·999,9984	·999,9992	·999,9996	·999,9998
·72	2·57143	·999,9917	·999,9957	·999,9978	·999,9989	·999,9994	·999,9997	·999,9998
·73	2·70370	·999,9937	·999,9968	·999,9984	·999,9992	·999,9996	·999,9998	·999,9999
·74	2·84615+	·999,9953	·999,9977	·999,9988	·999,9994	·999,9997	·999,9999	·999,9999
·75	3·00000	·999,9965−	·999,9983	·999,9992	·999,9996	·999,9998	·999,9999	1·000,0000
·76	3·16667	·999,9974	·999,9988	·999,9994	·999,9997	·999,9999	·999,9999	
·77	3·34783	·999,9981	·999,9991	·999,9996	·999,9998	·999,9999	1·000,0000	
·78	3·54545+	·999,9987	·999,9994	·999,9997	·999,9999	·999,9999		
·79	3·76190	·999,9991	·999,9996	·999,9998	·999,9999	1·000,0000		
·80	4·00000	·999,9994	·999,9997	·999,9999	·999,9999			
·81	4·26316	·999,9996	·999,9998	·999,9999	1·000,0000			
·82	4·55556	·999,9997	·999,9999	1·000,0000				
·83	4·88235+	·999,9998	·999,9999					
·84	5·25000	·999,9999	1·000,0000					
·85	5·66667	·999,9999						
·86	6·14286	1·000,0000						
·87	6·69231							
·88	7·33333							
·89	8·09091							
·90	9·00000							
·91	10·11111							
·92	11·50000							
·93	13·28571							
·94	15·66667							
·95	19·00000							
·96	24·00000							
·97	32·33333							
·98	49·00000							
·99	99·00000							
1·00	∞							

TABLE I (*continued*).

$n=$	$\frac{1}{2}(n-1)=$	23 11·0	24 11·5	25 12·0	26 12·5	27 13·0	28 13·5	29 14·0
$B(\frac{1}{2}, \frac{1}{2}(n-1))=$		·5405,2037	·5283,7848	·5170,1948	·5063,6271	·4963,3870	·4868,8722	·4779,5579
x	x^2							
·00	·00000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000
·01	·01010	·679,0034	·682,8193	·686,5326	·690,1497	·693,6760	·697,1166	·700,4760
·02	·02041	·745,1036	·749,9419	·754,6265−	·759,1664	·763,5700	·767,8448	·771,9976
·03	·03093	·790,8504	·796,1572	·801,2691	·806,1979	·810,9542	·815,5477	·819,9874
·04	·04167	·825,6221	·831,1037	·836,3566	·841,3951	·846,2322	·850,8797	·855,3482
·05	·05263	·853,2190	·858,6947	·863,9147	·868,8955+	·873,6523	·878,1986	·882,5472
·06	·06383	·875,6721	·881,0251	·886,1012	·890,9191	·895,4961	·899,8475+	·903,9876
·07	·07527	·894,2340	·899,3876	·904,2485−	·908,8376	·913,1738	·917,2745−	·921,1552
·08	·08696	·909,7485+	·914,6530	·919,2540	·923,5744	·927,6348	·931,4538	·935,0485−
·09	·09890	·922,8185−	·927,4435−	·931,7586	·935,7884	·939,5551	·943,0786	·946,3770
·10	·11111	·933,8934	·938,2221	·942,2386	·945,9688	·949,4362	·952,6619	·955,6650−
·11	·12360	·943,3191	·947,3448	·951,0593	·954,4899	·957,6611	·960,5947	·963,3106
·12	·13636	·951,3679	·955,0912	·958,5074	·961,6446	·964,5283	·967,1810	·969,6230
·13	·14943	·958,2584	·961,6852	·964,8115+	·967,6662	·970,2752	·972,6615−	·974,8457
·14	·16279	·964,1685+	·967,3087	·970,1570	·972,7428	·975,0924	·977,2291	·979,1737
·15	·17647	·969,2451	·972,1110	·974,6954	·977,0279	·979,1351	·981,0401	·982,7637
·16	·19048	·973,6101	·976,2160	·978,5520	·980,6480	·982,5303	·984,2219	·985,7435−
·17	·20482	·977,3658	·979,7270	·981,8311	·983,7077	·985,3828	·986,8794	·988,2174
·18	·21951	·980,5987	·982,7311	·984,6198	·986,2942	·987,7798	·989,0990	·990,2713
·19	·23457	·983,3819	·985,3017	·986,9917	·988,4807	·989,7937	·990,9526	·991,9761
·20	·25000*	·985,7780	·987,5011	·989,0085+	·990,3284	·991,4852	·992,4998	·993,3904
·21	·26582	·987,8403	·989,3823	·990,7228	·991,8893	·992,9051	·993,7906	·994,5629
·22	·28205+	·989,6146	·990,9906	·992,1792	·993,2069	·994,0963	·994,8665−	·995,5340
·23	·29870	·991,1404	·992,3648	·993,4156	·994,3184	·995,0946	·995,7625−	·996,3376
·24	·31579	·992,4515+	·993,5380	·994,4644	·995,2551	·995,9305−	·996,5079	·997,0018
·25	·33333	·993,5773	·994,5388	·995,3532	·996,0436	·996,6346	·997,1271	·997,5500+
·26	·35135+	·994,5430	·995,3915+	·996,1055−	·996,7067	·997,2135−	·997,6410	·998,0019
·27	·36986	·995,3706	·996,1174	·996,7415+	·997,2635+	·997,7006	·998,0667	·998,3737
·28	·38889	·996,0791	·996,7345+	·997,2786	·997,7305−	·998,1062	·998,4189	·998,6792
·29	·40845+	·996,6847	·997,2585−	·997,7314	·998,1215−	·998,4435+	·998,7097	·998,9297
·30	·42857	·997,2018	·997,7027	·998,1126	·998,4483	·998,7235+	·998,9493	·999,1347
·31	·44928	·997,6426	·998,0787	·998,4329	·998,7210	·998,9555+	·999,1465+	·999,3022
·32	·47059	·998,0179	·998,3964	·998,7017	·998,9482	·999,1474	·999,3084	·999,4387
·33	·49254	·998,3369	·998,6645+	·998,9268*	·999,1371	·999,3057	·999,4410	·999,5497
·34	·51515+	·998,6075−	·998,8903	·999,1150+	·999,2938	·999,4360	·999,5494	·999,6398
·35	·53846	·998,8366	·999,0800	·999,2720	·999,4235−	·999,5432	·999,6378	·999,7127
·36	·56250*	·999,0303	·999,2392	·999,4026	·999,5306	·999,6310	·999,7097	·999,7715−
·37	·58730	·999,1937	·999,3724	·999,5111	·999,6189	·999,7027	·999,7680	·999,8188
·38	·61290	·999,3312	·999,4836	·999,6010	·999,6915−	·999,7613	·999,8152	·999,8568
·39	·63934	·999,4468	·999,5763	·999,6753	·999,7510	·999,8089	·999,8532	·999,8872
·40	·66667	·999,5436	·999,6534	·999,7365+	·999,7996	·999,8475−	·999,8839	·999,9115+
·41	·69492	·999,6245+	·999,7172	·999,7869	·999,8393	·999,8787	·999,9084	·999,9308
·42	·72414	·999,6920	·999,7700	·999,8282	·999,8715+	·999,9039	·999,9280	·999,9461
·43	·75439	·999,7481*	·999,8136	·999,8619	·999,8977	·999,9241	·999,9437	·999,9582
·44	·78571	·999,7947	·999,8494	·999,8895−	·999,9188	·999,9403	·999,9561	·999,9677
·45	·81818	·999,8332	·999,8788	·999,9118	·999,9358	·999,9532	·999,9659	·999,9751
·46	·85185+	·999,8650−	·999,9027	·999,9299	·999,9494	·999,9635+	·999,9736	·999,9809
·47	·88679	·999,8911	·999,9223	·999,9445−	·999,9603	·999,9716	·999,9797	·999,9855−
·48	·92308	·999,9124	·999,9381	·999,9562	·999,9690	·999,9781	·999,9845−	·999,9890
·49	·96078	·999,9299	·999,9509	·999,9656	·999,9759	·999,9831	·999,9881	·999,9917
·50	1·00000	·999,9441	·999,9613	·999,9731	·999,9814	·999,9871	·999,9910	·999,9937

TABLE I (*continued*).

$n=$	$\frac{1}{2}(n-1)=$	23 11·0	24 11·5	25 12·0	26 12·5	27 13·0	28 13·5	29 14·0
$B(\frac{1}{2}, \frac{1}{2}(n-1))=$		·5405,2037	·5283,7848	·5170,1948	·5063,6271	·4963,3870	·4868,8722	·4779,5579
x	z^2							
·50	1·00000	·999,9441	·999,9613	·999,9731	·999,9814	·999,9871	·999,9910	·999,9937
·51	1·04081	·999,9556	·999,9696	·999,9791	·999,9856	·999,9901	·999,9932	·999,9958
·52	1·08333	·999,9649	·999,9762	·999,9838	·999,9890	·999,9925+	·999,9949	·999,9965+
·53	1·12766	·999,9724	·999,9815−	·999,9875+	·999,9916	·999,9944	·999,9962	·999,9974
·54	1·17391	·999,9784	·999,9856	·999,9905−	·999,9936	·999,9958	·999,9972	·999,9981
·55	1·22222	·999,9832	·999,9889	·999,9927	·999,9952	·999,9968	·999,9979	·999,9986
·56	1·27273	·999,9870	·999,9915+	·999,9945−	·999,9964	·999,9977	·999,9985−	·999,9990
·57	1·32558	·999,9899	·999,9935+	·999,9958	·999,9973	·999,9983	·999,9989	·999,9993
·58	1·38095+	·999,9923	·999,9951	·999,9969	·999,9980	·999,9987	·999,9992	·999,9995−
·59	1·43902	·999,9941	·999,9963	·999,9977	·999,9985+	·999,9991	·999,9994	·999,9996
·60	1·50000	·999,9956	·999,9973	·999,9983	·999,9989	·999,9993	·999,9996	·999,9997
·61	1·56410	·999,9967	·999,9980	·999,9988	·999,9992	·999,9995+	·999,9997	·999,9998
·62	1·63158	·999,9975+	·999,9985	·999,9991	·999,9995−	·999,9997	·999,9998	·999,9999
·63	1·70270	·999,9982	·999,9989	·999,9993	·999,9996	·999,9998	·999,9999	·999,9999
·64	1·77778	·999,9986	·999,9992	·999,9995+	·999,9997	·999,9998	·999,9999	1·000,0000
·65	1·85714	·999,9990	·999,9994	·999,9997	·999,9998	·999,9999	·999,9999	
·66	1·94118	·999,9993	·999,9996	·999,9998	·999,9999	·999,9999	1·000,0000	
·67	2·03030	·999,9995−	·999,9997	·999,9998	·999,9999	·999,9999		
·68	2·12500	·999,9996	·999,9998	·999,9999	·999,9999	1·000,0000		
·69	2·22581	·999,9997	·999,9999	·999,9999	1·000,0000			
·70	2·33333	·999,9998	·999,9999	·999,9999				
·71	2·44828	·999,9999	·999,9999	1·000,0000				
·72	2·57143	·999,9999	1·000,0000					
·73	2·70370	·999,9999						
·74	2·84615+	1·000,0000						
·75	3·00000							
·76	3·16667							
·77	3·34783							
·78	3·54545+							
·79	3·76190							
·80	4·00000							
·81	4·26316							
·82	4·55556							
·83	4·88235+							
·84	5·25000							
·85	5·66667							
·86	6·14286							
·87	6·69231							
·88	7·33333							
·89	8·09091							
·90	9·00000							
·91	10·11111							
·92	11·50000							
·93	13·28571							
·94	15·66667							
·95	19·00000							
·96	24·00000							
·97	32·33333							
·98	49·00000							
·99	99·00000							
1·00	∞							

TABLE I (*continued*).

$n=$	$\frac{1}{2}(n-1)=$	30 14·5	31 15·0	Normal Curve
$B(\frac{1}{2}, \frac{1}{2}(n-1))=$		·4694,9840	·4614,7455+	S.D. $=\frac{1}{\sqrt{28}}$
x	z^2			
·00	·00000	·500,0000	·500,0000	·500,0000
·01	·01010	·703,7584	·706,9675+	·702,5733
·02	·02041	·776,0346	·779,9617	·775,1541
·03	·03093	·824,2812	·828,4366	·823,9640
·04	·04167	·859,6477	·863,7871	·859,9564
·05	·05263	·886,7093	·890,6955+	·887,6174
·06	·06383	·907,9293	·911,6846	·909,3682
·07	·07527	·924,8302	·928,3127	·926,7120
·08	·08696	·938,4343	·941,6255−	·940,6649
·09	·09890	·949,4669	·952,3633	·951,9538
·10	·11111	·958,4627	·961,0707	·961,1200
·11	·12360	·965,6267	·968,1592	·968,5777
·12	·13636	·971,8725+	·973,9461	·974,6504
·13	·14943	·976,8464	·978,6801	·979,5952
·14	·16279	·980,9445−	·982,5582	·983,6187
·15	·17647	·984,3242	·985,7378	·986,8879
·16	·19048	·987,1129	·988,3462	·989,5393
·17	·20482	·989,4145−	·990,4861	·991,6847
·18	·21951	·991,3138	·992,2414	·993,4158
·19	·23457	·992,8807	·993,6307	·994,8083
·20	·25000	·994,1726	·994,8601	·995,9245−
·21	·26582	·995,2369	·995,8257	·996,8159
·22	·28205+	·996,1130	·996,6154	·997,5248
·23	·29870	·996,8333	·997,2606	·998,0860
·24	·31579	·997,4247	·997,7870	·998,5282
·25	·33333	·997,9097	·998,2157	·998,8749
·26	·35135+	·998,3068	·998,5645−	·999,1452
·27	·36986	·998,6313	·998,8476	·999,3547
·28	·38889	·998,8961	·999,0770	·999,5163
·29	·40845+	·999,1118	·999,2626	·999,6400
·30	·42857	·999,2871	·999,4123	·999,7340
·31	·44928	·999,4292	·999,5329	·999,8050−
·32	·47059	·999,5443	·999,6298	·999,8583
·33	·49254	·999,6371	·999,7074	·999,8979
·34	·51515+	·999,7119	·999,7694	·999,9271
·35	·53846	·999,7719	·999,8169	·999,9484

$n=$	$\frac{1}{2}(n-1)=$	30 14·5	31 15·0	Normal Curve
$B(\frac{1}{2}, \frac{1}{2}(n-1))=$		·4694,9840	·4614,7455+	S.D. $=\frac{1}{\sqrt{28}}$
x	z^2			
·35	·53846	·999,7719	·999,8169	·999,9484
·36	·56250	·999,8200	·999,8582	·999,9639
·37	·58730	·999,8584	·999,8893	·999,9750−
·38	·61290	·999,8890	·999,9140	·999,9828
·39	·63934	·999,9133	·999,9333	·999,9884
·40	·66667	·999,9325+	·999,9485+	·999,9922
·41	·69492	·999,9477	·999,9604	·999,9948
·42	·72414	·999,9596	·999,9697	·999,9966
·43	·75439	·999,9689	·999,9769	·999,9978
·44	·78571	·999,9762	·999,9825−	·999,9986
·45	·81818	·999,9819	·999,9867	·999,9992
·46	·85185+	·999,9862	·999,9900	·999,9995−
·47	·88679	·999,9896	·999,9925+	·999,9997
·48	·92308	·999,9922	·999,9944	·999,9998
·49	·96078	·999,9942	·999,9959	·999,9999
·50	1·00000	·999,9957	·999,9970	·999,9999
·51	1·04081	·999,9968	·999,9978	1·000,0000
·52	1·08333	·999,9976	·999,9984	
·53	1·12766	·999,9983	·999,9988	
·54	1·17391	·999,9987	·999,9992	
·55	1·22222	·999,9991	·999,9994	
·56	1·27273	·999,9994	·999,9996	
·57	1·32558	·999,9995+	·999,9997	
·58	1·38095+	·999,9997	·999,9998	
·59	1·43902	·999,9998	·999,9999	
·60	1·50000	·999,9998	·999,9999	
·61	1·56410	·999,9999	·999,9999	
·62	1·63158	·999,9999	1·000,0000	
·63	1·70270	1·000,0000	1·000,0000	

TABLE II. *Values of* $\mathscr{P}_x(n)$ *and* $\delta^2\mathscr{P}_x(n)$ *from* $x = \cdot 00$ *to* $\cdot 10$.

x	$n=2$	δ^2	$n=3$	δ^2	$n=4$	δ^2	$n=5$	δ^2	$n=6$	δ^2	$n=7$	δ^2
·00	·318,3099	46	·500,0000	0	·636,6198	−32	·750,0000	0	·848,8264	128	·937,5000	375
·01	·318,8428	48	·500,0000	0	·635,5572	−32	·747,5000	0	·844,5886	128	·931,2688	375
·02	·319,3806	50	·500,0000	0	·634,4913	−33	·745,0000	0	·840,3636	128	·925,0750	375
·03	·319,9233	50	·500,0000	0	·633,4222	−33	·742,5000	0	·836,1515−	129	·918,9188	375
·04	·320,4711	51	·500,0000	0	·632,3498	−33	·740,0000	0	·831,9522	129	·912,8000	375
·05	·321,0240	52	·500,0000	0	·631,2741	−34	·737,5000	0	·827,7658	130	·906,7188	375
·06	·321,5821	53	·500,0000	0	·630,1950+	−34	·735,0000	0	·823,5924	130	·900,6750	375
·07	·322,1456	54	·500,0000	0	·629,1125+	−34	·732,5000	0	·819,4320	131	·894,6688	375
·08	·322,7145+	55	·500,0000	0	·628,0266	−35	·730,0000	0	·815,2847	131	·888 7000	375
·09	·323,2889	57	·500,0000	0	·626,9372	−35	·727,5000	0	·811,1505+	132	·882,7688	375
·10	·323,8690	58	·500,0000	0	·625,8443	−36	·725,0000	0	·807,0295−	132	·876,8750	375

x	$n=8$	δ^2	$n=9$	δ^2	$n=10$	δ^2	$n=11$	δ^2	$n=12$	δ^2	$n=13$	δ^2
·00	1·018,5916	763	1·093,7500	1312	1·164,1047	2037	1·230,4688	2953	1·293,4497	4075−	1·353,5156	5415
·01	1·010,1415+	761	1·082,8780	1303	1·150,6250	2015+	1·214,2095−	2911	1·274,2505−	4002	1·331,2258	5299
·02	1·001,7675+	759	1·072,1362	1294	1·137,3468	1994	1·198,2413	2869	1·255,4514	3931	1·309,4659	5186
·03	·993,4694	756	1·061,5239	1284	1·124,2680	1972	1·182,5600	2828	1·237,0455−	3860	1·288,2246	5074
·04	·985,2468	753	1·051,0400	1275	1·111,3864	1951	1·167,1616	2787	1·219,0255	3790	1·267,4907	4964
·05	·977,0995+	750	1·040,6836	1266	1·098,6998	1929	1·152,0419	2746	1·201,3846	3721	1·247,2532	4856
·06	·969,0273	747	1·030,4538	1256	1·086,2062	1908	1·137,1968	2706	1·184,1157	3653	1·227,5014	4750+
·07	·961,0297	745	1·020,3495+	1247	1·073,9033	1887	1·122,6223	2666	1·167,2122	3586	1·208,2245+	4645+
·08	·953,1067	742	1·010,3700	1237	1·061,7891	1865+	1·108,3144	2626	1·150,6672	3519	1·189,4123	4543
·09	·945,2578	739	1·000,5142	1228	1·049,8615−	1844	1·094,2690	2587	1·134,4742	3454	1·171,0543	4441
·10	·937,4828	736	·990,7812	1219	1·038,1182	1823	1·080,4824	2548	1·118,6265+	3389	1·153,1404	4342

x	$n=14$	δ^2	$n=15$	δ^2	$n=16$	δ^2	$n=17$	δ^2	$n=18$	δ^2	$n=19$	δ^2
·00	1·411,0360	6986	1·446,3086	8799	1·519,5773	10866	1·571,0449	13198	1·620,8824	15808	1·669,2352	18698
·01	1·385,5134	6812	1·437,4182	8550−	1·487,1906	10521	1·535,0392	12734	1·581,1403	15195+	1·625,6438	17912
·02	1·360,6719	6642	1·409,3827	8307	1·455,8560	10186	1·500,3070	12284	1·542,9177	14606	1·583,8437	17156
·03	1·336,4947	6476	1·382,1780	8070	1·425,5400	9860	1·466,8031	11848	1·506,1557	14038	1·543,7591	16430
·04	1·312,9651	6313	1·355,7802	7839	1·396,2100	9543	1·434,4841	11426	1·470,7975−	13489	1·505,3176	15731
·05	1·290,0668	6153	1·330,1664	7613	1·367,8343	9234	1·403,3077	11017	1·436,7882	12960	1·468,4491	15060
·06	1·267,7839	5997	1·305,3138	7392	1·340,3820	8934	1·373,2330	10621	1·404,0749	12449	1·433,0868	14416
·07	1·246,1006	5843	1·281,2005−	7177	1·313,8232	8643	1·344,2204	10238	1·372,6065−	11957	1·399,1659	13797
·08	1·225,0017	5693	1·257,8049	6967	1·288,1287	8360	1·316,2317	9867	1·342,3338	11483	1·366,6248	13202
·09	1·204,4720	5546	1·235,1059	6762	1·263,2702	8085−	1·289,2296	9508	1·313,2094	11025+	1·335,4038	12631
·10	1·184,4970	5402	1·213,0832	6562	1·239,2201	7817	1·263,1782	9160	1·285,1876	10585−	1·305,4459	12083

x	$n=20$	δ^2	$n=21$	δ^2	$n=22$	δ^2	$n=23$	δ^2	$n=24$	δ^2	$n=25$	δ^2
·00	1·716,2284	21886	1·761,9705+	25379	1·806,5563	29182	1·850,0690	33310	1·892,5827	37768	1·934,1631	42566
·01	1·668,6793	20891	1·710,3591	24138	1·750,7817	27658	1·790,0339	31458	1·828,1926	35541	1·865,3265−	39913
·02	1·623,2193	19938	1·661,1615+	22955−	1·697,7730	26209	1·733,1444	29704	1·767,3564	33440	1·800,4812	37421
·03	1·579,7531	19026	1·614,2594	21827	1·647,3852	24833	1·679,2253	28044	1·709,8644	31460	1·739,3779	35081
·04	1·538,1894	18152	1·569,5399	20751	1·599,4806	23525+	1·628,1107	26473	1·655,5183	29594	1·681,7828	32884
·05	1·498,4410	17316	1·526,8955−	19725+	1·553,9286	22283	1·579,6433	24988	1·604,1316	27835+	1·627,4760	30822
·06	1·460,4243	16516	1·486,2236	18747	1·510,6049	21104	1·533,6748	23583	1·555,5285	26178	1·576,2515−	28887
·07	1·424,0591	15751	1·447,4264	17816	1·469,3916	19985+	1·490,0645+	22255−	1·509,5432	24619	1·527,9156	27072
·08	1·389,2691	15019	1·410,4108	16928	1·430,1769	18924	1·448,6798	21000	1·466,0199	23150+	1·482,2869	25370
·09	1·355,9809	14319	1·375,0880	16083	1·392,8545−	17917	1·409,3950−	19814	1·424,8115+	21768	1·439,1953	23775+
·10	1·324,1247	13650+	1·341,3735−	15278	1·357,3237	16961	1·372,0916	18693	1·385,7800	20468	1·398,4812	22280

x	$n=26$	δ^2	$n=27$	δ^2	$n=28$	δ^2	$n=29$	δ^2	$n=30$	δ^2	$n=31$	δ^2
·00	1·974,8689	47711	2·014,7532	53212	2·053,8637	59075−	2·092,2437	65310	2·129,9327	71925−	2·166,9667	78923
·01	1·901,4972	44578	1·936,7603	49541	1·971,1660	54805+	2·004,7599	60375−	2·037,5836	66252	2·069,6752	72442
·02	1·832,5834	41647	1·863,7216	46120	1·893,9489	50840	1·923,3135+	55809	1·951,8597	61026	1·979,6279	66493
·03	1·767,8342	38905+	1·795,2948	42932	1·821,8157	47160	1·847,4480	51587	1·872,2384	56213	1·896,2299	61034
·04	1·706,9755+	36341	1·731,1612	39962	1·754,3985−	43744	1·776,7412	47685−	1·798,2384	51780	1·818,9353	56028
·05	1·649,7509	33943	1·671,0237	37196	1·691,3557	40576	1·710,8029	44079	1·729,4164	47701	1·747,2434	51437
·06	1·595,9207	31703	1·614,6059	34622	1·632,3706	37638	1·649,2725+	40748	1·665,3645+	43947	1·680,6952	47229
·07	1·545,2607	29609	1·561,6503	32225+	1·577,1493	34915−	1·591,8169	37672	1·605,7073	40493	1·618,8699	43372
·08	1·497,5617	27654	1·511,9172	29996	1·525,4195−	32390	1·538,1286	34832	1·550,0994	37316	1·561,3818	39838
·09	1·452,6281	25828	1·465,1836	27921	1·476,9286	30050+	1·487,9234	32210	1·498,2231	34395+	1·507,8776	36601
·10	1·410,2772	24123	1·421,2422	25992	1·431,4428	27883	1·440,9392	29799	1·449,7862	31709	1·458,0335−	33636

ON THE APPLICATION OF CONTINUED FRACTIONS TO THE EVALUATION OF CERTAIN INTEGRALS, WITH SPECIAL REFERENCE TO THE INCOMPLETE BETA-FUNCTION.

BY J. H. MÜLLER, M.Sc.

(i) THE subject of the present short paper was suggested by the lectures of Professor Pearson on "Laplace."

Laplace considers $\int_t^\infty e^{-t^2}dt$ or the incomplete integral of a normal curve whose area $= \sqrt{\pi}$ and standard deviation $= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

Let
$$I_n = \int_t^\infty t^{-n}e^{-t^2}dt$$
$$= -\frac{1}{2}\int_t^\infty t^{-n-1}d\left(e^{-t^2}\right)$$
$$= \frac{1}{2}\frac{e^{-t^2}}{t^{n+1}} - \frac{1}{2}(n+1)I_{n+2},$$

$$I_0 = \int_t^\infty e^{-t^2}dt$$
$$= \frac{e^{-t^2}}{2t} - \frac{e^{-t^2}}{2^2t^3} + \frac{1.3.e^{-t^2}}{2^3t^5} - \frac{1.3.5.e^{-t^2}}{2^4t^7} + \frac{1.3.5.7}{2^4}I_8$$
$$= \frac{e^{-t^2}}{2t}\left\{1 - \frac{1}{2t^2} + \frac{1.3}{(2t^2)^2} - \frac{1.3.5}{(2t^2)^3} + \dots + \frac{(-1)^r 1.3.5\dots(2r-1)}{(2t^2)^r} + \dots \right.$$
$$\left. + \frac{(-1)^{r+t}1.3\dots(2\overline{r+t}-1)}{2^{r+t}e^{-t^2}}.2tI_{2(r+t)}\right\};$$

put $q = \frac{1}{2t^2}$,

$$\therefore\ I_0 = \frac{e^{-\frac{1}{2q}}}{\sqrt{\frac{1}{2q}}}\{1 - q + 1.3.q^2 - 1.3.5.q^3 + \dots\}.$$

This is the series considered by Laplace. It is ultimately divergent, but, for large values of t or small values of q, a very accurate approximation to I_0 may be obtained by summing the terms before the point of divergence is reached.

We write $$S = 1 - q + 1.3.q^2 - 1.3.5.q^3 + \dots.$$

Let $$\phi(t) = \frac{1}{1-t}\left(1 - \frac{q}{(1-t)^2} + \frac{1.3.q^2}{(1-t)^4} - \dots\right) \quad \text{.....................(A)},$$

$$= y_1 + y_2 t + y_3 t^2 + \dots + y_{x+1} t^x + y_{x+2} t^{x+1} + \dots \quad \text{............(B)}.$$

From (B) $$q\frac{d\phi}{dt} = qy_2 + 2qy_3 t + \dots + (x+1)\,qy_{x+2} t^x + \dots \quad \text{..................(C)}.$$

From (A) $$q\frac{d\phi}{dt} = \frac{q}{(1-t)^2} - \frac{1.3.q^2}{(1-t)^4} + \frac{1.3.5.q^3}{(1-t)^6} - \dots$$

$$= 1 - (1-t)\,\phi(t)$$

$$= 1 - (1-t)\{y_1 + y_2 t + y_3 t^2 + \dots + y_{x+1} t^x + \dots\} \quad \text{.........(D)},$$

equating coefficients of t^x in (C) and (D).

$$\therefore\ q(x+1)\,y_{x+2} = -y_{x+1} + y_x \quad \text{.........................(E)}.$$

$$\therefore\ \frac{y_x}{y_{x+1}} = 1 + q(x+1)\frac{y_{x+2}}{y_{x+1}}.$$

$$\therefore\ \frac{y_{x+1}}{y_x} = \frac{1}{1 + q(x+1)\dfrac{y_{x+2}}{y_{x+1}}}.$$

$$\therefore\ \frac{y_1}{y_0} = \cfrac{1}{1 + \cfrac{q}{1 + \cfrac{2q}{1 + \cfrac{3q}{1 + \dots}}}}$$

y_1 is the term in (A) or (B) independent of t

$$= 1 - q + 1.3.q^2 - \dots = S.$$

In (E) put $q = 0$. $\therefore\ y_0 = y_{1\,(q=0)} = 1$.

$$\therefore\ S = \frac{1}{1+}\,\frac{q}{1+}\,\frac{2q}{1+}\dots \quad \text{..........................(F)}.$$

This is the continued fraction obtained by Laplace:

$$S = \frac{2t\int_t^\infty e^{-t^2}dt}{e^{-t^2}}$$

$$= \frac{x\cdot\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_x^\infty e^{-\frac{1}{2}x^2}dx}{\dfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}}$$

$$= x\,\frac{\frac{1}{2}(1-\alpha_x)}{z_x} \text{ in the usual notation,}$$

or $$\frac{\frac{1}{2}(1-\alpha_x)}{z_x} = \frac{1}{x}.S, \text{ where } q = \frac{1}{x^2}.$$

This form of the continued fraction (F) has recently been employed in the Biometric Laboratory, in checking tables of

$$\frac{\frac{1}{2}(1-\alpha_x)}{z_x}.$$

When $x=4$, $\frac{1}{x}.S=\cdot 236{,}652{,}383$ correct to 9 places is obtained by using 15 convergents of (F).

A well-known method of converting series of such a type as S into a continued fraction is the following:

Let $$s_r=(-1)^{r+1}1.3\ldots(2r+1)q^{r+1}+(-1)^{r+2}1.3\ldots(2r+3)q^{r+2}+\ldots,$$

$$s_{r+1}=(-1)^{r+2}1.3\ldots(2r+3)q^{r+2}+(-1)^{r+3}1.3\ldots(2r+5)q^{r+3}+\ldots.$$

$\therefore\ s_r-s_{r+1}=(-1)^{r+1}1.3\ldots(2r+1)q^{r+1},$

and $\quad s_{r-1}-s_r=(-1)^r 1.3\ldots(2r-1)q^r.$

$$\therefore\ \frac{s_r-s_{r+1}}{s_{r-1}-s_r}=-(2r+1)q.$$

$$\therefore\ 1-\frac{s_{r+1}}{s_r}=-(2r+1)q\left(\frac{s_{r-1}}{s_r}-1\right).$$

$$\therefore\ \frac{s_r}{s_{r-1}}=\frac{(2r+1)q}{(2r+1)q-1+}\frac{s_{r+1}}{s_r}.$$

$$\therefore\ \frac{s_0}{s_{-1}}=\frac{q}{q-1+}\frac{3q}{3q-1+}\frac{5q}{5q-1+}\frac{7q}{7q-1+\ldots}\qquad\ldots\ldots\ldots\ldots\ldots(G).$$

$$s_0=\ -q+1.3.q^2-\ldots=S-1,$$

$$s_{-1}=1-q+1.3.q^2\ \ \ldots=S.$$

$\therefore\ \frac{S-1}{S}$ is given by (G).

$$\therefore\ S=\frac{1}{1-}\frac{q}{q-1+}\frac{3q}{3q-1+\ldots}$$

This type of continued fraction may be termed the "equivalent" fraction.

The n^{th} convergent of this continued fraction exactly reproduces n terms of the series.

(ii) Besides the Incomplete Normal Curve Integral, two other most interesting functions in statistical analysis are the incomplete Beta and Gamma functions.

Series and their equivalent fractions, analogous to (G), may easily be found for these functions.

Let $F(n)=\int_x^\infty e^{-x}x^n\,dx \qquad x>0$

$$=e^{-x}x^n+nF(n-1)$$

$$=e^{-x}x^n\left\{1+\frac{n}{x}+\frac{n(n-1)}{x^2}+\ldots+\frac{n(n-1)\ldots(n-r+1)}{e^{-x}x^n}\int_x^\infty e^{-x}x^r\,dx\right\}.$$

Consider $\qquad S=1+nt+n(n-1)t^2+\ldots,$ where $t=\frac{1}{x}$.

This is a terminating series if n be integral, but ultimately diverges if n be fractional. If x is large compared to n a very good approximation to $\frac{F(n)}{e^{-x}x^n}$ may be obtained by summation of the terms before the point of divergence is reached.

Let $\quad s_r = n(n-1)\dots(n-r+1)t^r + n(n-1)\dots(n-r)t^{r+1} + \dots.$

$$\therefore \quad \frac{s_r - s_{r+1}}{s_{r-1} - s_r} = (n-r+1)t.$$

$$\therefore \quad (n-r+1)t\frac{s_{r-1}}{s_r} = (n-r+1)t + 1 - \frac{s_{r+1}}{s_r}.$$

$$\therefore \quad \frac{s_r}{s_{r-1}} = \frac{(n-r+1)t}{1+(n-r+1)t - \frac{s_{r+1}}{s_r}}.$$

$$s_0 = 1 + nt + n(n-1)t^2 + \dots,$$

$$s_1 = s_0 - 1.$$

$\therefore$ put $r=1$.

$$\therefore \quad \frac{s_0 - 1}{s_0} = \frac{nt}{1+nt-}\,\frac{(n-1)t}{1+(n-1)t-}\,\frac{(n-2)t}{1+(n-2)t-}\;\dots\dots\dots\dots(\text{H}),$$

and
$$s_0 = \frac{1}{1-(H)}.$$

(iii) Let
$$B_x(u,v) = \int_0^x x^{u-1}(1-x)^{v-1}dx$$

$$= \frac{x^u}{u}(1-x)^{v-1} + \frac{v-1}{u}\int_0^x x^u(1-x)^{v-2}dx$$

$$= \frac{x^u}{u}(1-x)^{v-1} + \frac{v-1}{u}B_x(u+1, v-1);$$

put
$$1-x=y, \quad \frac{x}{y}=t \quad \text{and} \quad I_x(u,v) = \frac{B_x(u,v)}{B_1(u,v)}.$$

Hence
$$I_x(u,v) = \frac{x^u y^{v-1}}{uB_1(u,v)}\left\{1 + \frac{v-1}{u+1}t + \frac{(v-1)(v-2)}{(u+1)(u+2)}t^2 + \dots\right.$$
$$\left. + \frac{(v-1)(v-2)\dots(v-s+1)}{(u+1)(u+2)\dots(u+s-1)}t^{s-1}\right\} + I_x(u+s, v-s).$$

If v be an integer (s) the series terminates after s terms and $I_x(u,v)$ becomes

$$\frac{x^u y^{v-1}}{uB_1(u,v)}\left\{1 + \frac{v-1}{u+1}t + \dots + \frac{(v-1)(v-2)\dots 2.1}{(u+1)(u+2)\dots(u+v-1)}t^{v-1}\right\} \quad \dots\dots(\text{I});$$

put $k = u+v-1$, then the first v terms of

$$(x+y)^k = x^k + kx^{k-1}y + \frac{k!}{2!(k-2)!}x^{k-2}y^2 + \dots$$
$$+ \frac{k!\,x^{k-v+2}y^{v-2}}{(v-2)!(k-v+2)!} + \frac{k!}{(v-1)!(k-v+1)!}x^{k-v+1}y^{v-1}.$$

Writing the terms backwards, the first v terms

$$=x^u y^{v-1}\frac{k!}{(v-1)!\,u!}$$

$$\times\left\{1+\frac{v-1}{u+1}t+\frac{(v-1)(v-2)}{(u+1)(u+2)}t^2+\ldots+\frac{(v-1)!}{(u+1)(u+2)\ldots(u+v-1)}t^{v-1}\right\}.$$

Hence if v is an integer

$$I_x(u, v)=\text{first } v \text{ terms of } (x+y)^k,$$

where $$k=u+v-1,\quad y=1-x.$$

(See Karl Pearson, *Biometrika*, Vol. XVI. p. 202, and Soper in *Tracts for Computers*, No. VII.)

Let $$s'=\frac{v}{u}+\frac{v(v-1)}{u(u+1)}t+\ldots+\frac{v(v-1)\ldots(v-r)}{u(u+1)\ldots(u+r)}t^r+\ldots,$$

$$s_r=\frac{v(v-1)\ldots(v-r)}{u(u+1)\ldots(u+r)}t^r+\frac{v(v-1)\ldots(v-r-1)}{u(u+1)\ldots(u+r+1)}t^{r+1}+\ldots.$$

$$\therefore\quad s_r-s_{r+1}=\frac{v(v-1)\ldots(v-r)}{u(u+1)\ldots(u+r)}t^r,$$

$$s_{r-1}-s_r=\frac{v(v-1)\ldots(v-r+1)}{u(u+1)\ldots(u+r-1)}t^{r-1}.$$

$$\therefore\quad 1-\frac{s_{r+1}}{s_r}=\frac{v-r}{u+r}t\left(\frac{s_{r-1}}{s_r}-1\right),$$

or $$\frac{s_r}{s_{r-1}}=\frac{\frac{v-r}{u+r}t}{1+\frac{v-r}{u+r}t-\frac{s_{r+1}}{s_r}}.$$

$$\therefore\quad \frac{s_1}{s_0}=\frac{\frac{v-1}{u+1}t}{1+\frac{v-1}{u+1}t-\frac{s_2}{s_1}}=\frac{\frac{v-1}{u+1}t}{A}\text{ say,}$$

$$s_0=s'\quad\text{and}\quad s_1=s_0-\frac{v}{u}.$$

$$\therefore\quad 1-\frac{\frac{v-1}{u+1}t}{A}=\frac{\frac{v}{u}}{s_0}.$$

$$\therefore\ \frac{v}{u}+1-\frac{\frac{v-1}{u+1}t}{A}=\frac{v}{u}\left(\frac{s_0+1}{s_0}\right).$$

$$\therefore\quad \frac{s_0}{1+s_0}=\frac{\frac{v}{u}}{1+\frac{v}{u}-}\,\frac{\frac{v-1}{u+1}t}{1+\frac{v-1}{u+1}t-}\,\frac{\frac{v-2}{u+2}t}{1+\frac{v-2}{u+2}t-},\ \ldots\quad\ldots\ldots\ldots\ldots(\text{J}).$$

If $\frac{s_0}{1+s_0}=F$, then $s_0=\frac{F}{1-F}$.

It is therefore seen that continued fractions (G), (H) and (I) of the "Equivalent" type are obtained for the "Normal Integral," the "Gamma Function" and the "Beta Function" series. In the Normal series, another type of continued fraction (F) was obtained by Laplace. It appeared desirable to investigate whether a similar type to (F) also exists for the Gamma and Beta Function series.

In my investigations I found that another class had actually been found for the Gamma Function (De Morgan, *Differential Calculus*, p. 590).

De Morgan finds for

$$1+\frac{n}{x}+\frac{n(n-1)}{x^2}+\frac{n(n-1)(n-2)}{x^3}+\dots$$

the continued fraction

$$\frac{1}{1-}\,\frac{\frac{n}{x}}{1+}\,\frac{\frac{1}{x}}{1+}\,\frac{\frac{1-n}{x}}{1+}\,\frac{\frac{2}{x}}{1+}\,\frac{\frac{2-n}{x}}{1+}\,\frac{\frac{3}{x}}{1+}\,\frac{\frac{3-n}{x}}{1+}\dots \qquad \dots\dots\dots\dots\dots(\text{K}).$$

Finally I came across an important paper of Thomas Muir, "New General Formulae for the Transformation of Infinite Series into Continued Fractions," *Transactions of the Royal Society of Edinburgh*, Vol. XXVII. p. 467.

Assume $$1+B_1x+B_2x^2+\dots=\frac{1}{a_1+}\,\frac{x}{a_2+}\,\frac{x}{a_3+}\dots,$$

where $a_1, a_2, \dots$ are independent of x. $a_1, a_2, \dots$ are then obtained in the form of determinants in $B_1, B_2, \dots$ by equating coefficients of like powers of x. Thus

$$1+B_1x+B_2x^2+B_3x^3+\dots$$

$$=\frac{1}{1-}\,\frac{\beta_1 x}{1-}\,\frac{\beta_2 x}{\beta_1-}\,\frac{\beta_3 x}{\beta_2-}\,\frac{\beta_1\beta_4 x}{\beta_3-}\,\frac{\beta_2\beta_5 x}{\beta_4-}\,\frac{\beta_3\beta_6 x}{\beta_5-}\,\frac{\beta_4\beta_7 x}{\beta_6-}\dots \qquad \dots(\text{L}),$$

where the values of $\beta_1, \beta_2, \beta_3, \beta_4, \dots$ are given by

$$B_1,\ \begin{vmatrix}1 & B_1\\ B_1 & B_2\end{vmatrix},\quad \begin{vmatrix}B_1 & B_2\\ B_2 & B_3\end{vmatrix},\quad \begin{vmatrix}1 & B_1 & B_2\\ B_1 & B_2 & B_3\\ B_2 & B_3 & B_4\end{vmatrix},\quad \begin{vmatrix}B_1 & B_2 & B_3\\ B_2 & B_3 & B_4\\ B_3 & B_4 & B_5\end{vmatrix},$$

$$\begin{vmatrix}1 & B_1 & B_2 & B_3\\ B_1 & B_2 & B_3 & B_4\\ B_2 & B_3 & B_4 & B_5\\ B_3 & B_4 & B_5 & B_6\end{vmatrix},\quad \begin{vmatrix}B_1 & B_2 & B_3 & B_4\\ B_2 & B_3 & B_4 & B_5\\ B_3 & B_4 & B_5 & B_6\\ B_4 & B_5 & B_6 & B_7\end{vmatrix},\ \dots.$$

This method I applied to the Normal series

$$S=1-x+1.3x^2-1.3.5x^3+\dots,$$

and found
$$B_1=-1, \qquad B_2=1.3,$$
$$B_3=-1.3.5, \qquad B_4=1.3.5.7,$$
$$B_5=-1.3.5.7.9, \quad B_6=1.3.5.7.9.11, \dots.$$

The values found from the determinants were

$\beta_1=1,\ \beta_2=2,\ \beta_3=6,\ \beta_4=48,\ \beta_5=-45.16,\ \beta_6=45.48.16,\ \beta_7=2^8.3^4.5^2.7.$

On substituting these values and simplifying,

$$s = \frac{1}{1+}\,\frac{x}{1+}\,\frac{2x}{1+}\,\frac{3x}{1+}\,\frac{4x}{1+}\,\frac{5x}{1+}\cdots,$$

i.e. the same form as Laplace's. Compare (F).

This method was next applied to the Gamma Function series

$$S = 1 + nt + n(n-1)t^2 + n(n-1)(n-2)t^3 + \ldots.$$

Hence $\qquad B_1 = n,\quad B_2 = n(n-1),\quad B_3 = n(n-1)(n-2), \ldots.$

The values found from the determinants were

$$\beta_1 = n,\quad \beta_2 = -n,\quad \beta_3 = -n^2(n-1),\quad \beta_4 = -2n^2(n-1),\quad \beta_5 = -2n^3(n-1)^2(n-2).$$

Substituting these values, and putting $t = \frac{1}{x}$, we find

$$s = \frac{1}{1-}\,\frac{\frac{n}{x}}{1+}\,\frac{\frac{1}{x}}{1+}\,\frac{\frac{1-n}{x}}{1+}\,\frac{\frac{2}{x}}{1+}\,\frac{\frac{2-n}{x}}{1+}\cdots,$$

which is the same as De Morgan's form. Compare (K).

Tables of the Incomplete Beta Function $B_x(u, v)$ are at present being computed; such tables are of necessity limited, and it therefore seems highly important to investigate every channel which may lead to values of this function, as such may be of assistance in the actual computation of the tables, or in obtaining values outside their range.

Attempts to obtain a continued fraction of Laplace's or De Morgan's type, by employing their methods, were unsuccessful. Muir's method was then applied to the series (see (I) above)

$$S = 1 + \frac{v-1}{u+1}t + \frac{(v-1)(v-2)}{(u+1)(u+2)}t^2 + \ldots,$$

where $$t = \frac{x}{y} = \frac{x}{1-x}.$$

Further let $$k = u + v - 1,\quad u_1 = \frac{v-1}{u+1}, \ldots u_r = \frac{v-r}{u+r}.$$

The following values were found:

$$\beta_1 = u_1,$$

$$\beta_2 = -\frac{(k+1)u_1}{(u+1)(u+2)},$$

$$\beta_3 = -\frac{(k+1)u_1^2u_2}{(u+2)(u+3)},$$

$$\beta_4 = -\frac{2(k+1)^2(k+2)u_1^2u_2}{(u+1)(u+2)^2(u+3)^2(u+4)},$$

$$\beta_5 = -\frac{2(k+1)^2(k+2)u_1^3u_2^2u_3}{(u+2)(u+3)^2(u+4)^2(u+5)},$$

$$\beta_6 = \frac{2^2.3.(k+1)^3(k+2)^2(k+3)u_1^3u_2^2u_3}{(u+1)(u+2)^2(u+3)^3(u+4)^3(u+5)^2(u+6)},$$

$$\beta_7 = \frac{2^2.3.(k+1)^3(k+2)^2(k+3)u_1^4u_2^3u_3^2u_4}{(u+2)(u+3)^2(u+4)^3(u+5)^3(u+6)^2(u+7)}.$$

Substituting these values in (L) we find if

$$I_x(u, v) = \frac{B_x(u, v)}{B_1(u, v)} = \frac{\int_0^x x^{u-1}(1-x)^{v-1}\,dx}{\int_0^1 x^{u-1}(1-x)^{v-1}\,dx}$$

that $$I_x(u, v) = C\left[\frac{b_1}{1+}\,\frac{b_2}{1+}\,\frac{b_3}{1+}\,\frac{b_4}{1+}\cdots\right] \qquad \text{(M)},$$

where $$C = \frac{x^u y^{v-1}}{u \,.\, B_1(u, v)} = x^u y^{v-1}\frac{\Gamma(k+1)}{\Gamma(u+1)\Gamma(v)},$$

$$b_1 = 1,$$

$$b_2 = -u_1 t,$$

$$b_3 = \frac{(k+1)}{(u+1)(u+2)}\,t,$$

$$b_4 = -\frac{(u+1)(u+2)}{(u+2)(u+3)}\,u_2 t,$$

$$b_5 = \frac{2(k+2)}{(u+3)(u+4)}\,t,$$

$$b_6 = -\frac{(u+2)(u+3)}{(u+4)(u+5)}\,u_3 t,$$

$$b_7 = \frac{3(k+3)}{(u+5)(u+6)}\,t,$$

$$b_8 = -\frac{(u+3)(u+4)}{(u+6)(u+7)}\,u_4 t,$$

....

$$b_{2r} = -\frac{(u+r-1)(u+r)}{(u+2r-2)(u+2r-1)}\,u_r t,$$

$$b_{2r+1} = \frac{r(k+r)}{(u+2r-1)(u+2r)}\,t.$$

This I believe to be a new expression for the Incomplete B-function. In computing any value $I_x(u, v)$, there is an alternative given by $I_x(u, v) = 1 - I_y(v, u)$. From general considerations it appears that the particular form to be selected should be that which does not require the binomial $(x+y)^k$ to be summed through its largest term.

The largest term of this binomial is the $(r+1)^{\text{th}}$, where r is the greatest integer consistent with

$$\frac{y(k+1)}{x+y} > r \quad\text{or}\quad y(u+v) > r, \text{ since } x+y=1.$$

With integration to the mode, there are two courses, (*a*) from the left of the mode, (*b*) from the right of the mode. Consider

$$I_x(u, v) \text{ and } I_{x'}(v, u),\ u < v \qquad (u, v > 1).$$

In the associated frequency distribution of $I_x(u, v)$, the mode is between the origin and the mean, and the distance of the mode from the origin is

$$\frac{u-1}{u+v-2}=x<\frac{1}{2};$$

$$\therefore\ y(u+v)=\frac{(v-1)(u+v)}{u+v-2}=v-1+\frac{2(v-1)}{u+v-2}$$

$$=v+\text{a positive fraction.}$$

Therefore the $(v+1)^{\text{th}}$ term is the largest in the binomial.

Considering $I_{x'}(v, u)$, the mean is between the mode and the origin. If integration to the mode be attempted, the greatest term of the binomial is given by

$$u-1+\frac{2(u-1)}{u+v-2}>r,$$

i.e. $$u-\text{a positive fraction}>r.$$

Therefore the u^{th} term is the largest.

In computing the value of $I_x(u, v)$ from the continued fraction, it is necessary first to calculate the values of the b's; this is very simply done on a calculating machine. Since

$$(u+r)(u+r-1)=(u+r-1)(u+r-2)+2(u+r-1),$$

the denominators of $b_3, b_4, \ldots$ can be computed continuously for a long series, and each value so obtained multiplied by $1-x$, since $t=\frac{x}{1-x}$; this stage completes the calculation of denominators. u_r contains a factor $u+r$, which cancels out.

The numerators will be already in part found, or can be easily calculated.

If the n^{th} convergent of the continued fraction be $\frac{p_n}{q_n}$,

$$p_1=1,\quad p_2=1,\quad q_1=1,\quad q_2=1+b_2;$$

further values are given by $p_n=p_{n-1}+b_n p_{n-2}$ and $q_n=q_{n-1}+b_n q_{n-2}$.

Successive values of p_n and q_n are therefore continuously obtained. In the other or "equivalent" type of continued fraction

$$F=\frac{b_1}{1+b_1-}\ \frac{b_2}{1+b_2-}\ \frac{b_3}{1+b_3-},\ \ldots \quad \text{............compare (J).}$$

It is easily seen that $p_1=b_1$, $q_1=1+b_1$, $p_2=b_1(1+b_2)$, $q_2=1+p_2$, and generally $q_n=1+p_n$.

The n^{th} convergent $$F_n=\frac{p_n}{q_n}=\frac{p_n}{1+p_n},$$

and the series to n terms is $$\frac{F_n}{1-F_n}=p_n,$$

where $$p_n=(1+b_n)p_{n-1}-b_n p_{n-2}$$
$$=p_{n-1}+b_n(p_{n-1}-p_{n-2}),$$

and $$I_x(u, v)=\frac{x^u y^{v-1}}{v\,.\,B_1(u, v)}\times p_n,$$

if n terms of the series be taken to represent $I_x(u, v)$ approximately.

Illustration.

$I_{.908}(55, 91) = {\cdot}0401\ 5474$ by Weddle's formula.

New Type.

$C = {\cdot}0128\ 7968\ 94$; $I = C.\frac{p_r}{q_r}$.

$p_1 = b_1$; $p_2 = b_1$; $p_r = p_{r-1} + b_r p_{r-2}$.

$q_1 = 1$; $q_2 = 1 + b_2$; $q_r = q_{r-1} + b_r q_{r-2}$.

Equivalent Type.

$C = {\cdot}0077\ 8442\ 764$; $I = Cp_r$.

$p_1 = b_1$; $p_2 = b_1 + b_1 b_2$; $p_r = p_{r-1} + b_r \Delta p_{r-2}$

r	b_r	p_r	q_r	$I_x(u, v)$	b_r	p_r	Δp_r	$I_x(u, v)$
1	1·0	1·0	1·0	·0129	1·6545 4545 455	1·6545 4545 45	1·1835 2601 16	·0128 8
2	−·7153 1791 908	1·0	·2846 8208 09	·0452	·7153 1791 908	2·8380 7146 61	·8225 0316 98	·0220 9
3	+·0203 5797 586	1·0203 5797 59	·3050 4005 68	·0431	·6949 5994 321	3·6605 7463 59	·5554 3965 31	·0284 9
4	−·6709 9580 724	·3493 6216 86	·1140 1957 42	·0394 6	·6753 0396 651	4·2160 1428 90	·3645 4298 38	·0328 2
5	+·0382 3947 730	·3883 8012 43	·1256 8414 65	·0398 0	·6563 1429 411	4·5805 5727 28	·2325 6297 10	·0356 5
6	−·6306 6522 974	·1680 4955 20	·0537 7596 55	·0402 5	·6379 5761 079	4·8131 2024 38	·1442 3620 25	·0374 7
7	+·0539 9412 489	·1890 1979 69	·0605 6217 10	·0401 99	·6202 0278 594	4·9573 5644 63	·0869 7741 54	·0385 9
8	−·5938 4122 540	·0892 2504 50	·0286 2778 58	·0401 43	·6030 2069 737	5·0443 3386 17	·0510 0217 10	·0392 7
9	+·0679 1389 562	·1020 6211 58	·0327 4079 87	·0401 495	·5863 8407 193	5·0953 3603 27	·0290 8487 25	·0396 6
10	−·5601 1159 281	·0520 8613 37	·0167 0604 40	·0401 563	·5702 6734 104	5·1244 2090 52	·0161 3182 30	·0398 9
11	+·0802 4399 733	·0602 7600 58	·0193 3329 66	·0401 554	·5546 4650 956	5·1405 5272 82	·0087 0310 30	·0400 16
12	−·5291 2405 514	·0327 1597 96	·0104 9372 69	·0401 5462	·5394 9903 661	5·1492 5583 12	·0045 6742 08	·0400 84
13	+·0911 9144 164	·0382 1263 54	·0122 5675 80	·0401 5474	·5248 0372 703	5·1538 2325 20	·0023 3185 39	·0401 20
14					·5105 4063 244	5·1561 5510 59	·0011 5821 08	·0401 38
15					·4966 9096 088	5·1573 1331 67	·0005 5969 03	·0401 467
16					·4832 3699 422	5·1578 7300 70	·0002 6314 51	·0401 511
17					·4701 6201 254	5·1581 3615 21	·0001 2037 57	·0401 531
18					·4574 5022 479	5·1582 5652 78	·0000 5357 77	·0401 5407
19					·4450 8670 520	5·1583 1010 55	—	·0401 5449
20					·4331 0808 172	5·1583 3331 0	—	·0401 5467

In the Tables on pp. 293—4 I have compared the convergence of the two types of continued fractions (distinguished as the "New" and "Equivalent" Types*) to the values found by "Weddling." In one case the computation is illustrated in detail.

Other Illustrations of the Degree of Approximation of the method.

First values given by Weddling:

"*New*," "*Equiv.*" = values given by New and Equivalent Types of Continued Fractions.

(n) denotes number of convergents.

$I_{.5}$ (91, 55)	·0013 3410
New (3)	·0013 7
Equiv. "	·0011 1
New (4)	·0013 30
Equiv. "	·0012 2
New (7)	·0013 3425
Equiv. "	·0013 20
New (8)	·0013 3407
Equiv. "	·0013 28
New (9)	·0013 3409
Equiv. "	·0013 312
" (12)	·0013 339
" (16)	·0013 3409
$I_{.5}$ (113, 101)	·2055 0427
New (3)	·2687
Equiv. "	·097
New (4)	·1778
Equiv. "	·126
New (7)	·2108
Equiv. "	·163
New (8)	·2028
Equiv. "	·178
New (11)	·2059
Equiv. "	·196
New (12)	·2053
Equiv. "	·1988
New (15)	·2055 3
Equiv. "	·2036
New (16)	·2055 49
Equiv. "	·2042 6
New (19)	·2055 045
Equiv. "	·2052 1
" (25)	·2054 95
" (30)	·2055 040
$I_{.6}$ (81, 21)	·0000 1221
New (3)	·0000 1227
Equiv. "	·0000 1175
New (4)	·0000 1219
Equiv. "	·0000 1207
New (7)	·0000 1220
Equiv. "	·0000 1220

$I_{.2}$ (21, 81)	·4603 9850
New (3)	·710
Equiv. "	·257
New (4)	·361
Equiv. "	·318
New (7)	·4705
Equiv. "	·4219
New (8)	·4559
Equiv. "	·4376
New (11)	·4606
Equiv. "	·4567
New (12)	·4603 1
Equiv. "	·4585
New (15)	·4604 01
Equiv. "	·4602 1
New (16)	·4603 9773
Equiv. "	·4603 2
New (17)	·4603 9834
Equiv. "	·4603 66
" (22)	·4603 9824
$I_{.8}$ (81, 21)	·5396 0150
New (3)	1·245
Equiv. "	·284
New (4)	·263
Equiv. "	·3600
New (7)	·599
Equiv. "	·497
New (8)	·494
Equiv. "	·517
New (11)	·543
Equiv. "	·538
New (12)	·537
Equiv. "	·5388 6
New (15)	·5397 2
Equiv. "	·5395 86
New (16)	·5395 0
Equiv. "	·5395 98
New (19)	·5396 038
Equiv. "	·5396 0154
New (20)	·5395 9982
Equiv. "	·5396 0154
New (21)	·5396 0129

$I_{.33}$ (55, 91)	·5381 5900
New (3)	1·63
" (4)	·243
" (7)	·6426
" (8)	·4765
" (11)	·5476
" (12)	·5323 1
" (15)	·5387 4
" (16)	·5378 2
" (19)	·5381 83
" (20)	·5381 46
" (23)	·5381 595
" (24)	·5381 587
$I_{.83}$ (91, 55)	·4618 4100
New (3)	·98
" (4)	·25
" (7)	·5265
" (8)	·4189
" (11)	·4686
" (12)	·4571 5
" (15)	·4623 4
" (16)	·4615 0
" (19)	·4618 7
" (20)	·4618 24
" (23)	·4618 416
" (24)	·4618 402
" (26)	·4618 408
$I_{.672}$ (91, 55)	·1017 1400
New (3)	·1161
" (4)	·0966 5
" (7)	·1022 5
" (8)	·1014 9
" (11)	·1017 32
" (12)	·1017 033
" (15)	·1017 144
" (16)	·1017 1374
" (18)	·1017 1398

$I_{.332}$ (55, 91)	·1314 8814
New (3)	·152
" (4)	·1244
" (7)	·1322
" (8)	·1312
" (11)	·1315 1
" (12)	·1314 81
" (15)	·1314 788
" (16)	·1314 8805
" (19)	·1314 8820
$I_{.7}$ (81, 21)	·0141 902
New (3)	·0146 9
" (4)	·0140 8
" (7)	·0141 935
" (8)	·0141 892
" (11)	·0141 9018
" (12)	·0141 9015
$I_{.12}$ (21, 81)	·0081 6185
New (3)	·0083
" (4)	·0081 5
" (7)	·0081 6204
" (8)	·0081 6182 7
" (9)	·0081 6184 7
" (10)	·0081 6185 5
" (11)	·0081 6185 41
" (12)	·0081 6185 38

* The term "Equivalent" is used to denote that the continued fraction method applied is really equivalent to summing a series in which the function is expanded. It merely provides an orderly means of achieving this.

In the Equivalent Type, $b_1 = \frac{v}{u}$, $b_r = \frac{v-r+1}{u+r-1}t$, see (J).

The value of b_r in the New Type is given by (M).

The frequency distributions considered were given by

$$y = y_0 x^{u-1}(1-x)^{v-1},$$

where $u = 91$, $v = 55$, Mode $=$ ·625, Mean $=$ ·623, S.D. $=$ ·040,
$u = 113$, $v = 101$, Mode $=$ ·5283, Mean $=$ ·5280, S.D. $=$ ·034,
$u = 81$, $v = 21$, Mode $=$ ·800, Mean $=$ ·794, S.D. $=$ ·040.

For $I_x(u, v)$ as the sum of the first v terms of $(x+y)^{u+v-1}$, I have taken

x	·308	·332	·38	·5	·572	·62	·5	·2	·6	·7	·8	·12
u	55	55	55	91	91	91	113	21	81	81	81	21
v	91	91	91	55	55	55	101	81	21	21	21	81
No. of largest term of the binomial	102	98	91	73 and 74	63	56	112 and 113	82	62	31	21	90
Dist. from Mode in terms of S.D.	1·5	1	0	3	1	·1	·8	0	5	2·5	0	3

It will also be noticed that the values obtained by the New Type of C.F. are in pairs less and greater than $I_x(u, v)$.

The following convergents are above the true value: 2, 3, 6, 7, 10, 11, 14, 15, 18, 19, 22, 23, etc.

The other convergents are below the true value. This is a necessary consequence of the form of the continued fraction $\frac{b_1}{1+}\frac{b_2}{1+}\frac{b_3}{1+}\ldots$ when the even b's are negative and less than unity. As we do not integrate beyond the mode of the distribution $x \equiv \frac{u-1}{u+v-2}$, it follows that the maximum value of t or $\frac{x}{y} = \frac{u-1}{v-1}$.

$$\therefore \text{ Max. value of } |b_2| = \frac{v-1}{u+1}\frac{x}{y} = \frac{(v-1)(u-1)}{(u+1)(v-1)} = \frac{u-1}{u+1} < 1.$$

$$\therefore \text{ In all cases } |b_2| < 1.$$

$$\therefore \text{ All even } |b\text{'s}| < 1.$$

It is a great advantage of the New Type that narrow limits, within which I_x lies, are soon obtained.

To illustrate the application of the New Type of continued fraction, I have chosen the more difficult examples; in several cases I have integrated the distribution as far as the mode; in one case, $I_{\cdot 38}(55, 91)$, slightly beyond the mode.

My attempts to find a value of the remainder after computing n convergents have not been successful.

Turning to the expression (L) it will be noticed that if $B_n, B_{n+1}, B_{n+2}, \ldots = 0$ the continued fraction terminates. The influence of the vanishing of $B_n, B_{n+1}, \ldots,$

is felt in the $(n+1)^{\text{th}}$ convergent and onward, i.e. in $\beta_n, \beta_{n+1}, \ldots$, but not till we reach $\beta_{2n-1}, \beta_{2n-2}, \ldots$ are these constants actually zero, and the continued fraction only terminates after $2n-1$ convergents. Hence if the series has n terms, the New Type of continued fraction terminates after $2n-1$ convergents, though the effect after the n^{th} of zero β-terms is already felt in the $(n+1)^{\text{th}}$ convergent.

The continued fraction terminating after $2n-1$ convergents will exactly reproduce the series. In taking n convergents of the continued fraction, exactly n terms of the series are reproduced, together with an approximation for the further terms, this approximation depending on the form of the terms of the series, previous to n.

In the Equivalent Type, the n^{th} convergent exactly reproduces n terms and no approximation to further terms (supposing them to exist) is obtained. If the series has only n terms, n terms of this type of C.F. exactly reproduce the series.

In terminating series, a point is thus ultimately reached after which the Equivalent Type converges more rapidly to the true value than the New Type does. Such a point may, however, be so far off that less convergents of the New Type may be necessary in order to obtain the required degree of accuracy.

In the foregoing, it has been assumed that u and v are integral. If v be not integral the expansion of $I_x(u, v)$ (by integration by parts, raising u and lowering v) ultimately diverges for $x > \frac{1}{2}$, if continued *ad infinitum*.

Another expansion, "raising u," may be obtained,

$$I_x(u, v) = \frac{\Gamma(u+v)}{\Gamma(u+1)\Gamma(v)} x^u y^v \left\{1 + \frac{u+v}{u+1} x + \frac{(u+v)(u+v+1)}{(u+1)(u+2)} x^2 + \ldots\right\},$$

which converges for all values of x in the range 0 to 1.

Suppose after a certain number of reductions by "Parts" we are left with $I_x(u+s, v-s)$, before negative indices start entering. We are summing along the decreasing direction of the expansion, hence

$$\frac{v-s}{u+s}\frac{x}{1-x} < 1, \text{ or } \frac{u+s}{u+v} > x, \text{ or } \frac{u+v}{u+s} x < 1.$$

Put $u+s = u'-1$, $v-s = v'+1$. By "raising" u' we obtain

$$\frac{\Gamma(u'+v')}{\Gamma(u')\Gamma(v'+1)} x^{u'-1} y^{v'+1} \left[1 + \frac{u'+v'}{u'} x + \frac{(u'+v')(u'+v'+1)}{u'(u'+1)} x^2 + \ldots \infty\right]$$
$$= C \,.\, S' \text{ (say)},$$

$$\frac{u+v}{u+s} x < 1, \text{ i.e. } \frac{u'+v'}{u'-1} x < 1, \text{ or } \frac{u'+v'}{u'} x < 1.$$

Also $$\frac{u'+v'+w}{u'+w} = 1 + \frac{v'}{u'+w} < 1 + \frac{v'}{u'}, \text{ i.e. } < \frac{u'+v'}{u'};$$

put $$r = \frac{u'+v'}{u'} x.$$

$$\therefore\ C(1+r+r^2+ \ldots \infty) > C.S' > C(1+x+x^2+\ldots\infty),$$

or $$\frac{C}{1-r} > C.S' > \frac{C}{1-x},$$

or $I_x(u+s, v-s)$ lies between $\frac{C}{1-r}$ and $\frac{C}{1-x}$ in value. It is simple to estimate from tables of $\log \Gamma(x)$ and of ordinary logarithms whether $I_x(u+s, v-s)$ may be negligible or not, within the accuracy desired for $I_x(u, v)$.

Hence if v be not an integer, it may be advisable to estimate $I_x(u+s, v-s)$, $v-s>0$, if $v=s+$ a fraction.

It will be found in most cases that if $u, v > 20$, $I_x(u+s, v-s)$ will prove to be negligible, with the proviso that we do not integrate through the mode of the distribution.

If $I_x(u+s, v-s)$ be not negligible, it is not advisable to use the New Type as the remainder has not been found.

A concluding remark may be made here, the two expansions for evaluating the Incomplete B-function due to Wishart*, while good in the neighbourhood of the mode or for the tails, do not give very satisfactory results for the range 1·5 to 3 times the standard deviation from the mode, and it is for this range that the new continued fraction appears to give good results with fairly low convergents.

* *Biometrika*, Vol. XIX. p. 29, Formula (27) for areas near mode; p. 28, Formula (25) for areas near the tail.

FURTHER NOTES ON THE χ^2 DISTRIBUTION

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In a previous paper* we have discussed certain aspects of the (P, χ^2) Tests for Goodness of Fit; the following notes form an addition to that paper. They fall under three heads:

(1) Use of the previous sampling results to throw light on the way in which the distribution of χ_1^2 is modified when certain of the groups used in the process of fitting a theoretical distribution to the observations are combined together in calculating χ_1^2.

(2) An experimental examination of the adequacy of the χ^2 integral in a case of very small samples.

(3) The correction of an error introduced into the earlier paper in the section dealing with the comparison of two samples.

In the notation previously used it is supposed that a sample of N is classed into k groups containing frequencies $n_1, n_2, \ldots n_k$, while $m_1, m_2, \ldots m_k$ are a series of frequencies following the law

$$m_s = Nf(s;\, a_1, a_2, \ldots a_c) \qquad (s = 1, 2, \ldots k) \quad \ldots\ldots(1),$$

whose values depend upon the c constants $a_1, a_2, \ldots a_c$; these are determined by fitting (1) to the sample observations. Then

$$\chi_1^2 = \mathop{S}_{s=1}^{k} \frac{(n_s - m_s)^2}{m_s} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

If the method of fitting be such as to make χ_1^2 a minimum, then it may be shown that upon certain conditions the sampling distribution of this quantity will follow approximately the law

$$\phi(\chi_1^2)\, d\chi_1^2 = \text{constant}\, (\chi_1^2)^{\frac{k-c-3}{2}} e^{-\frac{1}{2}\chi_1^2}\, d\chi_1^2 \quad \ldots\ldots\ldots(3).$$

Two of these conditions are as follows:

(*a*) That none of the expected frequencies m_s shall be too small.

(*b*) That the number of groups used in the process of fitting shall be the same as those used in calculating χ_1^2.

* *Biometrika*, Vol. xx^{A}. pp. 263—294, "On the Use and Interpretation of Certain Test Criteria for purposes of Statistical Inference, Part II." Dr W. F. Sheppard in a paper of about the same date published in the *Philosophical Transactions of the Royal Society*, Vol. 228 A, pp. 115—150 has also discussed very fully the validity of the law of equation (3), and the assumptions upon which it is based.

It often happens that some of the frequency groups contain very few observations—as in the tails of many curves—yet for convenience in practice we use the full number of observed groups in the fitting. For simplicity this is almost essential when using the method of moments. We are therefore placed in a dilemma. It is true that if we regard our problem as that of testing the hypothesis that the observed sample has arisen in random sampling from a population whose group proportions are actually the values $p_s = m_s/N$ determined by fitting, then no error is involved in calculating χ^2 from a reduced number of groups, say k', and entering the (P, χ^2) Tables with $n' = k'$. But if we look at the problem from the point of view of testing the adequacy of the law of equation (1) then we must decide the following point. Is less error involved by taking χ_1^2 from the full number of groups of which some are very small (neglecting condition (*a*)), and referring it to the distribution (3), i.e. entering Elderton's Tables with $n' = k - c$; or by taking a χ_1^2 from a smaller number of clubbed groups, k', of which none is too small, and entering the Tables with $n' = k' - c$ (neglecting condition (*b*))? We do not propose to enter into the general theoretical problem, but believe as is so often the case that a discussion of some experimental results will throw light on the issues involved.

(1) *The Effect of combining Groups.*

In section (4) of our previous paper a sampling experiment was described in which the population law followed a cubic ($c = 3$), and the area under the curve was broken into 8 groups ($k = 8$). Random samples of 200 were drawn, the expected frequencies being

Group	1	2	3	4	5	6	7	8	Total
	10·4	12·8	17·0	22·2	27·8	33·0	37·2	39·6	200·0

A cubic was fitted by the method of moments to each sample, the frequencies $m_1, m_2, \ldots m_8$ obtained, and the resulting distribution of χ_1^2 found. Within the errors of sampling this agreed satisfactorily with the distribution of equation (3), putting $k = 8$, $c = 3$. Suppose now that we use the same values of m obtained by fitting to 8 groups, but in calculating χ_1^2 club the groups together as follows:

Case (*a*); $k' = 7$, groups, $m_1 + m_2, m_3, m_4, m_5, m_6, m_7, m_8$.

Case (*b*); $k' = 6$, groups, $m_1 + m_2, m_3 + m_4, m_5, m_6, m_7, m_8$.

Case (*c*); $k' = 5$, groups, $m_1 + m_2, m_3 + m_4, m_5 + m_6, m_7, m_8$.

The resulting χ_1^2 distributions are shown in Table I, together with the theoretical distributions which would hold if we might use equation (3) with $k = k'$, $c = 3$. First consider the mean values of χ_1^2; if the theory were adequate we should have

Case (*a*). Mean $\chi_1^2 = k' - c - 1 = 3·000$; standard error for 208 samples $= 0·170$.
Observed mean $= 3·289$.

Case (*b*). Mean $\chi_1^2 = k' - c - 1 = 2·000$ „ „ „ „ „ $= 0·139$.
Observed mean $= 2·311$.

Case (*c*). Mean $\chi_1^2 = k' - c - 1 = 1·000$ „ „ „ „ „ $= 0·098$.
Observed mean $= 1·471$.

The observed values are significantly and increasingly too great. If we examine Table I it is seen that this shows itself most clearly in a shortage of very small values of χ_1^2. At the other end of the distributions where a knowledge of the form of the curve is the more important in practical testing, the disagreement is not so marked. If we level up the most serious discrepancy by combining the first two groups and test for goodness of fit with the groups indicated by the bracketings, the values of P given at the bottom of the Table are obtained.

TABLE I. *Sampling Experiment; Effect of combining certain Groups.*

	7 Groups				6 Groups				5 Groups			
χ_1^2	Observation		Theory		Observation		Theory		Observation		Theory	
0–1	19	77	41·3	88·9	51	119·5	81·8	131·4	105·5	163	142·0	175·3
1–2	58		47·6		68·5		49·6		57·5		33·3	
2–3	38		37·6		36·5		30·1		22		15·4	
3–4	30		27·1		23		18·3		10		7·8	
4–5	21		18·7		7		11·1		4	13	4·2	9·5
5–6	16		12·5		5	13	6·7	10·8	2		2·3	
6–7	8	15	8·3	13·7	8		4·1		2		1·3	
7–8	7		5·4		5	9	2·5	6·3	5		1·7*	
8–9	5	11	3·5	9·5	1		1·5					
9–10	3		2·2		3		2·3*					
10–11	2		1·4									
11–12	0		·9									
12–13	1		1·5*									
Goodness of Fit χ^2	3·62				6·77				5·60			
n'	7				6				4			
P	·740				·241				·136			

These discrepancies are of course of the type we should expect to meet. An essential condition for entering the χ^2 Tables with $n' = k - c$ is that the χ_1^2 used shall be approximately the minimum value that can be obtained in fitting a distribution of the form of equation (1) to the observed frequencies classed into k groups. The observed values of χ_1^2 shown in Table I may now differ considerably from minimum values owing to our reducing the number of groups *after* the process of fitting. It is clear that some danger is involved in this procedure.

In the paper referred to above Sheppard has specially mentioned this point†, and suggested that the value of the constants found from the original tabulation may be used as first approximations in obtaining the values corresponding to the reduced grouping system. This is the ideal procedure, but it can rarely be followed in the course of ordinary work where a rough appreciation of the adequacy of the fit is all that is required. It should in fact be remembered that the groups we have combined in the illustration contain a large portion of the total frequency; in practice it is only the small tail groups of a fitted frequency curve that are usually

* These frequencies correspond to the remaining tail area of the theoretical curves.

† *Loc. cit.* p. 144.

clubbed together, and the χ_1^2 calculated from the resulting k' groups may still remain nearly the minimum χ_1^2 for the new system of grouping. If so, no serious error is involved in entering the Tables for Goodness of Fit with this χ_1^2 and $n' = k' - c$. In any case we may be certain on one point—that if this process shows a reasonable fit we may be content, since the true minimum χ_1^2 would show a better fit still.

(2) *The Case of very small Samples.*

The manner in which the χ^2 integral fails when the group frequencies become very small is a problem not yet fully explored. Each worker has no doubt his own lower limit—10, 8, 5 ?—for the size he will allow an expected frequency group, but he is probably not very clear why he has chosen that limit or what errors will be involved if it be exceeded. The following simple example is perhaps therefore of interest.

Suppose that repeated samples of 10 be taken from a population divided into 3 groups in proportions $p_1 = 0{\cdot}2$, $p_2 = 0{\cdot}5$, $p_3 = 0{\cdot}3$. The expected frequencies will be $\tilde{m}_1 = 2$, $\tilde{m}_2 = 5$, $\tilde{m}_3 = 3$. There will be 66 possible types of sample n_1, n_2, n_3, and for each of these it is easy to calculate

(*a*) The chance of occurrence, or

$$C = \frac{N!}{n_1!\, n_2!\, n_3!}(p_1)^{n_1}(p_2)^{n_2}(p_3)^{n_3} \quad \text{..........................(4)}.$$

(*b*) The value of χ^2, or

$$\chi^2 = \frac{(n_1 - \tilde{m}_1)^2}{\tilde{m}_1} + \frac{(n_2 - \tilde{m}_2)^2}{\tilde{m}_2} + \frac{(n_3 - \tilde{m}_3)^2}{\tilde{m}_3} \quad \text{..................(5)}.$$

(*c*) The likelihood as previously defined, or

$$\lambda = \left(\frac{\tilde{m}_1}{n_1}\right)^{n_1}\left(\frac{\tilde{m}_2}{n_2}\right)^{n_2}\left(\frac{\tilde{m}_3}{n_3}\right)^{n_3} \quad \text{..........................(6)}.$$

These possible samples may be represented by 66 discrete points in a two-dimensioned space, with each of which is associated a value of C. When dealing with large samples these points increase in number and become so closely packed, that we can represent them and their associated C's by a continuous density field,

$$D = D_0 e^{-\frac{1}{2}\chi^2} \quad \text{....................................(7)}.$$

In this field the contours of χ^2 correspond closely to the levels both of constant C and of constant λ. But we may ask how far in the case of samples of 10 is the χ^2 integral of any value? The position is indicated in Table II. The 66 types of sample have been arranged in descending order of C, and for each we give:

(*a*) P_c, the sum of the values of C *lower* than that associated with the sample, or the chance of drawing a less probable sample.

(*b*) P_{χ^2}, which for three groups is $e^{-\frac{1}{2}\chi^2}$, or the value of the χ^2 probability integral that would ordinarily be obtained from the Tables in the case of larger samples.

TABLE II. *Measures of Probability in very small Samples.*

n_1	n_2	n_3	P_e	P_{χ^2}	P_λ	n_1	n_2	n_3	P_e	P_{χ^2}	P_λ
3	5	2	·915	1·000	·915	7	2	1	·027	·022	·053
2	6	2	⎧·844	·766	·724	0	9	1	·023	·035	·006
3	6	1	⎩·844	·705	·653	7	3	0	·020	·017	·012
4	4	2	⎧·709	·766	·795	0	5	5	·018	·024	·010
4	5	1	⎩·709	·659	·532	6	1	3	⎧·015	·035 ×	·034
3	4	3	⎧·589	·705	·858	5	1	4	⎩·015	·038 ×	·043
2	5	3	⎩·589	·659	·596	1	3	6	·011	·006	·025
2	7	1	·482	·442	·448	2	2	6	·0090	·0066	·0266
5	4	1	·443	·362	·409	4	1	5	·0073	·0180 ×	·0229
4	3	3	·409	·442	·498	7	1	2	·0058	·0140 ×	·0155
1	7	2	·376	·344	·262	0	10	0	·0048	·0067	·0013
1	6	3	·344	·362	·319	0	4	6	·0039	·0037	·0036
5	3	2	·313	·344	·379	8	2	0	·0032	·0023	·0025
2	4	4	·285	·282	·350	3	1	6	·0025	·0037	·0050
4	6	0	·258	·282	·149	8	1	1	·0019	·0024	·0044
3	7	0	·233	·247	·114	1	2	7	·0015	·0004	·0033
3	3	4	·211	·247	·296	6	0	4	·0013	·0067	·0011
1	8	1	·189	·163	·195	7	0	3	·0011	·0044	·0007
5	5	0	·170	·189	·095	2	1	7	·0008 8	·0003 3	·0023 1
1	5	4	·151	·189	·244	5	0	5	·0006 8	·0044 4	·0008 9
2	8	0	·136	·127	·067	0	3	7	·0004 9	·0002 9	·0004 9
6	3	1	·120	·116	·228	8	0	2	·0003 7	·0012 7	·0002 6
5	2	3	·108	·163 ×	·216	4	0	6	·0002 6	·0012 7	·0003 8
4	2	4	·098	·127	·185	9	1	0	·0001 6	·0001 8	·0001 6
6	4	0	·088	·074	·057	3	0	7	·0001 2	·0001 6	·0000 9
6	2	2	·079	·091	·140	9	0	1	·0000 84	·0001 58	·0000 49
2	3	5	·070	·060	·176	1	1	8	·0000 49	·0000 13	·0001 30
1	4	5	·062	·049	·088	0	2	8	·0000 20	·0000 11	·0000 20
0	7	3	·055	·116 ×	·046	2	0	8	·0000 10	·0000 09	·0000 10
0	8	2	·048	·091 ×	·036	10	0	0	·0000 04	·0000 09	·0000 02
1	9	0	·042	·038	·017	0	1	9	·0000 02	·0000 002	·0000 07
3	2	5	·037	·043	·082	1	0	9	·0000 001	·0000 002	·0000 001
0	6	4	·031	·074 ×	·028	0	0	10	nil	·0000 0000 2	nil

(*c*) P_λ, the chance of obtaining a sample with a value of λ lower than that observed.

It will be seen that P_{χ^2} is on the whole a better approximation to P_e than to P_λ. In the most important region for tests of significance, namely between $P = {\cdot}10$ and ·01, a × indicates the cases of worst agreement between P_e and P_{χ^2}, but throughout the whole range the *order* of the values of the three P's can hardly be said to differ. Whether or no the χ^2 approximation will be considered here as satisfactory depends upon the degree of expectation entertained by the reader and the faith he has already placed in the test when dealing with very small samples, but the present authors must confess themselves pleasantly surprised to find so close an agreement in this rather extreme case.

(8) *The Value of Minimum χ_1^2 in the Case of two Samples.*

It is necessary to correct an error which was introduced into section (6) of our earlier paper. The problem was that of testing the hypothesis that two samples:

the first of size N with group frequencies $n_1, n_2, \ldots n_t$,
the second of size N' " " " $n_1', n_2', \ldots n_t'$,

TABLE III. *Two Sample Test; Comparison of values of χ_1^2 and P.*

Example	*I (a)*		*I (b)*		*I (c)*		*I (d)*		*II (a)*		*II (b)*		*II (c)*	
	n_s	n_s'	n_s	n_s'	n_s	n_s'	n_s	n_s'	n_s	n_s'	n_s	n_s'	n_s	n_s
Frequencies	24	52	24	58	24	64	24	67	10	12	10	15	10	17
	43	78	43	67	43	56	43	50	18	17	18	24	18	25
	33	70	33	75	33	80	33	83	22	21	22	11	22	8
Totals	100	200	100	200	100	200	100	200	50	50	50	50	50	50
χ_1^2 of (13)	·4473		2·6238		6·8473		10·1414		·2334		5·4864		9·3375	
P	·7996		·2693		·0326		·0063		·8899		·0644		·0094	
χ_1^2 of (14)	·4474		2·6257		6·8673		10·1969		·2336		5·5238		9·4877	
P	·7996		·2691		·0323		·0061		·8897		·0632		·0087	

Example	*III (a)*		*III (b)*		*III (c)*	
	n_s	n_s'	n_s	n_s'	n_s	n_s'
Frequencies	12	18	12	21	12	24
	15	16	15	17	15	17
	21	15	21	13	21	11
	46	55	46	59	46	65
	83	76	83	73	83	67
	51	60	51	63	51	65
	22	25	22	26	22	28
	20	12	20	9	20	7
	16	13	16	11	16	9
	14	10	14	8	14	7
Totals	300	300	300	300	300	300
χ_1^2 of (13)	7·2047		14·8771		24·7823	
P	·6159		·0944		·0032	
χ_1^2 of (14)	7·2406		15·0436		25·1712	
P	·6121		·0898		·0028	

have been drawn from the same population, and we considered the deduction of the usual test from the point of view of the method of likelihood. In the notation employed, Ω is the set of all possible pairs of populations with group proportions p_s and p_s' $(s=1, 2, \ldots t)$, while ω is the subset of Ω in which the pairs are identical, or $p_s = p_s' = q_s$ $(s=1, 2, \ldots t)$. Then the likelihood that the samples have come from some particular member of ω is

$$\lambda = \frac{C_g}{C(\Omega_{\max.})} = \prod_{s=1}^{t}\left(\frac{Nq_s}{n_s}\right)^{n_s}\left(\frac{N'q_s}{n_s'}\right)^{n_s'} \quad\ldots\ldots\ldots\ldots(8).$$

The expression (8) takes a maximum value when

$$q_s = (n_s + n_s')/(N + N') = Q_s \quad (s = 1, 2, \ldots t)\ldots\ldots(9).$$

If the sample group frequencies, however, are not too small, then the λ of (8) becomes approximately

$$\lambda = e^{-\frac{1}{2}\chi_1^2} \quad\ldots\ldots\ldots\ldots(10),$$

where

$$\chi_1^2 = \mathop{S}_{s=1}^{t}\left\{\frac{(n_s - Nq_s)^2}{Nq_s} + \frac{(n_s' - N'q_s)^2}{N'q_s}\right\} \quad\ldots\ldots\ldots\ldots(11).$$

The values of q_s which make χ_1^2 a minimum, and therefore the λ of (10) a maximum, are not as we stated by an oversight those of (9), but may be easily shown to be

$$q_s = \left(\sqrt{\frac{n_s^2}{N} + \frac{n_s'^2}{N'}}\right)\Big/\left(\mathop{S}_{s=1}^{t}\sqrt{\frac{n_s^2}{N} + \frac{n_s'^2}{N'}}\right)$$

$$= Q_s\sqrt{1 + \frac{x_s^2}{NN'Q_s^2}}\Big/\left\{\mathop{S}_{s=1}^{t}\left(Q_s\sqrt{1 + \frac{x_s^2}{NN'Q_s^2}}\right)\right\} = Q_s' \quad\ldots\ldots\ldots(12),$$

where

$$n_s = NQ_s + x_s, \quad n_s' = N'Q_s - x_s.$$

These values lead to

$$\text{Minimum } \chi_1^2 = \left\{\mathop{S}_{s=1}^{t}\sqrt{\frac{n_s^2}{N} + \frac{n_s'^2}{N'}}\right\}^2 - N - N' \quad\ldots\ldots\ldots\ldots(13).$$

If the sample group frequencies be not too small, x_s^2 will be small compared with $NN'Q_s^2$, at any rate for deviations lying within the region of significant frequency. It follows that Q_s (9), and Q_s' (12), will not differ greatly, and the true minimum χ_1^2 of (13) will be almost the same as that ordinarily used in applying this test, namely

$$\chi_1^2 = \mathop{S}_{s=1}^{t}\left\{NN'\left(\frac{n_s}{N} - \frac{n_s'}{N'}\right)^2\Big/(n_s + n_s')\right\} \quad\ldots\ldots\ldots\ldots(14),$$

which is obtained by taking $q_s = Q_s = (n_s + n_s')/(N + N')$, the value really maximising the λ of (8) not that of (10). The difference is of the order of approximation necessarily involved in any use of the χ^2 test. The numerical examples given in Table III, p. 303, illustrate this point; the difference between the two χ_1^2's is greatest when there are many groups, but it is clear that no error of importance would arise by using one value of P rather than the other.

Summary.

There are several ways by which to approach and to interpret the χ^2 Tests for Goodness of Fit; in all cases the use of the final integral can be considered only as an approximation. In our previous paper we discussed the use of the method of likelihood and emphasised the difference between testing a simple and testing a composite hypothesis. In the first case we obtain an answer to the question, "could this sample have come from a certain exactly specified population?" In the second, to a somewhat different question, "could it have come from a population whose distribution follows a law of a certain type depending on several undetermined parameters?" In the latter case the scheme of the test does not allow for the frequency groups being clubbed together *after* the process of fitting has been carried out. We have illustrated the effect of this clubbing on the distribution of χ^2 on the data of our previous sampling experiment. In general it would not be easy to gauge numerically the extent of the error involved, but it will probably not be large if, as is customary, the groups which are combined contain only a small portion of the total frequency.

The point at which the χ^2 distribution becomes inadequate to express the sampling variation when dealing with very small frequency groups has not yet been fully investigated. The result of an examination of the position in the case of a sample of 10 drawn from a population divided into 3 groups, suggests that the approximation is much more satisfactory than might have been expected.

In a final section we have corrected an earlier misstatement, in connection with the test for comparing two samples, and shown in a few numerical examples how the difference between minimum χ^2 and the χ^2 of maximum likelihood is not of real significance—is in fact of the general order of the χ^2 approximations.

TABLE OF THE VALUES OF THE DIFFERENCES OF THE POWERS OF ZERO.

By ETHEL M. ELDERTON, ASSISTED BY MARGARET MOUL.

At the time when the first quarter of this table was originally worked (for a special problem during the War) the authors* of it were unaware of Cayley's paper†. Later being informed of it, they checked the original work by reduction from Cayley's numbers, he tabling $\Delta^m 0^n/\Gamma(m+1)$, while they had tabled

$$\Delta^p 0^{p+s}/\Gamma(p+s+1).$$

Of the 100 entries six were found in error in the twelfth decimal place, two differed by one unit in that place, three differed by two units and one—the last entry in the table—by 5 ‡. Cayley takes in our notation $\Delta^p 0^{p+s}/\Gamma(p+1)$ from $p=1$ to 20 and $p+s$ from 1 to 20. Hence by reduction from Cayley's table E. M. Elderton and M. Moul were able to add half as much again to the original table in *Biometrika*, i.e. to take $p=11$ to 20 as long as $p+s$ did not exceed 20. Then, using Cayley's method, they extended his table and reduced from this extension a complete table for $p=1$ to 20 and $s=1$ to 20. Thus the table now includes all values of

$$\Delta^m 0^n/\Gamma(m+1)$$

from $m=0$ to 20 and $n=0$ to 40, and is probably the most extensive table of the differences of the powers of zero yet published.

Laplace has indicated the importance of the differences of the powers of zero in the theory of probability for problems allied to that of De Moivre, and further illustrations will be given in the forthcoming Part II of the *Book of Tables for Statisticians and Biometricians.*

* K. P. and E. M. E., *Biometrika*, Vol. XVII. p. 200.

† *Trans. Camb. Phil. Soc.* Vol. XIII. Part I (1881), pp. 1—4.

‡ 8·826,886,089,207 instead of the correct value 8·826,886,089,212.

Values of the Differences of the Powers of Zero.

$$\textit{Table of } q(p, s) = \frac{\Delta^p 0^{p+s}}{\Gamma(p+s+1)} \textit{ from } p = 1 \textit{ to } 20 \textit{ and } s = 0 \textit{ to } 20.$$

Values of p.

Values of s.

s	1	2	3	4	5
0	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000
1	·500,000,000,000	1·000,000,000,000	1·500,000,000,000	2·000,000,000,000	2·500,000,000,000
2	·166,666,666,667	·583,333,333,333	1·250,000,000,000	2·166,666,666,667	3·333,333,333,333
3	·041,666,666,667	·250,000,000,000	·750,000,000,000	1·666,666,666,667	3·125,000,000,000
4	·008,333,333,333	·086,111,111,111	·358,333,333,333	1·012,500,000,000	2·298,611,111,111
5	·001,388,888,889	·025,000,000,000	·143,750,000,000	·513,888,888,889	1·406,250,000,000
6	·000,198,412,698	·006,299,603,175	·050,016,534,392	·225,562,169,312	·741,732,804,233
7	·000,024 801,587	·001;405,423,280	·015,426,587,302	·087,632,275,132	·345,568,783,069
8	·000,002,755,732	·000,281,635,802	·004,284,060,847	·030,638,778,660	·144,695,216,049
9	·000,000,275,573	·000,051,256,614	·001,083,829,365	·009,760,802,469	·055,162,863,757
10	·000,000,025,052	·000,008,546,944	·000,252,086,841	·002,860,825,517	·019,341,229,758

s	6	7	8	9	10
0	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000
1	3·000,000,000,000	3·500,000,000,000	4·000,000,000,000	4·500,000,000,000	5·000,000,000,000
2	4·750,000,000,000	6·416,666,666,667	8·333,333,333,333	10·500,000,000,000	12·916,666,666,667
3	5·250,000,000,000	8·166,666,666,667	12·000,000,000,000	16·875,000,000,000	22·916,666,666,667
4	4·529,166,666,667	8·079,166,666,667	13·386,111,111,111	20·950,000,000,000	31·333,333,333,333
5	3·237,500,000,000	6·601,388,888,889	12·300,000,000,000	21·375,000,000,000	35·138,888,888,889
6	1·989,616,402,116	4·625,925,925,926	9·671,957,671,958	18·628,174,603,175−	33·604,414,682,540
7	1·077,777,777,778	2·851,851,851,852	6·679,365,079,365+	14·235,491,071,429	28·141,121,031,746
8	·523,916,997,354	1·575,358,796,296	4·127,361,937,831	9·721,510,416,667	21·034,795,249,118
9	·231,631,944,444	·790,558,449,074	2·314,315,476,190	6·017,912,946,429	14·238,267,471,340
10	·094,114,940,326	·364,277,277,988	1·190,485,668,524	3·414,504,607,083	8·826,386,039,212 *

s	1	2	3	4	5
11	·000,000,002,088	·000,001,315,236	·000,054,300,445−	·000,777,366,923	·006,287,061,463
12	·000,000,000,161	·000,000,187,914	·000,010,897,672	·000,197,066,149	·001,907,096,356
13	·000,000,000,011	·000,000,025,057	·000,002,048,012	·000,046,850,391	·000,542,762,985+
14	·000,000,000,001	·000,000,003,132	·000,000,361,967	·000,010,491,635−	·000,145,593,321
15	·000,000,000,000	·000,000,000,368	·000,000,060,389	·000,002,221,479	·000,036,953,700
16	·000,000,000,000	·000,000,000,041	·000,000,009,542	·000,000,446,204	·000,008,904,739
17	·000,000,000,000	·000,000,000,004	·000,000,001,432	·000,000,085,264	·000,002,043,183
18	·000,000,000,000	·000,000,000,000	·000,000,000,205−	·000,000,015,540	·000,000,447,548
19	·000,000,000,000	·000,000,000,000	·000,000,000,028	·000,000,002,707	·000,000,093,803
20	·000,000,000,000	·000,000,000,000	·000,000,000,000	·000,000,000,452	·000,000,018,851

s	6	7	8	9	10
11	·035,436,000,631	·155,444,052,796	·566,707,251,082	1·791,545,336,174	5·056,157,797,803
12	·012,447,698,996	·061,854,855,924	·251,424,842,802	·875,558,648,133	2·696,234,748,153
13	·004,102,251,152	·023,084,987,476	·104,575,173,440	·400,963,836,098	1·346,608,080,109
14	·001,274;353,342	·008,119,780,273	·040,979,983,168	·172,934,537,974	·633,142,757,534
15	·000,374,659,155−	·002,702,776,182	·015,194,003,252	·070,548,202,960	·281,476,384,198
16	·000,104,608,335−	·000,854,421,374	·005,349,474;876	·027,323,164,021	·118,769,057,007
17	·000,027,822,135−	·000,257,321,024	·001,794,174,688	·010,079,078,784	·047,721,531,774
18	·000,007,067,421	·000,074,028,765−	·000,574,831,831	·003,551,303,538	·018,311,726,897
19	·000,001,718,694	·000,020,393,546	·000,176,363,075−	·001,198,178,554	·006,727,553,604
20	·000,000,400,972	·000,005,391,171	·000,051,929,785−	·000,387,964,657	·002,371,839,420

* The table thus far was computed by K. P. and E. M. E. from the formula

$$q(p, s) = \frac{p}{p+s}\{q(p, s-1) + q(p-1, s)\}$$

and checked by the formula

$$\Delta^p 0^{p+10} = p^{p+10} - p(p-1)^{p+10} + \frac{p(p-1)}{2!}(p-2)^{p+10} - \ldots + (-1)^r \frac{p!}{(p-r)!\,r!}(p-r)^{p+10} + \ldots$$

It was calculated for a special investigation, but it was thought that it might be of value to other computers, and was accordingly published in *Biometrika*.

Table of the Differences of the Powers of Zero (continued).

Values of *p*.

Values of *s*.

s	*11*	*12*	*13*	*14*	*15*
0	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000
1	5·500,000,000,000	6·000,000,000,000	6·500,000,000,000	7·000,000,000,000	7·500,000,000,000
2	15·583,333,333,333	18·500,000,000,000	21·666,666,666,667	25·083,333,333,333	28·750,000,000,000
3	30·250,000,000,000	39·000,000,000,000	49·291,666,666,667	61·250,000,000,000	75·000,000,000,000
4	45·161,111,111,111	63·120,833,333,333	85·962,500,000,000	114·498,611,111,111	149·604,166,666,667
5	55·206,250,000,000	83·625,000,000,000	122·407,638,888,889	174·562,500,000,000	243·125,000,000,000
6	57·465,724,206,349	93·993,816,137,566	148·064,153,439,153	225·838,657,407,407	334·974,041,005,291
7	52·315,294,312,169	92·405,753,968,254	156·305,439,814,815−	254·762,731,481,481	402·093,253,968,254
8	42·465,841,324,956	80·922,957,175,926	146·856,074,327,601	255·575,349,151,235−	428·914,306,382,275+
9	31·187,259,837,963	64·062,981,150,794	124·633,750,964,506	231·431,626,157,407	412·716,207,837,302
10	20·959,528,792,806	46·375,914,514,691	96·657,637,009,981	191·385,403,514,310	362·460,966,810,967
11	13·007,843,295,304	30·082,830,161,736	69·138,586,384,680	145·893,434,343,434	293·281,385,281,385+
12	7·510,646,020,764	19·246,738,091,260	45·960,368,727,489	103·305,893,061,266	220·326,266,245,918
13	4·059,574,796,242	11·187,030,186,001	28·573,699,456,745+	68·382,011,401,932	154·665,148,739,919
14	2·064,795,723,662	6·116,227,342,921	16·702,557,347,988	42·542,284,374,960	102·003,844,714,593
15	·992,653,584,094	3·159,502,634,229	9·221,670,706,029	24·989,495,556,339	63·496,670,135,466
16	·452,801,816,745−	1·548,130,478,989	4·827,841,910,525+	13·914,757,484,537	37·457,142,396,776
17	·196,634,172,633	·721,971,579,981	2·404,919,179,220	7·370,176,557,826	21·012,805,759,969
18	·081,531,203,270	·321,401,113,301	1·143,295,606,541	3·724,044,071,910	11·244,295,378,127
19	·032,361,544,187	·130,940,383,544	·520,095,870,972	1·800,798,763,647	5·755,188,591,959
20	·012,324,749,022	·055,974,424,712	·226,936,783,148	·834,949,931,033	2·824,345,081,282

s	*16*	*17*	*18*	*19*	*20*
0	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000
1	8·000,000,000,000	8·500,000,000,000	9·000,000,000,000	9·500,000,000,000	10·000,000,000,000
2	32·666,666,666,667	36·833,333,333,333	41·250,000,000,000	45·916,666,666,667	50·833,333,333,333
3	90·666,666,666,667	108·375,000,000,000	128·250,000,000,000	150·416,666,666,667	175·000,000,000,000
4	192·216,666,666,667	243·336,111,111,111	304·025,000,000,000	375·408,333,333,333	458·673,611,111,111
5	331·688,888,888,889	444·337,500,000,000	585·675,000,000,000	760·857,638,888,889	975·625,000,000,000
6	484·845,767,195,767	686·787,632,275,132	954·346,974,206,349	1303·555,505,952,381	1753·215,773,809,524
7	617·001,058,201,058	923·516,989,087,302	1352·062,053,571,429	1940·643,601,190,476	2736·192,129,629,630
8	697·276,909,722,222	1102·130,951,190,476	1699·062,857,142,857	2561·274,915,123,457	3783·905,031,966,490
9	710·395,595,238,095+	1185·119,330,357,143	1922·788,125,000,000	3042·757,062,940,917	4708·042,824,074,074
10	660·219,422,799,423	1161·879,955,691,171	1983·000,009,015,763	3292·737,981,626,784	5333·853,870,467,239
11	565·037,515,899,738	1048·485,607,751,623	1881·612,320,752,165−	3277·088,524,840,001	5555·446,706,649,832
12	448·779,304,083,232	877·707,017,282,501	1655·591,602,820,789	3023·255,562,114,684	5361·688,917,977,822
13	332·934,870,523,118	680·030,403,089,851	1359·651,487,302,958	2602·351,060,591,725+	4826·690,898,102,756
14	231·967,314,793,446	503·418,103,355,350	1047·976,644,745,302	2101·703,830,345,561	4075·526,309,675,481
15	152·497,540,608,471	348·455,185,855,783	761·690,089,418,773	1600·131,806,338,893	3243·233,260,579,840
16	94·977,341,502,623	228·434,938,330,140	524·183,838,223,194	1153·190,970,190,847	2442·462,906,983,605−
17	56·237,647,157,621	142·336,292,746,885−	342·781,781,641,755−	789·545,924,578,318	1747·031,800,303,742
18	31·756,208,252,117	84·559,214,770,943	213·670,498,206,349	515·165,190,078,613	1190·629,994,938,081
19	17·148,067,128,720	48·028,438,674,841	127·312,090,320,579	321·239,093,109,596	775·317,481,096,245−
20	8·876,627,648,890	26·145,571,013,606	72·690,900,316,193	191·914,612,225,641	483·616,046,660,943

ON THE RELATION OF THE DURATION OF PREGNANCY TO SIZE OF LITTER AND OTHER CHARACTERS IN BITCHES*.

By MARGARET AND KARL PEARSON.

(1) (i) THE following data relate to the duration of pregnancy, the age of the bitch, the size of litter, the order of the pregnancy, etc. in small dogs bred in the Biometric Laboratory, partly pure Pekinese and partly hybrids from the cross Pekinese × Pomeranian. The material is more sparse than we had hoped for, since about half the whole series of dogs were bred by Dr Usher in Scotland, and we found on examining the Scottish schedules that most of the dates of mating had not been entered, only the dates of littering; the mating books themselves had disappeared during Dr Usher's absence on war-service in Greece. It was therefore only possible to use for this enquiry the data for dogs bred in England.

The data must necessarily be of an approximate nature, because (i) if a bitch be lined only once there is less chance of obtaining a litter than if she be served twice, and in our experimental work the chief aim is to obtain a litter. The cost of keeping dogs which fail to litter, and there are many slips, is already too heavy for a poor institution. And (ii) the date of littering is that of the day when the bitch was found to have puppies. With our Pekinese and Pekinese hybrids we have noted a marked objection to littering in the presence of anyone; they cry and whine and will not attend to business†. The bitches as a rule litter during the night, most probably in the early morning, for this is the time the attendant usually finds that the bitch has just littered but has not yet cleaned up, or is just littering. Accordingly the date of littering is in this paper taken to be the day on which she is known to have littered, or the day on which she is found with puppies, although these might in some cases have been born actually twelve hours earlier.

As to the date of mating, when it has occurred twice, for the most part the mid-date between the two matings has been taken. By duration of pregnancy we understand accordingly here the time which has elapsed between this mid-date and the day on which the bitch is known to have littered, or has been found to have puppies. This is not of course the period of true pregnancy, for we do not know the time at which the spermatozoon comes into conjunction with the ovum, nor to a few hours the time of the littering, indeed the latter sometimes lasts several hours. But it is as close as we can get by aid of experimental work, not intended solely for the present investigation, and it is close enough to give results of value for practical breeding.

* Acknowledgment must be made of assistance from the Royal Society Government Grant received on several occasions during the course of these experiments.

† I have sat up in my early days hours with a bitch, but to no purpose. Half-an-hour after I had left her in despair, she would have her pups without more fuss! K. P.

The purpose of the experiments being to study hybridisation we had relatively few pure-bred dogs, and rarely bred Pekinese with Pekinese, or Pomeranian with Pomeranian. The hybrid was termed a "Pompek," and for shortness we may speak of Poms and Peks. It is possible that Peks in pure breeding and Poms in pure breeding have different durations of pregnancy, but our experiments are not adequate to determine a slight difference if it exists.

Pure Pom bitches had an average duration of 60·1 days of pregnancy, whatever the sire. When mated with Pom sires 60·2 days' duration. When mated with Pek sires 60·0 days. When mated with Pompeks 61·0 days. These are all on relatively few numbers, and the results do not suggest that the period of pregnancy of a pure Pom bitch is influenced by the race of the sire.

Turning to Peks the pure Pek bitch, whatever the sire, had an average duration of 61·4 days. When mated with a pure Pek the duration was 62·6 days, with a pure Pom 61·3, and with a Pompek 59·9 days. It may be asked why the sire should affect the period of pregnancy of the bitch? The answer is that the period of pregnancy is influenced by the nature of the litter, e.g. the larger the litter (and the heavier probably) the shorter is the pregnancy. It may be that the number of the litter depends entirely on the bitch, but it is not impossible that it depends in part also on the sire*. Hence it by no means follows that the duration of pregnancy will be the same with a cross and with a pure mating. Our results do not indicate any such relation in the averages for pure Pom bitches. More might be read into the case of the pure Pek bitches, but when we see that the duration of pregnancy can vary from 55 to 68 days, we are not inclined to lay any stress on differences such as the above, which have in fact probable errors of the order of one day. We shall see that the mean duration of pregnancy for all available material is 60·76 days, and we are not able on the basis of our material to lay any stress on the difference involved in a Pom 60·4 and a Pek 61·4 days' duration. It seems probable that the age of the bitch, the order of the pregnancy and the size of the litter have more to do with the duration of pregnancy than the race of bitch or dog in these small dogs.

(ii) One grave difficulty in our breeding work has been the lengthy period which in certain cases has elapsed before the bitch showed the smallest sign of heat. In one case 4½ years, and in seven cases three or more years out of a total of 54 first litters for which data were available. It will be observed how very much it adds to the cost of experiments of this nature, if a mating which is desired has to be postponed for two or even more years. As a rule after the first pregnancy the bitch comes into season twice a year, but by no means at fixed intervals; to what extent these are varied by (i) the absence of mating†, (ii) the length of suckling, (iii) the failure to have a litter after mating, or (iv) the age of the bitch, has not been adequately determined. The period of suckling, 4 to 6 weeks, depends largely on the size of the litter and the age of the bitch, but also on the condition at littering

* There is some evidence to indicate that in man twinning may arise from the Father's side.

† In some cases it was thought desirable owing to the youth of the bitch, or her non-recovery after the previous litter to full health and strength, to allow the heat to pass without mating.

and the food she will consent to take*. As a rule, however, there is a season at the end of spring and another at the beginning of winter. The following is a typical example:

Setie: born Oct. 8, 1923. 1st heat, Oct. 1924; 2nd, June 1925; 3rd, Dec. 1925; 4th, July 1926; 5th, March 1927; 6th, Oct. 1927; 7th, June 1928; 8th, Nov. 1928; 9th, July 1929. She was mated on all nine occasions and gave birth to 29 puppies. She was parted with after the 9th litter.

Here is another illustration:

Meg bhan: born May 3, 1913. 1st heat, Feb. 1914; 2nd, Aug. 1914 (no litter); 3rd, March 1915; 4th, April 1916; 5th, Nov. 1916; 6th, June 1917; 7th, Nov. 1918; 8th, Feb. 1920; 9th, Oct. 1921. She had to be destroyed in 1922, having also given birth to 29 puppies. The occurrence of heat is here more irregular, but may reasonably be associated with difficulties as to food during the War.

One last case:

Siri: born July 28, 1915. 1st heat, Aug. 1916 (no litter); 2nd, March 1917; 3rd, Oct. 1917; 4th, May 1918; 5th, Nov. 1919; 6th, May 1920; 7th, June 1921. Total number of puppies born 23.

Irregularities chiefly occur when the bitch is very young or old, but a general discussion of the intervals between heats would require more data, especially with regard to suckling period and food, than our records provide.

(2) We will deal in the first place with the first litter, which usually, but not invariably, corresponds with the first heat and the first mating. We have only

TABLE I. *Age at First Littering and Duration of Pregnancy.*

Age of Bitch at First Littering (Central Values in Months).

Duration of Pregnancy in Days.	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	Totals
56	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1
57	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	2
58	—	3	2	1	1	—	—	—	—	—	—	—	—	—	—	—	7
59	—	1	—	2	—	2	—	—	—	—	—	—	—	—	—	—	5
60	—	—	1	1	1	—	—	—	1	1	1	—	—	—	—	—	6
61	—	—	1	1	1	1	—	—	—	—	—	—	—	—	—	—	4
62	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	2
63	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	2
64	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
65	—	—	3	—	—	—	—	—	—	—	—	—	—	—	—	—	3
66	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
67	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	2
68	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	1
Totals	1	7	9	6	4	5	—	—	1	2	1	—	—	—	—	1	37

* Pekinese and Pekinese hybrids will often both before and after littering refuse cows' milk, or can only be induced to take it, if sponge cake be soaked with it!

TABLE II. *Age at First Littering and Size of Litter.*

Age of Bitch at First Littering (Central Values in Months)

Size of Litter.	*9*	*12*	*15*	*18*	*21*	*24*	*27*	*30*	*33*	*36*	*39*	*42*	*45*	*48*	*51*	*54*	Totals
1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
2	1	2	4	2	2	1	—	—	1	3	1	—	—	—	—	—	17
3	2	—	4	1	2	2	—	—	—	1	—	1	—	—	—	1	14
4	—	4	—	3	—	—	2	—	1	—	—	—	—	—	—	—	10
5	1	2	3	—	1	2	—	—	—	-	—	—	—	—	—	—	9
6	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	1
7	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	2
Totals	4	9	12	7	5	6	2	—	2	4	1	1	—	—	—	1	54

37 cases in which the duration of pregnancy is provided for the age of mother at first litter. We have 54 cases in which the number of puppies is known for age of mother at first litter (see Tables I and II).

The average age of the mother at first litter is 20·33 months* in the first table and 19·95 in the second table. The average duration of first pregnancy is 60·86 days, while the average duration of all pregnancies is 60·76 days. There is nothing very different in the first pregnancy as far as its average duration is concerned from the average duration of later pregnancies.

The number of puppies in the first litter averages 3·37, while the average number for all litters is 3·22. This does not, of course, prove that the first litter is the most numerous, but only that it has somewhat more than the average number of puppies. We shall return to this point later.

The constants of the two tables are as follows:

Table I.	Mean age of Mother at first litter 20·33 months Standard Deviation = 9·156 months Mean Length of Pregnancy 60·86 days Standard Deviation = 3·112 days	Correlation ·143 ± ·109
Table II.	Mean age of Mother at first litter 19·95 months Standard Deviation = 9·567 months Mean number of Puppies 3·37 Standard Deviation = 1·365 puppies	Correlation − ·347 ± ·081

The latter correlation is significant, the former cannot be said to be. The general meaning if both were significant would be that:

The older the bitch at first pregnancy the fewer puppies she will have, and the longer the pregnancy.

* By a "month" in this paper is to be understood an average calendar month of 30·4 days.

That the pregnancy is longer may merely arise from the fact that the litter is smaller, the fertility of the bitch depending upon her age. The following results are suggestive:

Age of Bitch in Months	Mean Length of 1st Pregnancy	Mean Number of Puppies
8—13	60·75	3·38
14—19	60·80	3·57
20—25	60·56	3·55
Above 25	62·40	2·81

The average age at first litter being almost exactly twenty months, and the duration of first pregnancy almost exactly two months, we conclude that in these bitches the first heat occurred on the average at 18 months with a variability of 9·4 months, the distribution of this onset of puberty being very skew.

(3) After the above consideration of the first pregnancy, based admittedly on very slender data, we turn to the general relations between the four variates: Age of Mother (a), Size of Litter (l), Order of Pregnancy (ω), and Duration of Pregnancy (d). Our data are arranged in the six correlation tables, Tables III—VIII, to be found on this and the following pages.

TABLE III. *Order of Pregnancy and Size of Litter.*

Order of Pregnancy.

Size of Litter.	I	II	III	IV	V	VI	VII	VIII	IX	Totals
1	2	4	4	5	2	—	2	—	—	19
2	17	9	6	2	3	2	1	2	—	42
3	14	15	7	5	2	—	2	1	—	46
4	11	·6	10	2	4	3	1	—	—	37
5	9	6	3	3	—	2	—	—	1	24
6	2	1	1	3	1	1	—	—	—	9
7	2	—	—	—	—	—	—	—	—	2
Totals	57	41	31	20	12	8	6	3	1	179

Table III provides the relation between the order of pregnancy and the size of the litter. Matings not followed by pregnancy are omitted; one first pregnancy which was a miscarriage, and one second pregnancy in which the total number of puppies born was not recorded, have been disregarded. Table III contains 179 pregnancies leading to 577 puppies. The following are the constants of the table:

$\bar{l}$ = Mean No. of Puppies $= 3{\cdot}22 \pm {\cdot}071$,

σ_l = Standard Deviation of No. of Puppies $= 1{\cdot}4084 \pm {\cdot}050$,

$\bar{\omega}$ = Mean No. of Pregnancies $= 2{\cdot}77 \pm {\cdot}093$,

σ_ω = Standard Deviation of No. of Pregnancies $= 1{\cdot}8402 \pm {\cdot}066$,

$r_{l\omega}$ = Correlation of Order of Pregnancy and Size of Litter $= -{\cdot}0509 \pm {\cdot}0503$.

Accordingly if we could trust to the regression being linear, there would appear to be no significant relation between the order of the pregnancy and the size of the litter. We must accordingly investigate the means of each array. We have:

Order of Pregnancy	I	II	III	IV	V	VI—IX	Mean all Pregnancies
Size of Litter ...	3·39	3·10	3·16	3·25	3·00	3·22	3·22

Although this appears to make the litter at first pregnancy the largest, the actual value in this case is 3·387 ± ·126, which does not indicate any significant difference from the general mean 3·223.

This result appears to contradict the ordinary impression, which we ourselves have shared, that the litters of the first and of the last one or two pregnancies are smaller than the average. The source of this apparent paradox may lie in the fact that we are not dealing with nine successive pregnancies of the same bitches; we are clubbing bitches of varied degrees of fertility together, and the horizontal margin shows that many bitches drop out after the first two or three matings. It was only those of the greater experimental interest that could be preserved to the last stages of their reproductive powers, and this was peculiarly the case during the War years, when, owing to the scarcity and cost of food, the sole aim was to keep enough dogs alive to continue the work when peace came*.

We pass next to the duration of the pregnancy and the size of the litter. We have already drawn attention to the fact that the dates of mating have not always been recorded. Further, there were not always two matings, and if there were, they might be on successive days, or there might be an intervening day. We have the following results according as we measure the pregnancy from the day of the first mating, from the day of the second mating or from the midday between:

	1st Mating	Midday	2nd Mating
Mean Duration of Pregnancy	61·41	60·76	59·74
Standard Deviation ...	2·0846	3·1825	3·1969

all in days.

It is clear that there was an interval of about 1·7 days between the two matings. The midday is not midway between the first and second matings, because the first mating includes all those cases in which there was only a single mating. Examining the standard deviations, it will be seen that the duration of pregnancy varies least about the mean duration from first mating to littering. It seems probable therefore that in most cases the first mating is successful. For practical purposes therefore we may say that a bitch of these breeds will litter after an interval of 61·41 ± 2·14 days from first mating, or that a bitch is very unlikely to have a litter at all if it

* Even at present the size of our Animal House and the extent of our funds do not permit of more than 15 to 20 adult dogs being kept at one time.

does not occur between the 55th and 68th* days from first mating, the 61st to 62nd days being the most probable days for littering. If there has been a double mating then the bitch will litter most probably $60{\cdot}76 \pm 2{\cdot}15$ days from the midday between the two matings. If a bitch does not litter between the 54th and 67th days from the midday of mating, she is very unlikely to have a litter†. Our actual experience has been one bitch littering on the 55th day and two on the 68th day.

TABLE IV. *Duration of Pregnancy and Size of Litter.*

Duration of Pregnancy in Days.

Size of Litter.	*55*	*56*	*57*	*58*	*59*	*60*	*61*	*62*	*63*	*64*	*65*	*66*	*67*	*68*	Totals
1	—	—	—	1	—	1	1	—	—	3	—	—	2	—	8
2	—	—	—	2	1	2	3	4	2	1	4	4	2	—	25
3	—	—	2	2	2	8	4	3	3	3	4	—	2	2	35
4	—	1	2	5	5	1	1	1	1	1	1	—	—	—	19
5	—	1	—	2	4	3	—	2	1	1	1	—	—	—	15
6	1	—	1	—	2	2	—	—	—	—	1	—	—	—	7
7	—	—	—	1	—	—	—	—	—	—	—	—	—	—	1
Totals	1	2	5	13	14	17	9	10	7	9	11	4	6	2	110

In Table IV the length of pregnancy is measured from the first mating and the constants of the table are as follows:

$\bar{l}$ = Mean Size of Litter = $3{\cdot}300$,

σ_l = Standard Deviation = $1{\cdot}3655$,

$\bar{d}$ = Mean Duration of Pregnancy = $61{\cdot}409$,

σ_d = Standard Deviation = $3{\cdot}0846$,

r_{ld} = Correlation of Size of Litter and Duration of Pregnancy = $-{\cdot}4479 \pm {\cdot}0514$.

There is thus a significant and quite considerable negative correlation between size of litter and duration of pregnancy. This correlation may be illustrated by the following mean values:

Size of Litter	Mean Duration of Pregnancy in days
1	63·13
2	62·96
3	61·97
4	59·47
5	60·27
6	59·29
7	58·00

* This is based on plus and minus three times the probable error from the mean.

† In some cases a bitch after mating makes up her mind that she will litter, she develops, and sometimes shows signs of milk, and finally may even prepare her lair, without having any puppies.

The corresponding prediction formulae are:

d_l = probable duration of pregnancy for given size of litter $l = 64{\cdot}75 - 1{\cdot}012\, l$,

or pregnancy is delayed about one day for each decrease of one in the litter.

l_d = probable litter for given duration of pregnancy $d = 15{\cdot}48 - {\cdot}198\, d$,

or five days' delay in littering would on the average denote a reduction of two in the litter.

We now turn to Table V, which gives the relation between the duration and order of pregnancy.

TABLE V. *Duration of Pregnancy and Order of Pregnancy.*

Duration of Pregnancy in Days.

Order of Pregnancy.	*55*	*56*	*57*	*58*	*59*	*60*	*61*	*62*	*63*	*64*	*65*	*66*	*67*	*68*	Totals
I	—	1	2	7	5	6	4	2	2	1	3	1	2	1	37
II	—	1	2	3	2	3	—	6	2	3	4	2	1	—	29
III	—	—	—	2	2	4	3	1	—	2	1	1	1	—	17
IV	—	—	—	1	3	1	1	1	—	1	1	—	1	—	10
V	1	—	1	—	1	2	—	—	—	1	—	—	—	—	6
VI	—	—	—	—	1	1	1	—	1	—	—	—	—	—	4
VII	—	—	—	—	—	—	—	—	1	1	2	—	1	—	5
VIII	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
IX	—	—	—	—	—	—	—	—	1	—	—	—	—	—	1
Totals	1	2	5	13	14	17	9	10	7	9	11	4	6	2	110

The constants of the table are as follows:

$\bar{d}$ = Mean Duration of Pregnancy $= 61{\cdot}409 \pm {\cdot}198$,
σ_d = Standard Deviation of $d = 3{\cdot}0849 \pm {\cdot}1403$,
$\bar{\omega}$ = Mean Order of Pregnancy $= 2{\cdot}6546 \pm {\cdot}1187$,
σ_ω = Standard Deviation of $\omega = 1{\cdot}8461 \pm {\cdot}0839$,
$r_{d\omega}$ = Correlation of Duration with Order of Pregnancy $= {\cdot}1780 \pm {\cdot}0623$.

There is thus a positive correlation between the order of pregnancy and its duration; it is rather small but is probably significant. As the regression is unlikely to be linear we determined the correlation ratio of duration of pregnancy on order of pregnancy and found

$$\eta_{d.\omega} = {\cdot}3785,$$

indicating an association more than double that determined for the correlation coefficient.

Our data are too scant to give a close approximation to the manner in which the duration changes with the order of pregnancy, but the following series of mean durations:

Order of Pregnancy	Mean Duration
I	60·86
II	61·76
III	61·47
IV	61·40
V	59·17
VI	60·75
VII—IX	64·71
All Pregnancies	61·41

suggest that the first pregnancy has a duration rather below the mean; the duration rises above the mean in the second and third pregnancies, sinks below the mean again in the fourth and fifth to become very protracted in the extreme pregnancies. This is only a *suggestion*, but it seems not out of accord with probable physiological changes.

The partial correlations $r_{wd.l}$, $r_{wl.d}$ and $r_{ld.w}$ are not without some interest, although they will not bear much stressing. Thus

$$r_{wd} = \cdot 1780, \text{ but } r_{wd.l} = \cdot 1738,$$

and we see that the observed relation between the order and duration of pregnancy is little influenced by the fact that the duration depends upon the size of the litter. Again,

$$r_{wl} = -\cdot 0509, \text{ but } r_{wl.d} = +\cdot 0323;$$

accordingly such little relation as there exists between the order of the pregnancy and the size of the litter is reversed, or is practically zero, when we take a constant duration of pregnancy.

Finally,

$$r_{ld} = -\cdot 4479, \text{ but } r_{ld.w} = -\cdot 4465;$$

thus the association of a long duration of pregnancy with a small litter is practically independent of the order of pregnancy. These are all points concerning which it would be desirable to collect more ample data.

(4) We will now consider what effect the age of the bitch has on the size of the litter and the duration of pregnancy; it will clearly be of necessity fairly highly correlated with the order of pregnancy. Now the age of the bitch may be considered with relation to the mating or the littering. Table VI^A (see p. 317) provides the relation of the size of the litter (l) to the age of the bitch (a) at mating, and Table VI^B that of the size of the litter with the age of the mother at littering (a').

The constants of these two tables are given below:

Table VI^A.

$\bar{a} = 34\cdot 205 \pm 1\cdot 241$ months,
$\sigma_a = 19\cdot 4772 \pm \cdot 8778$ months,
$\bar{l} = 3\cdot 277 \pm \cdot 088$,
$\sigma_l = 1\cdot 3772 \pm \cdot 0620$,
$r_{al} = -\cdot 1722 \pm \cdot 0618$.

Table VI^B.

$\bar{a}' = 37\cdot 115 \pm 1\cdot 040$ months,
$\sigma_{a'} = 19\cdot 2670 \pm \cdot 7462$ months,
$\bar{l} = 3\cdot 160 \pm \cdot 074$,
$\sigma_l = 1\cdot 3753 \pm \cdot 0525$,
$r_{a'l} = -\cdot 1468 \pm \cdot 0528$.

TABLE VI^A. *Age of Bitch at Date of Mating and Size of Litter.*

Age of Bitch at Date of Mating (Central Values in Months).

Size of Litter.	*6*	*9*	*12*	*15*	*18*	*21*	*24*	*27*	*30*	*33*	*36*	*39*	*42*	*45*	*48*	*51*	*54*	*57*	*60*	*63*	*66*	*69*	*72*	*75*	*78*	*81*	Totals
1	—	—	—	—	1	—	—	1	—	—	—	—	—	—	1	—	—	—	1	2	—	1	1	—	1	—	9
2	2	2	2	2	2	1	—	—	2	3	—	2	1	2	—	1	—	—	—	1	—	1	—	1	—	1	26
3	—	—	2	4	3	5	1	3	—	1	4	2	2	—	—	3	—	1	1	1	—	1	—	—	—	—	34
4	—	3	—	3	—	—	—	3	1	1	1	—	2	—	1	1	1	1	—	—	2	—	—	—	—	—	20
5	—	2	2	—	1	4	1	—	1	1	—	—	—	—	—	1	—	—	—	—	1	1	—	—	—	—	15
6	—	—	—	—	—	1	—	1	—	1	—	—	—	1	1	—	—	2	—	—	—	—	—	—	—	—	7
7	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
Totals	2	7	7	9	7	11	2	8	4	7	5	4	5	3	3	6	1	4	2	4	3	4	1	1	1	1	112

TABLE VI^B. *Age of Bitch at Date of Littering and Size of Litter.*

Age of Bitch at Date of Littering (Central Values in Months).

Size of Litter.	*9*	*12*	*15*	*18*	*21*	*24*	*27*	*30*	*33*	*36*	*39*	*42*	*45*	*48*	*51*	*54*	*57*	*60*	*63*	*66*	*69*	*72*	*75*	*78*	*81*	*84*	*87*	*90*	*93*	Totals
1	—	1	—	1	1	—	—	1	2	—	—	1	1	—	1	1	—	1	2	1	—	1	1	1	—	—	—	—	—	17
2	2	2	3	1	3	1	—	1	5	4	3	2	2	2	—	1	1	—	—	1	—	1	—	1	—	1	—	—	1	38
3	1	—	3	5	3	6	5	—	—	4	2	1	2	—	2	1	—	1	1	1	—	1	1	—	1	—	—	—	—	41
4	1	3	1	3	—	—	2	2	1	1	2	4	2	1	3	1	2	1	—	—	2	1	—	—	—	—	—	—	—	33
5	1	3	1	—	2	4	2	—	1	1	—	—	—	—	—	1	—	—	—	—	2	—	—	—	—	—	—	—	—	18
6	—	—	—	—	—	1	—	1	1	1	—	—	1	—	1	—	—	2	—	—	—	—	—	—	—	—	—	—	—	8
7	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
Totals	5	9	9	10	9	12	9	5	10	11	7	8	8	3	7	5	3	5	3	3	4	4	2	2	1	1	—	—	1	156

The relation between the age at mating and the size of the litter, the number of puppies being smaller the older the bitch, is probably significant, but is not very considerable; it is larger than the correlation between order of pregnancy and size of the litter (− ·0509). It is probably reduced in experimental breeding because when the bitch's fertility is reduced, i.e. when she, although mated, produces no litter or only one or two puppies, she is discarded for stud purposes*. The difference between the bitches' ages at mating and pregnancy $= \bar{a}' - \bar{a} = 2{\cdot}910$ months $= 88{\cdot}5$ days. This is *not* the average duration of pregnancy because the second series of dogs is not identical with the first, there are 44 additional entries principally due to the records of C. H. Usher, which provide dates of littering but not those of mating. Even allowing for an average period of 60·8 days for pregnancy, it will be seen that the Aberdeen dogs were on the whole mated to greater ages than the London dogs. As to the remainder of the constants there is no difference of practical importance between them. Accordingly, as the only advantage of taking age of bitch at litter over age at mating lies in the increase of entries, and as this involves a risk of heterogeneity (as Usher introduced new Pom blood while Pearson, after the first cross of Pekinese with Pompeks, continued to inbreed), we shall for the remainder of this paper confine our attention to Age of Mother at Mating.

Table VII (p. 320) shows the relationship between Age of Bitch at mating and Order of Pregnancy. The very appearance of the table indicates how considerable the correlation is, a result which it was easy to predict.

The constants of Table VII are as follows:

$\bar{a}$ = Mean Age at Mating $= 34{\cdot}289 \pm 1{\cdot}237$ months,

σ_a = Standard Deviation of Age $= 19{\cdot}5756 \pm {\cdot}8744$ months,

$\bar{\omega}$ = Mean Order of Pregnancy $= 2{\cdot}632 \pm {\cdot}1167$,

σ_ω = Standard Deviation of Order $= 1{\cdot}8461 \pm {\cdot}0825$,

$r_{a\omega}$ = Correlation of Age and Order $= {\cdot}7967 \pm {\cdot}0231$.

The following table shows the average age at each pregnancy:

Order of Pregnancy	Observed Age	Smoothed Values from Regression Line
I	17·76 months	20·50 months
II	29·36 "	28·95 "
III	40·65 "	37·40 "
IV	51·33 "	45·85 "
V	53·50 "	54·29 "
VI	57·60 "	62·74 "
VII	64·80 "	71·19 "
VIII and IX	68·00 "	83·86 "

That the observed ages at later pregnancies fall so much below those calculated from the correlation formula is no doubt due to the fact that the more fecund bitches had litters earlier and rarely missed a mating. There are, however, only

* Matings leading to no litters have been excluded from these tables. It is not possible in such cases to determine whether the dog or bitch is at fault.

TABLE VII. *Age of Bitch at Date of Mating and Order of Pregnancy.*

Age of Bitch at Date of Mating (Central Values in Months).

Order of Pregnancy.	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63	66	69	72	75	78	81	Totals
I	2	7	7	8	2	5	—	—	2	1	2	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	38
II	—	—	—	2	5	6	1	3	1	1	2	1	3	1	—	—	—	1	—	—	—	1	—	—	—	—	28
III	—	—	—	—	—	—	1	5	1	3	1	—	1	1	1	2	—	—	1	2	1	—	—	—	—	—	20
IV	—	—	—	—	—	—	—	—	—	2	—	2	—	1	—	—	—	—	—	2	—	1	—	—	1	—	9
V	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	2	—	1	—	—	—	—	1	—	—	—	6
VI	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	1	—	—	2	—	—	—	—	—	5
VII	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	1	—	1	—	—	—	1	5
VIII	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1	—	—	2
IX	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1
Totals	2	7	7	10	7	11	2	8	4	7	5	4	5	3	3	6	1	4	2	5	3	4	1	1	1	1	114

TABLE VIII. *Age of Bitch at Date of Mating and Duration of Pregnancy.*

Age of Bitch at Date of Mating (Central Values in Months).

Duration of Pregnancy in Days.	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63	66	69	72	75	78	81	Totals
54	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	1
55	—	—	—	—	—	—	—	—	—	—	0·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0·5
56	—	—	2	—	0·5	—	—	—	—	—	0·5	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	4
57	—	3·5	—	0·5	0·5	2·5	—	0·5	1	0·5	—	—	2	—	—	1·5	—	1	—	—	—	—	—	—	—	—	13·5
58	—	0·5	1	1·5	1	1·5	—	1	0·5	1	—	1	—	0·5	—	0·5	—	—	—	—	0·5	—	—	—	—	—	10·5
59	—	1	2	2	—	2	—	1·5	1	0·5	1	—	1	0·5	—	1	—	1	—	—	1·5	—	—	—	—	—	16
60	—	—	—	2	1	0·5	—	2	0·5	1	—	1	—	—	—	—	—	—	—	—	—	—	1	—	—	—	9
61	—	—	—	—	1	2	—	1	—	1	1	—	—	1	—	1	—	—	—	1	1	2	—	—	—	—	12
62	—	1	—	3	—	1·5	—	—	1	—	—	—	—	—	0·5	—	—	—	—	—	—	1	—	—	0·5	0·5	9
63	—	—	—	—	2	—	1	0·5	—	—	—	—	—	—	0·5	0·5	1	—	1	—	—	—	—	—	0·5	0·5	7·5
64	1	—	2	—	1	—	—	1·5	—	1	—	—	2	1	—	0·5	—	—	—	0·5	—	—	—	0·5	—	—	11
65	—	0·5	—	—	—	—	—	—	—	0·5	1	0·5	—	—	—	—	—	1	—	1·5	—	—	—	0·5	—	—	5·5
66	—	0·5	—	—	—	—	1	—	—	0·5	1	0·5	—	—	—	—	—	—	—	1	—	—	—	—	—	—	4·5
67	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	0·5	—	—	—	—	—	—	—	2·5
68	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0·5	—	—	—	—	—	—	—	0·5
Totals	2	7	7	9	7	11	2	8	4	6	5	3	5	3	2	5	1	4	2	4	3	3	1	1	1	1	107

three bitches who reached the VIII and IX pregnancies, and these count very little in the determination of the correlation. The correlation-ratio of ω on a does not differ sensibly from $r_{\omega a}$.

The average interval between pregnancies is 8·45 months. The physiological interval is about 6 months, and the observed increase is due to matings which were omitted or failed when the bitch was in season. It has been observed also that an aged bitch may occasionally omit one or more heats.

The regression equation giving the probable age ($\check{a}$) at a given pregnancy ω is

$$\check{a} = 12{\cdot}054 + {\cdot}8{\cdot}448\omega.$$

The other constants of Table VII, considering how the total numbers vary from table to table owing to one or another omission in the record, are in reasonable accordance with those of Tables III and V.

We now turn to Table VIII, associating the Age of the Bitch at mating with the Duration of Pregnancy.

The constants of this table are as follows:

$\bar{a}$ = Mean Age of Bitch at Mating = 33·56 ± 1·272 months,
σ_a = Standard Deviation of Age = 19·5119 ± ·8997 months,
$\bar{d}$ = Duration of Pregnancy = 60·66 ± ·198 days,
σ_d = Standard Deviation of Duration = 3·0434 ± ·1403 days,
r_{ad} = Correlation between Age at Mating and Duration of Pregnancy = ·1547 ± ·0636.

The first four constants are within their probable errors of the like characters previously determined. The correlation is small but probably just significant. It is noteworthy that while the correlations of duration of pregnancy with age and with order of pregnancy are both small and positive the latter appears to be somewhat the larger.

The difficulty, however, with practical breeding lies in the economic factor, that matings in the case of such expensive animals as dogs will no longer be made (unless the bitch is of especial value or interest) after the fecundity has begun seriously to diminish.

The regression equation of Duration of Pregnancy on Age at Mating is

$$\check{d} = {\cdot}02413a + 59{\cdot}85.$$

Hence if we take the lowest age of the first heat at 9 months and the highest age of last pregnancy at 84 months = 7 years*, we have for the corresponding durations 60·07 and 61·88 days, or age would have a maximum range of influence of 2 days only on period of pregnancy.

If we take the correlation of Age and Duration of Pregnancy for constant Order, and the correlation of Order and Duration of Pregnancy for constant Age, we have

$$r_{ad.\omega} = {\cdot}0111,$$
$$r_{\omega d.a} = {\cdot}0918,$$

* Our experience seems to show that these dogs have on *the average* a life of nine years or even less, and that few bitches are of use for breeding purposes beyond six or seven years.

and these seem to indicate, taken at their face values, that the number of the pregnancy is more important than the age of the bitch for the duration. But the data are too slender, and the artificial selection of bitches for stud purposes too great, for any stress—other than that of suggestion—to be placed on this result.

(5) *Conclusions.* It seems worth while publishing these results. It is true that the disappearance during the War of the mating books of the Scottish bred dogs, before the dates of mating had been recorded on the schedules, has much reduced the available material. Further, the experiments were not made directly to determine problems regarding gestation in dogs; their primary purpose was to investigate as economically as possible the inheritance of certain characters in dogs.

Thus the bitches were not retained in the kennels long after their period of maximum fecundity was passed. Again, in London kennels with only yard exercise, general fitness is far less than can be maintained in the country, or even in a London home with daily walks. However, the general results seem suggestive enough to make further research worth while. The principal correlations are:

$$r_{dw} = +\cdot 1780,$$

$$r_{ld} = -\cdot 4479, \quad r_{lw} = -\cdot 0509, \quad r_{la} = -\cdot 1722,$$

$$r_{aw} = +\cdot 7967, \quad r_{ad} = +\cdot 1547.$$

Thus the three factors, increasing duration of pregnancy, increasing number of pregnancies, increasing age, all tend to decrease the size of the litter. In the first case it is probable that it is the size of the litter which is the causal factor and hastens the end of gestation. This gives the most marked correlation, and it would be of interest to determine—size of litter being associated with weight—whether in other mammals the average period of gestation is less for male than for female offspring, and less for twins than for single births. We have seen that the correlation coefficient between the duration and order of pregnancy is small, because the relationship is not linear. It is hardly possible to account for the small coefficient of age and size of litter on similar grounds*, but it may be possible to do so on the ground of artificial selection. Probably the fertility of the bitch is not diminished until she is over five years of age. Further, we cannot attribute the small relationship between age and duration of pregnancy to markedly curved regression†. The

* The relationship is as follows:

Age in months	6—22	23—40	41—58	59—82
Size of Litter	3·42	3·30	3·64	2·41

† There is a fairly continuous increase thus:

Age in months	6—16	17—28	29—46	47—64	65 and over
Duration of Pregnancy	60·06	60·63	60·69	61·03	61·20

multiple-regression equation of Duration of Pregnancy on Order of Pregnancy and Age at start of Pregnancy is

$$\frac{d-\bar{d}}{\sigma_d} = \cdot 1499\,\frac{\omega-\bar{\omega}}{\sigma_\omega} + \cdot 0353\,\frac{a-\bar{a}}{\sigma_a},$$

which indicates that with equally likely deviations from the mean order of pregnancy and mean age, the former, the order of the pregnancy, will be more than four times as influential as the age*

The fact remains that none of the factors we have taken into consideration suffices to provide a causal explanation for the duration of pregnancy varying from 55 to 68 days. Can it be that this duration is individual and possibly an inherited character? If so it would be of evolutionary importance. The evidence as to this possibility must be discussed on another occasion.

No one can recognise more clearly than the writers the paucity of their data, but this field of investigation is of considerable interest. It is possible that an appeal to large breeders of dogs might produce more ample data as the variates we are dealing with must have been recorded in many cases. We shall be content if the present paper leads others to collect and reduce material on a wider basis, dealing if possible with small dogs of a single species; for comparative purposes Pekinese or Pomeranians would be most serviceable.

* The actual numerical equation to determine the probable duration of pregnancy $\check{d}$, in days, for a bitch in her ωth pregnancy and of age a months is

$$\check{d} = 60\cdot 557 + \cdot 2505\omega + \cdot 005{,}581a.$$

Thus a bitch in her fifth pregnancy and four years old—i.e. $\omega=5$ and $a=48$—would have a probable duration of pregnancy $\check{d}$ given by

$$\check{d} = 60\cdot 557 + 1\cdot 2525 + \cdot 2679 = 62\cdot 08 \text{ days}.$$

ON THE ASYMMETRY OF THE HUMAN SKULL.

By T. L. WOO, Ph.D. Lond., Research Fellow of the China Foundation for the Promotion of Education and Culture.

(1) Much has been written about the quantitative asymmetry of the brain, on the assumption that differentiated functioning of the right and left hemispheres might (or must) be manifested by differentiated *size*. On such a hypothesis the bony skull developing so as to fit the growing brain should exhibit significant evidence of this asymmetry. Reasoning in this way there is nothing in the least absurd in the fundamental conceptions of phrenology. What has been the misfortune of that science was the premature localisation of certain mental and sensory activities before any adequate statistical evidence was forthcoming (or had at least been published) for each such local assignment. Since in the case of a sensory or mental activity we might on the above hypothesis anticipate an exaggerated or at least a marked development of the brain and a correlated development of the skull, the question of the asymmetry of the latter becomes one of great importance. If it be possible to demonstrate that some well-used mental or sensory activity is controlled generally from a centre on one side of the brain and there is no correlated increase in skull size in that region, i.e. that there is no resulting asymmetry, we shall have a strong argument—it may not necessarily be a conclusive one—that this emphasised local brain activity is not highly correlated with *size*. To the same extent we weaken the standpoint of the phrenologist that a cranial "bump" which to a large extent connotes asymmetry* marks the special development of a local centre of brain activity.

(2) Most measurements of the skull have hitherto been taken in the service of anthropology, i.e. with a view to finding the differentiated characteristics of various races. Racial differences were first approached from the standpoint of appearance, in other words from the conception of portraiture. Anthropometricians endeavoured to give quantitative value to the differences that were obvious to them at first sight. They observed the roundness of the head, the breadth of the forehead, the height of the face, the ellipticity of the orbit and so forth. Such measurements are usually composite, covering more than one bone of the skull, and are generally far from suitable for testing the asymmetry of the skull. Indeed they often cover *both* the homologous bones the difference in the sizes of which leads to the asymmetry, or again are worthless for our purpose because the measurements are taken in the median sagittal plane.

* Always supposing that the homologous region—i.e. from the phrenologist's standpoint an independent mental or emotional trait—does not chance to be equally developed.

However valuable the present measures of the anthropologists may be for the purpose for which they were devised, it is clear that they can give no final answer to many important problems, and one of these is the asymmetry of the skull. For this purpose we need measurements on the individual bones of the skull, taken in homologous pairs. A special advantage of taking measurements on the individual bones of the skull is that we thus get some idea of the size and shape of relatively small regions, not indeed coinciding with the phrenological areas, but giving us a better appreciation of *local* asymmetries than the run of anthropometric measurements can. I think it might be useful to distinguish the two types of measurements as ethnometric and morphometric, for both are actually anthropometric. The division really refers to the purposes which they are to serve; for while some few ethnometric characters have morphometric value, the bulk of the latter could be used for ethnometric distinctions, and will undoubtedly be more and more so used in the future developments of anthropometry, i.e. we shall gradually come to the study of the ethnic differences of the individual bones of the skull, rather than those of its composite characters—just as a study of the individual long bones has greater ethnic value than a study merely of stature.

(3) Having need for one of the important morphometric problems to which I have referred to study the characters of the individual cranial bones, I took a number of measurements of each of these. I did this on the long series of Egyptian skulls, 26th to 30th dynasties (Series E), in the Biometric Laboratory, confining my attention to those classed as male, amounting to about 800 in number*. On the separate bones I took 63 measurements, partly chordal and partly arcual. Of these 63 measurements, 50 were corresponding measurements on homologous bones, and accordingly of value for determining the degree of asymmetry in the two sides of the skull, and for measuring what regions were in excess on the right or on the left side with the amount of that excess.

The following are the 25 measurements which were taken bilaterally (Figs. 1 and 2):

(*a*) *Frontal Bone.* F_1 = minimum arc from a point on the coronal suture equidistant from the bregma and stephanion to the upper border of the orbit immediately outside the supra-orbital notch. The line of the coronal suture is marked in pencil to indicate its general direction, so no account is taken of a local indentation in determining the terminal of this measurement. The point equidistant from the bregma and stephanion can be found with the aid of coordinate callipers, or with small dividers. Although the supra-orbital notch is very variable in form, there is no difficulty in making the steel-tape pass immediately outside it.

F_2 = arc from ophryon to stephanion. The ophryon is defined, for this purpose, to be the intersection of the minimum arc from nasion to bregma and the minimum arc (marked in pencil) between the temporal lines.

* Some 58,400 measurements were taken, and as each skull took over an hour to measure, it required a year's work to complete the series.

(*b*) *Parietal Bone.* P_2 = arc from bregma to sphenion* along the line of the coronal suture.

P_3 = minimum arc from bregma to asterion.

P_4 = minimum arc from sphenion to lambda, avoiding the temporal squama so that the arc falls entirely on the parietal bone†. This measurement is generally close to, but not identical with, the geodesic line.

(*c*) *Occipital Bone.* O_7 = arc from lambda to asterion along the line of the lambdoid suture. This may diverge appreciably from the geodesic line between the points.

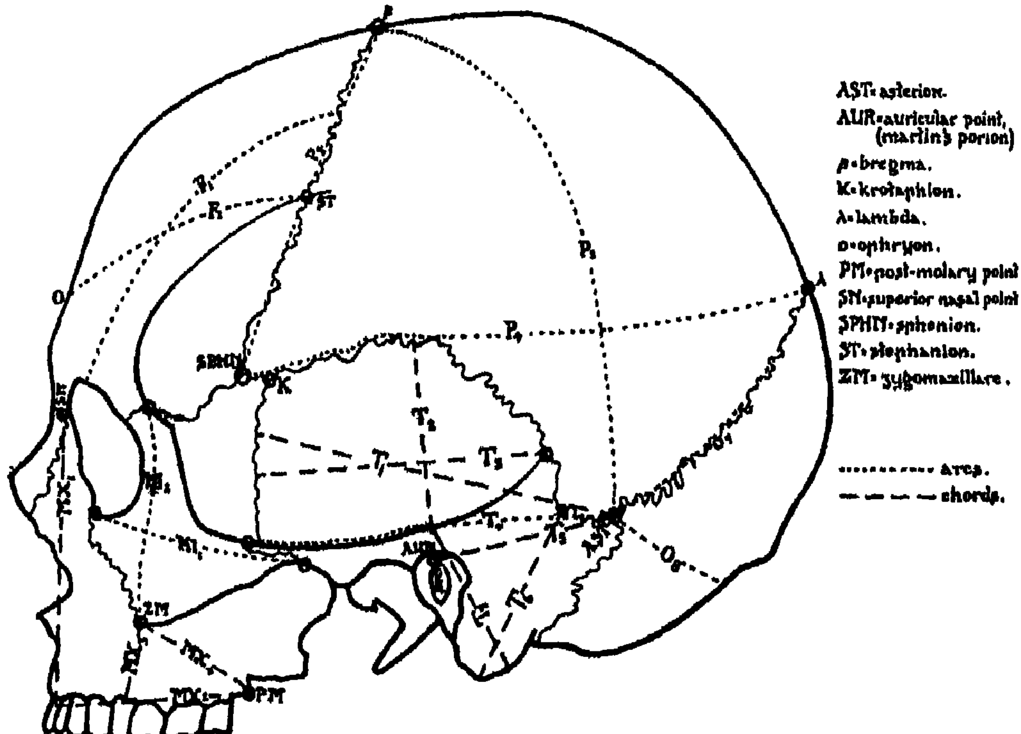

Fig. 1.

O_8 = arc from the median line of the occipital bone to the asterion. The median line is defined, for this purpose, by the minimum arc from opisthion to lambda, and the join of this line with the geodesic between the asteria gives the terminal. The measurement is then taken along that geodesic. The opisthion is here defined to be the point where the extension of the external occipital crest meets the border of the *foramen magnum*.

* If there be an epipteric bone at the pterion in contact with the frontal bone, the sphenion is supposed indeterminate.

† If there be an ossicle of the bregma, that "point" is accepted as the intersection of the lines (traced in pencil) marking the general direction of the coronal and sagittal sutures. The lambda and asterion are defined in a similar way, if supernumerary bones are present, though less exactly since each is defined to be the join of three sutural lines and, if it is necessary to continue them, they may not meet in a unique point. If the sutures round the lambda are very complex it may be necessary to use the same method.

O_9 = chord from basion to asterion.

(*d*) *Temporal Bone.* T_1 = maximum chord from the asterion to the anterior border of the temporal bone. When the anterior border—i.e. the spheno-squamous suture—is deeply dentated, but not otherwise, the anterior terminal is taken on the pencil line which marks its general direction. The point appears to be invariably above the zygomatic arch and it may be close to the pterion.

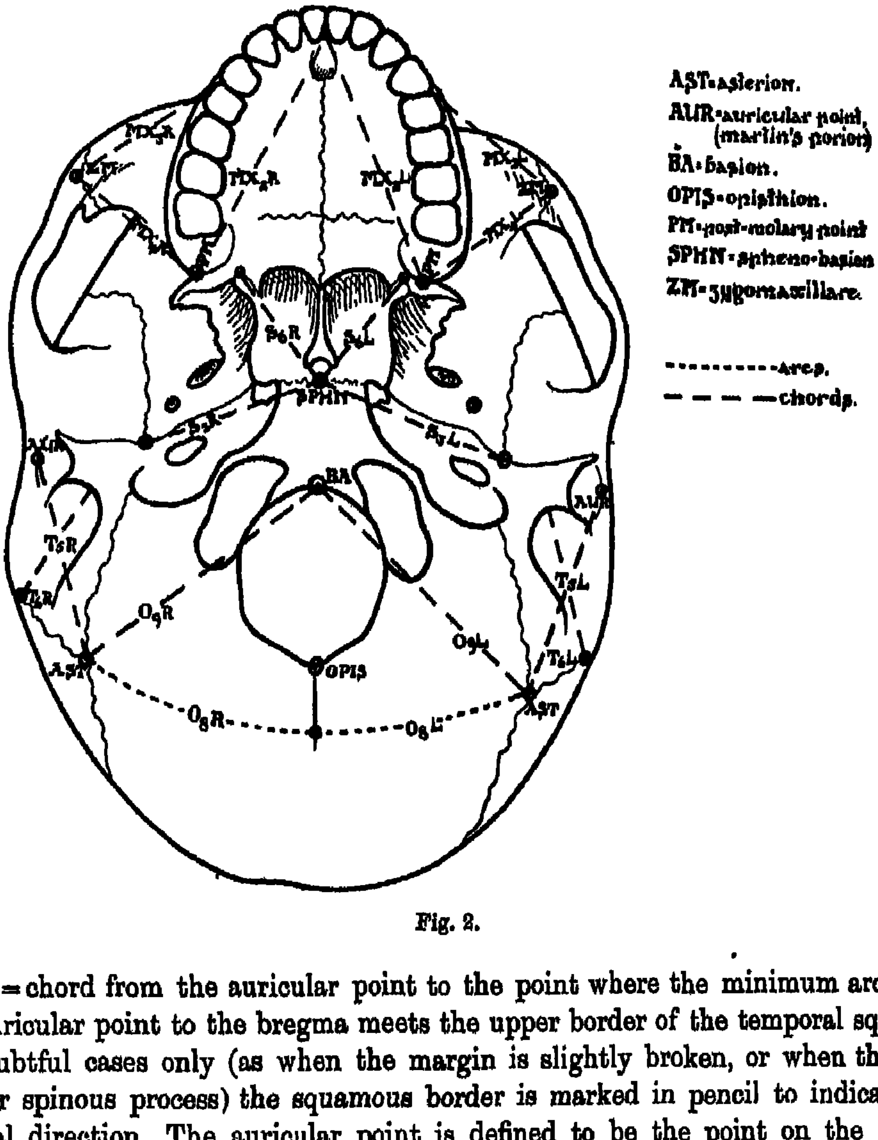

Fig. 2.

T_2 = chord from the auricular point to the point where the minimum arc from the auricular point to the bregma meets the upper border of the temporal squama. In doubtful cases only (as when the margin is slightly broken, or when there is a clear spinous process) the squamous border is marked in pencil to indicate its general direction. The auricular point is defined to be the point on the upper margin of the auricular passage which lies in the plane bisecting the orifice transversely*. It can generally be found in practice by continuing forward the curve of the thin lip of bone which terminates posteriorly in a well-marked notch on the upper part of the posterior wall of the passage.

* This point is Martin's "porion."

T_3 = maximum chord from the point where the backward extension of the temporal ridge meets the parietal bone to the anterior border of the squama. This measurement is not entirely satisfactory, but it would be difficult to devise a better measure of the antero-posterior length of the temporal squama. The temporal ridge is generally blunt, and in continuing it as a pencil line to meet the parieto-squamous suture considerable differences might be made by different workers*. When the spheno-squamous suture is deeply dentated, but not otherwise, the anterior terminal is taken on the pencil line which marks its general direction.

T_4 = minimum arc from asterion, above the auricular passage, along the upper border of the zygomatic ridge to the suture with the malar bone. This passes through the point on the temporal ridge, at the root of the zygomatic process, which is in the plane bisecting the auricular orifice transversely, i.e. the "auriculare" of Martin, the "point sus-auriculaire" of the French.

T_5 = chord from the asterion to the auricular point.

T_6 = maximum chord from a point on the suture with the parietal bone, equidistant from the asterion and the point where the temporal ridge meets the parietal bone, to the tip of the mastoid process. The suture in question is made up of parieto-squamous and parieto-mastoid portions and it is often irregular. The terminal is a point on the pencil line which indicates its general direction without regard to local indentations. Its position is somewhat uncertain owing to the fact that the point where the temporal ridge meets the parietal bone in some cases cannot be found precisely (see the definition of T_3).

T_7 = maximum chord from Martin's "auriculare" (see the definition of T_4) to the most remote part of the mastoid process. The mastoid terminals of the measurements T_6 and T_7 are not coincident.

(*e*) *Maxilla.* Mx_1 = chord from the point where the frontal, nasal and maxillary bones meet (the superior nasal point) to the lowest point on the alveolar process between the central incisors. If the alveolar processes of the two maxillae are completely fused the lower terminal will coincide with the alveolar point, but if they are slightly separated at the tips, as is often found, the two points will be distinct.

Mx_2 = chord from the lowest point on the alveolar process between the central incisors (as for Mx_1) to the "postreme point" on the alveolar process behind the last molar—the "post-molary point." The last molar is normally the third, but fully adult specimens for which no third molars have erupted are measured. The measurement cannot be taken, however, if the third molar was lost before death, or if the alveolar process was appreciably deformed by the loss of other teeth.

Mx_3 = chord from the lowest point on the malar-maxillary suture (Martin's "zygomaxillare") to the mid-point of the alveolar margin of the second premolar.

* For the purpose of the present study male skulls only were dealt with and the point in question would certainly be more difficult to determine on female specimens.

Mx_4 = chord from the lowest point on the malar-maxillary suture to the post-molary point.

(*f*) *Malar Bone.* Ml_1 = minimum arc from the point where the malar-maxillary suture crosses the lower border of the orbit to the lowest point on the zygomatic suture which is still on the lateral surface of the arch.

Ml_2 = minimum arc from the point where the malar ridge meets the fronto-malar suture to the lowest point on the malar-maxillary suture.

(*g*) *Sphenoid Bone.* S_2 = chord from the point where the frontal, sphenoid and temporal bones meet (Martin's "krotaphion") to the point in the median sagittal plane on the union of the basi-occipital and sphenoid bones. (Martin's "spheno-basion.") The synchondrised basal suture can be marked by a pencil line with a close approach to accuracy in most cases.

S_3 = chord from the most posterior point of the sphenoid exposed on the base of the skull to the spheno-basion. The point is on the *spina angularis* which occupies the angle between the petrous and squamous portions of the temporal bone. This process is extremely variable in form, but the most posterior point on it can almost invariably be found without ambiguity.

S_5 = chord from the postreme point of the sphenoid exposed on the base of the skull to the krotaphion.

S_6 = chord from the spheno-basion to the lowest point on the suture between the medial pterygoid plate and the palate bone.

By a *minimum* arc a geodesic is to be understood. By a "suture" when much indented is to be understood a smooth line drawn midwise across the indentations. Arcs were measured to the nearest half millimetre and chords to the nearest $\frac{1}{10}$th millimetre.

(4) Having reduced my measurements I computed the means, standard deviations and coefficients of variation of each of the 50 measurements with their probable errors; also the coefficients of correlation of each of the 25 pairs of homologous measurements. The latter I obtained in two different ways as a check on my results, namely (i) by the usual product moment method for which the 25 correlation tables are given below, and (ii) by the well-known formula:

$$r_{xy} = \frac{\sigma_x^2 + \sigma_y^2 - \sigma^2_{x-y}}{2\sigma_x\sigma_y},$$

which involves a knowledge of the standard deviation of the difference of the two characters. The two methods should give identical results if we do not group x, y and $x - y$, but doing so and correcting for grouping we get slight differences in our results for r_{xy}. The two methods, however, give results sufficiently close to check the arithmetic.

Table I (p. 330) contains the values of the constants thus determined, the units being millimetres. In Table II (p. 333), I have arranged in order of significance those measurements in which the right and the left sides respectively are dominant.

TABLE I. *Constants of the Distributions an*

Measurements in millimetres.

Bone	Measurement	No.	Means	Standard Deviations	Coefficients of Variation	Correlation Coefficients *
	Constants of the Distributions					
Frontal	F_1 R	885	98·4381 ± ·1156	5·0988 ± ·0817	5·1707 ± ·0833	·9059 ± ·0041
	F_1 L		98·0428 ± ·1146	5·0567 ± ·0811	5·1577 ± ·0829	·9043 ± ·0041
	F_2 R	887	88·0070 ± ·1083	4·7796 ± ·0765	5·4309 ± ·0872	·9384 ± ·0027
	F_2 L		87·4366 ± ·1127	4·9700 ± ·0797	5·6910 ± ·0914	·9366 ± ·0028
Parietal	P_2 R	754	112·5789 ± ·1457	5·9322 ± ·1030	5·2694 ± ·0918	·6738 ± ·0134
	P_2 L		111·2487 ± ·1400	5·7017 ± ·0990	5·1251 ± ·0893	·6726 ± ·0134
	P_3 R	873	165·3960 ± ·1351	5·9180 ± ·0955	3·5781 ± ·0578	·7112 ± ·0113
	P_3 L		163·1612 ± ·1380	6·0431 ± ·0975	3·7038 ± ·0599	·7100 ± ·0113
	P_4 R	738	177·0305 ± ·1451	5·8437 ± ·1026	3·3010 ± ·0580	·7909 ± ·0093
	P_4 L		175·6199 ± ·1480	5·9608 ± ·1047	3·3942 ± ·0597	·7897 ± ·0093
Temporal	T_1 R	860	86·9157 ± ·0892	3·8021 ± ·0631	4·4780 ± ·0727	·8133 ± ·0078
	T_1 L		86·0242 ± ·0879	3·8341 ± ·0622	4·4570 ± ·0724	·8105 ± ·0079
	T_2 R	856	46·6262 ± ·0872	3·7813 ± ·0610	8·1098 ± ·1331	·8145 ± ·0078
	T_2 L		46·7956 ± ·0855	3·7090 ± ·0605	7·9260 ± ·1300	·8116 ± ·0079
	T_3 R	871	66·2204 ± ·0976	4·2700 ± ·0690	6·4482 ± ·1046	·8503 ± ·0064
	T_3 L		65·7371 ± ·0972	4·2537 ± ·0687	6·4709 ± ·1050	·8480 ± ·0064
	T_4 R	726	99·7293 ± ·1075	4·2950 ± ·0760	4·3076 ± ·0764	·8061 ± ·0088
	T_4 L		99·4200 ± ·1121	4·4767 ± ·0792	4·5024 ± ·0799	·8039 ± ·0089
	T_5 R	876	45·9612 ± ·0627	2·7499 ± ·0443	5·9831 ± ·0967	·6729 ± ·0125
	T_5 L		46·0000 ± ·0630	2·7664 ± ·0446	6·0096 ± ·0972	·6716 ± ·0125
	T_6 R	827	45·5266 ± ·0889	3·7915 ± ·0629	8·3281 ± ·1391	·7748 ± ·0094
	T_6 L		45·8761 ± ·0886	3·7788 ± ·0627	8·2370 ± ·1375	·7718 ± ·0095
	T_7 R	840	36·1359 ± ·0684	2·9405 ± ·0484	8·1374 ± ·1348	·7937 ± ·0086
	T_7 L		35·9714 ± ·0665	2·8565 ± ·0470	7·9410 ± ·1315	·7925 ± ·0087
Sphenoidal	S_2 R	719	75·0542 ± ·0863	3·4328 ± ·0611	4·5738 ± ·0815	·8315 ± ·0078
	S_2 L		74·7774 ± ·0881	3·5017 ± ·0623	4·6828 ± ·0835	·8280 ± ·0079
	S_3 R	866	36·2992 ± ·0443	1·9344 ± ·0314	5·3289 ± ·0866	·7771 ± ·0091
	S_3 L		36·5860 ± ·0462	2·0146 ± ·0327	5·5065 ± ·0895	·7733 ± ·0092
	S_5 R	722	56·6454 ± ·0864	3·4406 ± ·0611	6·0739 ± ·1082	·7962 ± ·0092
	S_5 L		56·3102 ± ·0865	3·4481 ± ·0612	6·1234 ± ·1091	·7926 ± ·0093
	S_8 R	808	35·5364 ± ·0569	2·3965 ± ·0402	6·7438 ± ·1137	·8776 ± ·0055
	S_8 L		35·5015 ± ·0579	2·4399 ± ·0409	6·8727 ± ·1159	·8750 ± ·0056
Malar	Ml_1 R	817	49·3853 ± ·0734	3·1118 ± ·0519	6·3011 ± ·1056	·9219 ± ·0035
	Ml_1 L		49·9642 ± ·0757	3·2068 ± ·0535	6·4182 ± ·1076	·9177 ± ·0037
	Ml_2 R	718	59·4213 ± ·1102	4·3794 ± ·0780	7·3701 ± ·1319	·9399 ± ·0029
	Ml_2 L		59·5829 ± ·1146	4·5542 ± ·0811	7·6435 ± ·1368	·9378 ± ·0030
Maxillary	Mx_1 R	692	66·5954 ± ·1007	3·9276 ± ·0712	5·8978 ± ·1078	·9766 ± ·0012
	Mx_1 L		66·3685 ± ·1001	3·9045 ± ·0708	5·8830 ± ·1070	·9738 ± ·0013
	Mx_2 R	517	55·6924 ± ·0851	2·8695 ± ·0602	5·1524 ± ·1084	·9278 ± ·0041
	Mx_2 L		55·9226 ± ·0857	2·8899 ± ·0606	5·1677 ± ·1087	·9228 ± ·0043
	Mx_3 R	451	40·8182 ± ·0997	3·1396 ± ·0705	7·6917 ± ·1738	·9134 ± ·0053
	Mx_3 L		40·8803 ± ·1031	3·2453 ± ·0729	7·9385 ± ·1794	·9093 ± ·0055
	Mx_4 R	545	38·3780 ± ·0749	2·5924 ± ·0530	6·7549 ± ·1386	·8616 ± ·0074
	Mx_4 L		38·3541 ± ·0795	2·6823 ± ·0548	6·9936 ± ·1436	·8556 ± ·0076
Occipital	O_7 R	864	97·3113 ± ·1225	5·3384 ± ·0866	5·4859 ± ·0893	·7954 ± ·0084
	O_7 L		98·6551 ± ·1271	5·5363 ± ·0899	5·6118 ± ·0914	·7940 ± ·0085
	O_8 R	858	63·7943 ± ·0849	3·6869 ± ·0600	5·7793 ± ·0944	·5379 ± ·0164
	O_8 L		63·3409 ± ·0758	3·2930 ± ·0536	5·1988 ± ·0849	·5345 ± ·0165
	O_9 R	858	74·7284 ± ·0791	3·4344 ± ·0559	4·5959 ± ·0750	·7928 ± ·0086
	O_9 L		74·4242 ± ·0760	3·2980 ± ·0537	4·4313 ± ·0723	·7892 ± ·0087

* The upper of the two correlations is found by the direct product moment method, the lower by the formula based on the three standard deviations.

Measurements in millimetres.

Constants of the Differences of Means			Constants of the Differences of Absolute Variabilities			Constants of the Differences of Relative Variabilities		
Mean Differences Δ_{R-L}	Standard Deviations σ_Δ	Ratio Δ/(p.e. of Δ)	Δ $\sigma_R-\sigma_L$	Standard Deviation of $\sigma_R-\sigma_L$	Ratio Δ/(p.e. of Δ)	V V_R-V_L	Standard Deviation of V_R-V_L	Ratio Δ/(p.e. of Δ)
+ ·3955 ± ·05036	2·2216 ± ·03561	+ 7·85	+ ·0421	·07228	+0·66	+ ·0220	·073,669	+0·44
+ ·5704 ± ·03959	1·7478 ± ·02798	+14·41	− ·1964	·05678	−5·13	− ·2601	·064,928	−5·94
+1·3302 ± ·11572	4·7177 ± ·08204	+11·49	+ ·2305	·15662	+2·18	+ ·1443	·140,115	+1·53
+2·2348 ± ·10402	4 5562 ± ·07354	+21·49	− ·1251	·14230	− 1·30	− ·1257	·086,747	−2·15
+1·4106 ± ·09508	3·8292 ± ·06724	+14·84	− ·1171	·13300	−1·31	− ·0932	·075,484	−1·83
+ ·8915 ± ·05453	2·3791 ± ·03858	+16·35	+ ·0580	·07641	+1·13	+ ·0210	·088,438	+0·35
− ·1694 ± ·05302	2·2489 ± ·03666	− 3·20	+ ·0723	·07425	+1·44	+ ·1838	·159,601	+1·71
+ ·4833 ± ·05369	2·3752 ± ·03838	+ 9·00	+ ·0163	·07600	+0·32	− ·0227	·115,443	−0·29
+ ·3003 ± ·06889	2·7524 ± ·04872	+ 4·36	− ·1808	·09643	−2·78	− ·1948	·096,973	−2·98
− ·0388 ± ·05095	2·2326 ± ·03597	− 0·76	− ·0165	·06891	−0·36	− ·0265	·150,184	−0·26
− ·3495 ± ·05996	2·5570 ± ·04240	− 5·83	+ ·0127	·08320	+0·23	+ ·0911	·182,488	+0·74
+ ·1645 ± ·04349	1·8690 ± ·03076	+ 3·78	+ ·0840	·06088	+2·05	+ ·1964	·169,394	+1·72
+ ·2768 ± ·05117	2·0345 ± ·03619	+ 5·41	− ·0689	·07186	−1·42	− ·1090	·096,031	−1·68
− ·2868 ± ·03052	1:3317 ± ·02159	− 9·40	− ·0802	·04227	−2·81	− ·1775	·116,113	−2·27
+ ·3352 ± ·05568	2·2183 ± ·03937	+ 6·02	− ·0075	·07755	−0·14	− ·0495	·137,625	−0·53
+ ·0349 ± ·02871	1·2098 ± ·02030	+ 1·22	− ·0434	·04080	− 1·58	− ·1289	·115,154	−1·66
− ·5789 ± ·03033	1·2851 ± ·02145	−19·09	− ·0950	·04289	−3·28	− ·1171	·086,439	−2·01
− ·1616 ± ·03989	1·6497 ± ·02936	− 4·05	− ·1748	·05710	−4·54	− ·2734	·096,210	−4·21
+ ·2269 ± ·02298	·8961 ± ·01625	+ 9·87	+ ·0231	·03202	+1·07	+ ·0148	·048,190	+0·46
− ·2302 ± ·03358	1·1320 ± ·02375	− 6·86	− ·0204	·04727	−0·64	− ·0153	·084,790	−0·27
− ·0621 ± ·04331	1·3637 ± ·03063	− 1·43	− ·1057	·06129	−2·56	− ·2468	·150,509	−2·43
+ ·0239 ± ·04103	1·3006 ± ·02443	+ 0·58	− ·0899	·05741	−2·32	− ·2387	·150,022	−2·36
−1·3438 ± ·08022	3·4955 ± ·05673	−16·75	− ·1979	·11223	−2·61	− ·1259	·114,654	−1·63
+ ·4534 ± ·07796	3·3850 ± ·05511	+ 5·82	+ ·3939	·10071	+5·80	+ ·5805	·158,784	+5·42
+ ·3042 ± ·05043	2·1897 ± ·03565	+ 6·03	+ ·1364	·07010	+2·89	+ ·1646	·094,086	+2·59

It will be seen that of the 25 characters we can only say of four that no definite asymmetry is indicated. Further, of these four measurements none is of first-class importance so far as the brain is concerned, that of most interest being the chord from asterion to auricular point. A noteworthy fact is that none of the measurements gives us differences of right and left measurements lying between two and three times their probable errors—the region in which significance is doubtful. No less than 14 of the measurements fall into the markedly significant group, while another seven are with high probability significant. We conclude therefore that *the human skull from its very nature (like the internal organs of the human body) is asymmetrical*; it is not a question of asymmetry in the individual, but of asymmetry in the type. The sculptor who desires to form not a portrait, but a typical representative of man (or of a god in the image of man) must model the head *asymmetrically**. The leading feature of this asymmetry is the predominance of the *right-hand* side. Examining Table II we find that the right side bones are predominant in 16 of the 25 measurements, as against nine on the left side. The table indicates further that the average measure of significance is 8·66 on the right as against 7·49 only on the left. Of the eight most significant differences, six are on the right side, only two on the left. *All* the measurements of the frontal and parietal bones show marked excess on the right side. They thus confirm the conclusion already reached in this journal† that the right cerebral hemisphere is the larger. Even with the sphenoid bone three out of the four measurements are predominant on the right, but the fourth measurement, the distance from the postreme point of the sphenoid to the spheno-basion, is markedly significant, and predominant on the left. The malar bone, so far as we can judge from two measurements, is predominant on the left side. The marked right predominance of the fundamental vertical measurement (Mx_1) of the maxillary bone possibly accounts for the nasal wryness which is so common, i.e. the slight drawing up of the nasal ala or even the mouth on the left. On the other hand, the horizontal Mx_2 is larger on the left, indicating that the left upper jaw is larger than the right. It is therefore possible that a correlated predominance of the left side of the mandible exists, and this point would be worth investigating.

Of the seven measurements of the temporal bone, one difference is practically of no significance, the distance from asterion to auricular point, T_8 being practically symmetrical.

Of the remaining six measurements four are predominant on the right side and two of these very markedly so. The two measurements predominant on the left—both vertical measurements—are significant but neither very markedly so. On the whole the temporal bone while not entirely dominant on the right side must be considered as part of that system of frontal and parietal bones which gives pre-eminence to the right side.

* Quite recently an obtuse writer laboriously measured the heads of Greek statues, and accused the sculptors of the Periclean age of making their gods asymmetrical!

† Hoadley and Pearson: "On Measurement of the Internal Diameters of the Skull." *Biometrika*, Vol. XXI. pp. 85—123.

TABLE II. *Dominance of Right and Left Cranial Bones, estimated by Average Size.*

	Right Dominance			Left Dominance		
	Bone	Length	$\frac{\Delta_{R-L}}{\text{p.e. of } \Delta_{R-L}}$	Bone	Length	$\frac{\Delta_{R-L}}{\text{p.e. of } \Delta_{R-L}}$
Markedly significant	Parietal	P_3	+21·49	—	—	—
	—	—	—	Malar	Ml_1	−19·09
	—	—	—	Occipital	O_7	−16·75
	Temporal	T_1	+16·35	—	—	—
	Parietal	P_4	+14·84	—	—	—
	Frontal	F_2	+14·41	—	—	—
	Parietal	P_2	+11·49	—	—	—
	Maxillary	Mx_1	+ 9·87	—	—	—
	—	—	—	Sphenoid	S_3	−9·40
	Temporal	T_3	+ 9·00	—	—	—
	Frontal	F_1	+ 7·85	—	—	—
	—	—	—	Maxillary	Mx_2	−6·86
	Occipital	O_9	+ 6·03	—	—	—
	Sphenoid	S_5	+ 6·02	—	—	—
Significant	—	—	—	Temporal	T_6	−5·83
	Occipital	O_8	+ 5·82	—	—	—
	Sphenoid	S_2	+ 5·41	—	—	—
	Temporal	T_4	+ 4·36	—	—	—
Probably significant	—	—	—	Malar	Ml_2	−4·05
	Temporal	T_7	+ 3·78	—	—	—
	—	—	—	Temporal	T_2	−3·20
Non-significant	—	—	—	Maxillary	Mx_3	−1·43
	Sphenoid	S_6	+ 1·22	—	—	—
	Maxillary	Mx_4	+ 0·58	—	—	—
	—	—	—	Temporal	T_5	−0·76
	Mean Right		+8·66	Mean Left		−7·49

Now it is somewhat difficult to realise how predominance of one side can arise without a counterbalancing predominance somewhere else on the other. We might possibly anticipate a greater predominance of the left side on the cerebellar and basal portions of the skull. The occipital arc, from lambda to asterion, O_7, is very significantly greater on the left, but not so the lower arc, O_8, nor the chord from basion to asterion, O_9. S_3 is again, however, greater on the left. It is clear that we cannot state any rule as to left predominance compensating for right predominance owing to their balancing on the skull. The asymmetries of the cranial bones do not equalise each other, so as to produce a symmetrical total head form, rather they tend to give a distorted form to the skull as a whole.

We can examine the problem from another standpoint, that of the percentage of cases on either side in which the right or left measurement is in excess. The

two methods, that of predominance of mean size, and that of predominance in number of individuals, need not necessarily lead to the same results. The reduced data will be found in Table III. It is needful, however, to consider first what is the probable error of the difference of two percentages in a population. Let the numbers corresponding to the two percentages p_s and p_t be n_s and n_t in a population of size N; then if p_{s-t} be the percentage difference

$$p_s = 100n_s/N,\quad p_t = 100n_t/N,$$

$$p_{s-t} = \frac{100}{N}(n_s - n_t) \text{ and } \bar{p}_{s-t} = \frac{100}{N}(\bar{n}_s - \bar{n}_t);$$

thus

$$\delta p_{s-t} = \frac{100}{N}(\delta n_s - \delta n_t),$$

$$\sigma^2_{p_{s-t}} = \left(\frac{100}{N}\right)^2 (\sigma^2_{n_s} + \sigma^2_{n_t} - 2\sigma_{n_s}\sigma_{n_t} r_{n_s n_t}).$$

But

$$\sigma^2_{n_s} = \bar{n}_s\left(1 - \frac{\bar{n}_s}{N}\right),\quad \sigma^2_{n_t} = \bar{n}_t\left(1 - \frac{\bar{n}_t}{N}\right),$$

and

$$\sigma_{n_s}\sigma_{n_t} r_{n_s n_t} = -\frac{\bar{n}_s \bar{n}_t}{N},$$

where $\bar{n}_s$, $\bar{n}_t$ are the reduced parent population values which for want of better information we put equal to the sample values. Thus

$$\sigma^2_{p_{s-t}} = \left(\frac{100}{N}\right)^2 \left(\bar{n}_s + \bar{n}_t - \frac{(\bar{n}_s - \bar{n}_t)^2}{N}\right)$$

$$= \frac{100}{N}\left(\bar{p}_s + \bar{p}_t - \frac{1}{100}(\bar{p}_s - \bar{p}_t)^2\right).$$

$$\text{Probable error of } p_{s-t} = \frac{6{\cdot}7449}{\sqrt{N}}\left(p_s + p_t - \frac{1}{100}(p_s - p_t)^2\right)^{\frac{1}{2}}$$

approximately, substituting the observed values.

We can now form Table IV (p. 336), corresponding to Table II, and arranged according to the significance of percentage differences. There is not much change in the order or magnitude of the significance of the several cranial lengths, whether we judge dominance by average size or relative percentage of excess. The tendency when using percentage excess of size is to somewhat reduce the position of the measurements. Taking, however, the "significant" and "markedly significant" differences 17 out of 18 remain in the same group; only the sphenoidal length S_1 has dropped out of the "significant" into the "probably significant" category. Mx_2 has passed from "non-significant" dominance on the left to the same category on the right; no other measurement has changed its dominance. In other words, whether we judge by percentage of excess in size, or by mean size, the bones on the right side of the skull possess a dominance in the ratio of about 12 to 5 in the classes where significance may be taken to be certain. Why the antero-posterior lengths of malar and occipital bones should be so markedly greater on the left, I am unable

TABLE III. *Percentages and Significance of their Differences.*

Character and number		Number: Greater on Right	Equal*	Greater on Left	Δ_p percentage difference	Probable error of Δ_p	Ratio
F_1 (885)	Nos.	404	210	271	+15·03	1·9503	+ 7·71
	%	45·65 ± 1·13	23·73 ± ·96	30·62 ± 1·04			
F_2 (887)	Nos.	475	192	220	+28·75	1·8963	+15·16
	%	53·55 ± 1·13	21·65 ± ·93	24·80 ± ·98			
P_2 (754)	Nos.	443	61	250	+25·60	2·2691	+11·28
	%	58·75 ± 1·21	8·09 ± ·67	33·16 ± 1·16			
P_3 (873)	Nos.	561	89	223	+38·72	1·9976	+19·38
	%	64·26 ± 1·09	10·19 ± ·69	25·55 ± 1·00			
P_4 (738)	Nos.	439	80	219	+29·82	2·2245	+13·40
	%	59·49 ± 1·22	10·84 ± ·77	29·67 ± 1·13			
T_1 (866)	Nos.	499	135	232	+30·83	1·9837	+15·54
	%	57·62 ± 1·13	15·59 ± ·83	26·79 ± 1·02			
T_2 (856)	Nos.	321	147	388	− 7·83	2·0899	− 3·75
	%	37·50 ± 1·12	17·17 ± ·87	45·33 ± 1·15			
T_3 (871)	Nos.	423	173	275	+16·99	2·0083	+ 8·46
	%	48·57 ± 1·14	19·86 ± ·91	31·57 ± 1·06			
T_4 (726)	Nos.	339	121	266	+10·06	2·2710	+ 4·43
	%	46·69 ± 1·25	16·67 ± ·93	36·64 ± 1·21			
T_5 (876)	Nos.	345	175·5	355·5	− 1·20	2·0378	− 0·59
	%	39·38 ± 1·11	20·04 ± ·91	40·58 ± 1·12			
T_6 (827)	Nos.	293	144	390	−11·73	2·1133	− 5·55
	%	35·43 ± 1·12	17·41 ± ·89	47·16 ± 1·17			
T_7 (840)	Nos.	341	203	296	+ 5·36	2·0226	+ 2·65
	%	40·60 ± 1·14	24·16 ± 1·00	35·24 ± 1·11			
S_2 (719)	Nos.	293	194	232	+ 8·48	2·1385	+ 3·97
	%	40·75 ± 1·24	26·98 ± 1·12	32·27 ± 1·18			
S_3 (866)	Nos.	237	268	361	−14·28	1·8767	− 7·61
	%	27·40 ± 1·02	20·91 ± 1·06	41·69 ± 1·13			
S_5 (722)	Nos.	338	138	246	+12·74	2·2347	+ 5·70
	%	46·82 ± 1·25	19·11 ± ·99	34·07 ± 1·19			
S_6 (808)	Nos.	262	287	259	+ 0·37	1·9055	+ 0·20
	%	32·43 ± 1·11	35·52 ± 1·14	32·05 ± 1·11			
Ml_1 (817)	Nos.	153	255	409	−31·33	1·8123	−17·29
	%	18·73 ± ·92	31·21 ± 1·09	50·06 ± 1·18			
Ml_2 (718)	Nos.	239	197	282	− 5·99	2·1388	− 2·80
	%	33·29 ± 1·19	27·44 ± 1·12	39·27 ± 1·23			
Mx_1 (692)	Nos.	246	315	131	+16·62	1·8439	+ 9·01
	%	35·55 ± 1·23	45·52 ± 1·28	18·93 ± 1·00			
Mx_2 (517)	Nos.	124	197	196	−13·93	2·2966	− 6·06
	%	23·98 ± 1·27	38·11 ± 1·44	37·91 ± 1·44			
Mx_3 (451)	Nos.	149	156	146	+ 0·67	1·8711	+ 0·36
	%	33·04 ± 1·49	34·59 ± 1·51	32·37 ± 1·49			
Mx_4 (545)	Nos.	193	175	177	+ 2·94	2·3789	+ 1·23
	%	35·41 ± 1·38	32·11 ± 1·35	32·48 ± 1·39			
O_7 (864)	Nos.	262	77	525	−30·44	2·0761	−14·66
	%	30·33 ± 1·06	8·91 ± ·65	60·76 ± 1·12			
O_8 (858)	Nos.	410	125	323	+10·14	2·1158	+ 4·79
	%	47·79 ± 1·15	14·57 ± ·81	37·64 ± 1·12			
O_9 (858)	Nos.	388	166	304	+ 9·79	2·0559	+ 4·76
	%	45·22 ± 1·15	19·35 ± ·91	35·43 ± 1·10			

* These are the percentages of the characters equal, not to the unit of measurement, but to the unit of grouping used in the correlation tables. The grouping unit was 1 mm. in 21 cases, 0·6 mm. in 2 cases (S_2 and S_6) and 0·5 mm. in 2 cases (T_5 and T_7).

TABLE IV. *Dominance of Right and Left Cranial Bones, estimated by Percentage Excess.*

	Right Percentage Excess			Left Percentage Excess		
	Bone	Length	$\frac{\Delta_{p_R - p_L}}{\text{p.e. of } \Delta}$	Bone	Length	$\frac{\Delta_{p_R - p_L}}{\text{p.e. of } \Delta}$
Markedly significant	Parietal	P_3	+19·38	—	—	—
	—	—	—	Malar	Ml_1	−17·29
	Temporal	T_1	+15·54	—	—	—
	Frontal	F_2	+15·16	—	—	—
	—	—	—	Occipital	O_7	−14·66
	Parietal	P_4	+13·40	—	—	—
	Parietal	P_2	+11·28	—	—	—
	Maxillary	Mx_1	+ 9·01	—	—	—
	Temporal	T_3	+ 8·46	—	—	—
	Frontal	F_1	+ 7·71	—	—	—
	—	—	—	Sphenoid	S_3	− 7·61
	—	—	—	Maxillary	Mx_2	− 6·06
Significant	Sphenoid	S_5	+ 5·70	—	—	—
	—	—	—	Temporal	T_6	− 5·55
	Occipital	O_8	+ 4·79	—	—	—
	Occipital	O_9	+ 4·76	—	—	—
	Temporal	T_4	+ 4·48	—	—	—
Probably significant	Sphenoid	S_2	+ 3·97	—	—	—
	—	—	—	Temporal	T_2	− 3·75
	—	—	—	Malar	Ml_2	− 2·80
	Temporal	T_7	+ 2·65	—	—	—
Non-significant	Maxillary	Mx_4	+ 1·23	—	—	—
	—	—	—	Temporal	T_5	− 0·59
	Maxillary	Mx_3	+ 0·36	—	—	—
	—	S_6	+ 0·20	—	—	—
	Mean Dominance Right		+ 7·54	Mean Dominance Left		− 7·29

to say. The value of $Ml_1 + T_4 + O_7$, which is very nearly the whole arc from sub-orbital point to lambda, via the asterion, is 1·62 mm. greater on the left than on the right side, while the parietal arc, P_4, from lambda to sphenion, is greater by 1·41 mm. on the right; this would seem to indicate that the dominance on the left side, if due at all to brain growth, is cerebellar, as the malar length can be less influenced by such growth.

It is one thing for homologous lengths to differ in mean size, another for homologous lengths to be highly correlated, which signifies that their deviations from their respective means are closely related. It will be now of interest to arrange the twenty-five pairs of homologous lengths in their order of correlation. This is done in Table V.

The main feature of this table is that the facial lengths are those most highly correlated, while those of the temporal, occipital and parietal regions are less closely associated. Suppose a type skull formed with the average asymmetries we have shown to exist, then if any individual skull deviated from these type asymmetries on the right side, there would be a correlated deviation in the same sense on the left side, and these deviations would be in closer accordance on the anterior or facial portion of the cranium than on parts posterior to the coronal suture.

TABLE V. *Correlations of Homologous Lengths in order of Intensity.*

Length	Correlation	Length	Correlation	Length	Correlation
Mx_1	·9766	T_3	·8503	P_4	·7909
Ml_2	·9399	S_2	·8315	S_3	·7771
F_2	·9384	T_2	·8145	T_6	·7748
Mx_2	·9278	T_1	·8133	P_3	·7112
Ml_1	·9219	T_4	·8061	P_2	·6738
Mx_3	·9134	S_5	·7962	T_5	·6729
F_1	·9059	O_7	·7954	O_8	·5379
S_6	·8776	T_7	·7937	—	—
Mx_4	·8616	O_9	·7928	—	—

(5) *Variation.*

If the right side of the cranium is on the whole significantly dominant in size, it remains to consider the distribution of the variability, absolute and relative, of the skull. Is the right or the left side the more subject to limitation in its variation; is either by reason of its functions more stringently bound to type than its opposite? Or, shall we find equality of variability in homologous lengths notwithstanding their divergence in size?

The data for answering this problem are provided by the last six columns of Table I. In the first three of these columns absolute variabilities are dealt with. We have the difference of the standard deviations on the right and the left, then the standard error of this difference, which is provided by the formula*

$$\sigma_{\sigma_R-\sigma_L} = \frac{1}{\sqrt{2n}}\sqrt{\sigma_R^2 + \sigma_L^2 - 2r_{RL}^2\sigma_R\sigma_L},$$

and in the third column we have the ratio of the difference of the standard deviations to its probable error (i.e. ·67449 × standard error). In the last three columns relative variabilities are dealt with. In the first we have the difference of the coefficients of variation for the right and left homologous lengths; in the second of these three columns we have the standard error of the difference of these coefficients provided by the formula*

$$\sigma_{V_R-V_L} = \frac{1}{\sqrt{2n}}\left\{V_R^2 + V_L^2 - 2r^2 V_R V_L + \frac{2}{(100)^2}(V_R^4 + V_L^4 - 2r V_R^2 V_L^2)\right\}^{\frac{1}{2}},$$

* Here r is the correlation of the right and left homologous lengths. Of course these formulae are only approximations suitable to large samples, such as they are in our case.

and the last column gives the ratio of the difference of the coefficients of variation to the probable error of that difference.

From the ratio columns, Table VI has been drawn up. We can draw at once certain conclusions from Table VI. It will be seen that the cases of marked significance, which were so noteworthy when we considered dominance of size, do not occur at all. In other words, laterality is not a marked feature of variability either relative or absolute. Again, while the size dominance was on the *right* in the proportion of 16 to 9, the dominance of variability is on the *left* in the

TABLE VI. *Significance of the differences of Relative and Absolute Variability on the two sides of the skull.*

	Absolute Variability				Relative Variability			
	Bone	Length	Ratio: Right in excess	Ratio: Left in excess	Bone	Length	Ratio: Right in excess	Ratio: Left in excess
Significant	Occipital	O_8	+5·80	—	Frontal	F_2	—	−5·94
	Frontal	F_2	—	−5·13	Occipital	O_8	+5·42	—
	Malar	Ml_2	—	−4·54	Malar	Ml_3	—	−4·21
Possibly significant	Malar	Ml_1	—	−3·28	Temporal	T_4	—	−2·98
	Occipital	O_9	+2·89	—	Occipital	O_9	+2·59	—
	Sphenoid	S_3	—	−2·81	Maxillary	Mx_3	—	−2·43
	Temporal	T_4	—	−2·78	—	—	—	—
	Occipital	O_7	—	−2·61	—	—	—	—
	Maxillary	Mx_3	—	−2·56	—	—	—	—
Very doubtful significance	Maxillary	Mx_4	—	−2·32	Maxillary	Mx_4	—	−2·36
	Parietal	P_2	+2·18	—	Sphenoid	S_3	—	−2·27
	Temporal	T_7	+2·05	—	Parietal	P_3	—	−2·15
	—	—	—	—	Malar	Ml_1	—	−2·01
Non-significant	Sphenoid	S_6	—	−1·58	Parietal	P_4	—	−1·83
	Temporal	T_2	+1·44	—	Temporal	T_7	+1·72	—
	Sphenoid	S_1	—	−1·42	Temporal	T_2	+1·71	—
	Parietal	P_4	—	−1·31	Sphenoid	S_2	—	−1·68
	Parietal	P_3	—	−1·30	Sphenoid	S_6	—	−1·66
	Temporal	T_1	+1·13	—	Occipital	O_7	—	−1·63
	Maxillary	Mx_1	+1·07	—	Parietal	P_2	+1·53	—
	Frontal	F_1	+0·86	—	Temporal	T_6	+0·74	—
	Maxillary	Mx_2	—	−0·64	Sphenoid	S_5	—	−0·53
	Temporal	T_5	—	−0·36	Maxillary	Mx_1	+0·46	—
	Temporal	T_3	+0·32	—	Frontal	F_1	+0·44	—
	Temporal	T_6	+0·23	—	Temporal	T_1	+0·35	—
	Sphenoid	S_5	—	−0·14	Temporal	T_3	—	−0·29
	—	—	—	—	Maxillary	Mx_2	—	−0·27
	—	—	—	—	Temporal	T_5	—	−0·26

proportion of 15 to 10. Out of those cases for which dominance is either significant or possibly significant the ratio in *favour* of the *left side* is 7 to 2 for absolute and 8 to 2 for relative variability, giving a 15 to 4 proportion instead of 15 to 10. There seems little doubt therefore that the left side is somewhat less limited to type than the right side of the skull*. Confining our attention to the really significant group we remark that the chance of equalling or exceeding $\pm 4{\cdot}21$ times the probable error in one trial is only about $\frac{1}{350}$, and therefore the probability that in 50 trials we should get six such values is exceedingly small.

We see that the undoubtedly significant group consists solely of three lengths, one from the occipital with right dominance, two with left dominance from the frontal and malar bones. With one exception, T_3, the lengths with dominance for absolute variability have the same laterality for relative variability, so that we need not distinguish between the two. Of the 25 characters the dominances in size and absolute variability have the same laterality in 15 cases, the opposite laterality in 10 cases. Of these 10 cases (judged by variability) the difference is markedly significant in one, F_2, possibly significant in two, T_4 and Mx_4, and non-significant in seven. In Mx_4 the size difference is non-significant, but it is significant in F_2 and T_4. Of the 15 cases in which the dominance in size and in variability has the same laterality, O_8 has significance for both, the malar bone measurement, Ml_2, has significance for both, Ml_1 has marked significance for size, and doubtful significance for variability; of the six quantities which are possibly or just possibly significant for variability, P_2, S_3, O_7 and O_9 are markedly significant for size, T_7 is probably significant for size and Mx_3 is non-significant for size. There are six cases in which the variability dominance has no significance. Thus in the case where the dominances are of unlike sense there are only two measurements, F_2 and T_4, in which it is almost certainly significant for both size and variability. In the case where the dominance is of like sense, the lengths O_8 and Ml_2 have adequate significance for both size and variability; O_7, O_9, S_3 and Ml_1 have doubtful significance for one or other character, and P_2, T_7 have extremely doubtful significance for variability. Accordingly we have left four lengths distributed over four bones, F_2, T_4, Ml_2 and O_8, two of which have unlike and two like dominance in size and variability. Thus it seems idle to argue from these as to any correlation, positive or negative, existing between dominance in size and in variability†. It is clear that laterality has far less influence on variability than it has on size, and less on relative variability than on absolute variability.

(6) The conclusions of this paper are of the following kind:

(i) The human skull is definitely and markedly asymmetrical. It is not a question of the bones of individual crania differing from a symmetrical type, but the type cranium is itself asymmetrical.

* The odds against such an excess of dominance on the left are about 29 to 1.

† The ratios of significance for size and for absolute variability were correlated and gave the result $0{\cdot}2939 \pm {\cdot}1232$. Thus significant greater variability was associated with significant greater size, and not, as one might *a priori* suppose, a stringent predominance of size with a lesser variability. But the correlation is under $2{\cdot}5$ times its probable error, and it is too doubtful in itself for one to say more than that there is not sufficient evidence to indicate a relation between dominance in size and variability.

(ii) Some dimensions of the cranial bones have dominance on the right side, some on the left, but on the whole the right side for size has dominance over the left. This is especially true for the frontal and parietal bones; the malar bone is the only case in which the left side has dominance for all measurements taken, and this bone has less relation to brain development.

(iii) The anterior homologous lengths, particularly those of the face and forehead, are those most highly correlated, right and left.

(iv) The order of absolute variability is much the same as that of relative variability. There are no cases of markedly significant differences in variability of right and left bones. There are only three cases of definitely significant differences of variability, one on the right and two on the left. No relation of any importance was discovered between dominance in size and dominance in variability.

(v) Whatever causes, associated with brain growth, or otherwise, lead to dominance in size of certain lateral portions of the skull, these do not appear to restrict the variability of those portions in any sensible degree. That is to say, type is differentiated laterally, but not deviations from type.

TABLE VII.

Frontal Arc Measurement F_1.

Frontal Bone, F_1. (Central Values.) Right.

Frontal Bone, F_1. (Central Values.) Left.

	82·75	83·75	84·75	85·75	86·75	87·75	88·75	89·75	90·75	91·75	92·75	93·75	94·75	95·75	96·75	97·75	98·75	99·75	100·75	101·75	102·75	103·75	104·75	105·75	106·75	107·75	108·75	109·75	110·75	111·75	112·75	113·75	114·75	115·75	116·75	117·75	118·75	
80·75	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
81·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
82·75	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
83·75	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
84·75	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
85·75	—	—	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
86·75	—	—	—	—	1	2	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5
87·75	—	—	—	—	—	1	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
88·75	—	—	—	1	—	1	5	3	4	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	16
89·75	—	—	1	—	1	2	1	5	4	4	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	21
90·75	—	—	—	—	—	—	—	2	5	6	3	2	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	20
91·75	—	—	—	—	—	—	—	1	2	4	5	5	5	3	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	26
92·75	—	—	—	—	—	—	—	2	3	7	9	8	4	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	35
93·75	—	—	—	—	—	—	—	1	3	4	7	14	13	4	4	1	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	53
94·75	—	—	—	—	—	1	—	1	—	2	3	7	11	10	9	9	2	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	57
95·75	—	—	—	—	—	—	—	—	1	2	1	6	7	22	18	9	5	5	3	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	80
96·75	—	—	—	—	—	—	—	—	—	—	—	5	6	8	25	13	10	5	4	3	2	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	83
97·75	—	—	—	—	—	—	—	—	—	—	1	2	3	9	18	22	8	8	7	5	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	85
98·75	—	—	—	—	—	—	—	—	—	—	—	—	1	1	2	4	9	14	10	5	3	2	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	53
99·75	—	—	—	—	—	—	—	—	—	—	—	1	1	1	2	3	7	11	12	4	4	4	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	52
100·75	—	—	—	—	—	—	—	—	—	—	—	—	1	1	2	3	6	6	12	7	5	2	3	2	1	—	—	—	—	—	—	—	—	—	—	—	—	51
101·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	3	5	12	15	7	6	3	—	—	1	—	—	—	—	—	—	—	—	—	—	—	53
102·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	2	7	10	13	6	4	3	3	1	—	1	—	—	—	—	—	—	—	—	—	52
103·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	2	4	6	6	7	7	3	—	1	—	1	—	—	—	—	—	—	—	—	—	38
104·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	2	1	2	3	4	3	3	1	—	—	—	—	—	—	—	—	—	—	—	20
105·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	—	1	2	3	3	3	4	4	—	—	—	—	—	—	—	—	—	—	—	23
106·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	3	3	4	2	1	1	1	—	—	—	—	—	—	—	—	17
107·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	1	1	2	3	2	1	—	1	—	—	—	—	—	—	—	13
108·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	1	—	1	3	—	—	—	—	—	—	—	—	—	7
109·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	—	2	—	—	—	—	—	—	—	—	4
110·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	2	1	—	—	—	—	—	—	—	—	—	4
111·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	2
112·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1
113·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
114·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
115·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
116·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
117·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1
118·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
	1	2	4	3	2	7	8	15	23	31	33	51	53	61	62	66	55	61	76	59	45	38	32	22	18	13	7	8	3	1	1	1	—	—	1	1	1	885

TABLE VIII.

Frontal Arc Measurement F_2.

Frontal Bone, F_2. (Central Values.) Right.

Frontal Bone, F_2. (Central Values.) Left.

	73·75	74·75	75·75	76·75	77·75	78·75	79·75	80·75	81·75	82·75	83·75	84·75	85·75	86·75	87·75	88·75	89·75	90·75	91·75	92·75	93·75	94·75	95·75	96·75	97·75	98·75	99·75	100·75	101·75	102·75	103·75	104·75	105·75	106·75	
72·75	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
73·75	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
74·75	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
75·75	—	—	—	—	1	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
76·75	—	—	2	2	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
77·75	—	—	—	1	1	2	3	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	9
78·75	—	—	—	—	2	2	4	7	4	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	22
79·75	—	—	1	—	1	3	3	2	3	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	16
80·75	—	—	—	—	—	—	2	6	5	5	4	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	23
81·75	—	—	—	—	—	—	2	1	11	11	13	7	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	45
82·75	—	—	—	—	—	—	1	—	4	9	10	18	8	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	52
83·75	—	—	—	—	—	—	—	2	3	6	8	11	13	6	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	52
84·75	—	—	—	—	—	—	—	—	1	2	5	15	16	14	7	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	63
85·75	—	—	—	—	—	—	—	—	2	2	6	12	12	21	15	5	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	76
86·75	—	—	—	—	—	—	—	—	—	1	3	4	8	14	18	8	7	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	65
87·75	—	—	—	—	—	—	—	—	—	—	—	1	4	8	16	11	12	10	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	63
88·75	—	—	—	—	—	—	—	—	—	—	—	3	3	3	6	28	21	9	3	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	78
89·75	—	—	—	—	—	—	—	—	—	—	—	1	—	2	6	3	22	11	10	8	—	—	—	—	—	—	—	—	—	—	—	—	—	—	63
90·75	—	—	—	—	—	—	—	—	—	—	—	—	1	1	4	11	11	9	9	11	8	—	—	—	—	—	—	—	—	—	—	—	—	—	65
91·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	4	6	5	5	5	5	—	—	—	—	—	—	—	—	—	—	—	—	33
92·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	2	5	9	8	7	2	—	—	—	—	—	—	—	—	—	—	—	34
93·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	3	3	4	8	3	1	1	—	—	—	—	—	—	—	—	—	25
94·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	7	6	7	6	3	1	—	—	—	—	—	—	—	—	—	33
95·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	4	3	4	2	2	—	—	—	—	—	—	—	—	18
96·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	4	5	3	1	—	—	—	—	—	—	—	—	15
97·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	1	—	1	2	—	1	—	—	—	—	—	—	—	7
98·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	2	1	—	—	1	—	—	—	—	—	—	5
99·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	1	—	1	—	—	—	—	—	—	—	4
100·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	1
101·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1	—	2
102·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	—	—	2
103·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
	1	2	3	4	7	9	17	18	34	42	50	73	65	71	74	73	80	51	39	46	35	35	19	16	11	4	2	1	1	2	—	—	1	1	887

TABLE IX.

Parietal Arc Measurement P_2.

Parietal Bone, P_2. (Central Values.) Right.

Parietal Bone, P_2. (Central Values.) Left.	90·75	91·75	92·75	93·75	94·75	95·75	96·75	97·75	98·75	99·75	100·75	101·75	102·75	103·75	104·75	105·75	106·75	107·75	108·75	109·75	110·75	111·75	112·75	113·75	114·75	115·75	116·75	117·75	118·75	119·75	120·75	121·75	122·75	123·75	124·75	125·75	126·75	127·75	128·75	129·75	
89·75	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
90·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
91·75	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
92·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
93·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
94·75	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
95·75	—	—	—	—	—	1	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
96·75	—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
97·75	—	—	—	—	—	—	—	1	—	—	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
98·75	—	—	—	—	—	—	—	—	—	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
99·75	—	—	—	—	—	—	—	—	—	—	—	—	—	2	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
100·75	—	—	—	—	—	—	—	—	1	—	1	—	—	—	1	1	1	2	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	9
101·75	—	—	—	—	—	—	—	1	—	—	—	—	3	2	2	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	10
102·75	—	—	—	—	—	—	—	—	—	—	1	—	1	—	1	1	2	4	—	1	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	13
103·75	—	—	—	—	—	—	—	—	—	—	—	1	1	—	1	2	4	—	3	1	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	16
104·75	—	—	—	—	—	—	—	—	—	—	—	1	2	2	2	3	4	3	3	—	1	2	2	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	26
105·75	—	—	—	—	—	—	—	1	1	—	—	2	—	2	2	4	4	5	3	3	2	2	3	2	2	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	40
106·75	—	—	—	—	—	—	—	—	—	—	1	—	—	—	3	3	2	6	4	6	5	2	3	2	1	1	1	1	1	—	—	—	—	—	—	—	—	—	—	—	42
107·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	8	5	8	7	9	—	3	2	2	2	1	—	—	—	—	—	—	1	—	1	—	—	—	—	51
108·75	—	—	—	—	—	—	1	—	—	—	—	1	1	—	3	1	4	6	5	6	2	3	3	4	3	3	3	3	1	1	1	—	—	—	—	—	—	—	—	—	55
109·75	—	—	—	—	—	—	—	—	—	—	1	—	—	2	1	2	2	5	8	4	3	6	3	5	1	8	2	—	2	—	—	1	—	1	—	—	—	—	—	—	57
110·75	—	—	—	—	—	—	—	—	1	—	—	—	1	—	2	2	1	4	4	3	—	5	4	6	6	8	7	1	—	1	—	1	—	1	1	—	—	—	—	—	59
111·75	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	2	2	1	—	2	3	6	2	5	5	6	4	2	1	1	1	—	1	3	—	—	—	—	—	—	48
112·75	—	—	—	—	—	—	—	—	—	—	—	1	—	2	1	3	1	2	—	2	6	2	3	4	2	1	4	3	—	1	—	1	—	—	—	—	—	—	—	—	39
113·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	1	2	1	1	6	4	5	6	1	2	5	3	—	—	3	—	1	—	—	—	—	—	—	43
114·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	3	2	5	1	2	1	4	4	5	6	2	2	2	5	—	—	—	—	—	—	—	—	47
115·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	3	1	1	1	2	4	6	3	6	2	5	3	—	—	—	1	—	—	—	—	—	40
116·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	2	1	2	6	2	4	3	2	2	—	—	1	1	1	—	—	—	—	—	29
117·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	2	3	—	—	2	4	1	2	4	1	1	1	3	1	—	—	—	—	—	—	27
118·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	6	1	2	2	2	3	—	1	1	—	1	—	1	—	—	21
119·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	2	—	—	—	1	1	2	1	1	1	2	2	—	—	—	—	—	1	—	17
120·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	1	—	1	1	1	2	1	1	1	1	—	—	1	2	—	1	—	—	—	—	15
121·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	1	2	—	—	—	2	—	1	—	—	—	8
122·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	1	1	1	—	2	—	1	1	—	—	8
123·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	2	1	—	—	—	1	—	—	—	—	—	1	—	—	—	6
124·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	—	—	—	—	—	—	—	1	3
125·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	—	—	—	2
126·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	1
127·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	1
128·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1	2
	1	—	—	—	—	1	1	4	5	1	4	10	9	12	25	32	36	46	46	44	46	46	36	44	46	58	44	38	22	21	17	16	10	13	7	5	3	2	1	2	754

TABLE X.

Parietal Arc Measurement P_3.

Parietal Bone, P_3. (Central Values.) Right.

Parietal Bone, P_3. (Central Values.) Left.	149·75	150·75	151·75	152·75	153·75	154·75	155·75	156·75	157·75	158·75	159·75	160·75	161·75	162·75	163·75	164·75	165·75	166·75	167·75	168·75	169·75	170·75	171·75	172·75	173·75	174·75	175·75	176·75	177·75	178·75	179·75	180·75	181·75	182·75	183·75	184·75	185·75	
142·75	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
143·75	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
144·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
145·75	—	—	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
146·75	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
147·75	—	—	—	—	—	—	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
148·75	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
149·75	—	1	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
150·75	—	—	1	1	—	1	1	2	—	2	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	9
151·75	—	2	1	1	2	1	1	1	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	11
152·75	—	—	—	1	—	1	—	3	4	1	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	12
153·75	—	—	2	1	1	1	2	2	1	2	1	2	3	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	19
154·75	—	—	—	—	—	2	1	2	1	3	3	3	2	—	1	4	—	—	—	1	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	24
155·75	—	—	—	—	2	1	2	5	3	4	—	3	3	1	—	1	—	—	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	28
156·75	—	—	—	—	1	—	—	3	3	3	3	5	2	3	1	1	2	1	1	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	31
157·75	1	—	—	3	—	—	—	—	3	3	4	6	4	2	2	1	2	1	2	4	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	39
158·75	—	—	—	—	—	—	1	—	3	3	3	6	3	1	5	3	2	3	3	2	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	41
159·75	—	—	—	—	—	2	1	—	1	3	6	4	2	4	5	6	7	4	1	1	—	2	1	1	1	—	—	1	—	—	—	—	—	—	—	—	—	51
160·75	—	—	—	—	1	1	—	2	2	4	2	2	7	6	6	2	6	1	2	1	1	1	1	—	1	1	—	1	—	—	—	—	—	—	—	—	—	59
161·75	—	—	—	—	1	1	—	2	5	1	1	2	8	8	1	4	8	4	3	5	2	—	2	—	—	—	—	1	—	—	—	—	—	—	—	—	—	58
162·75	—	—	—	—	—	—	—	—	1	2	2	3	3	6	4	2	4	2	6	2	4	2	3	—	1	—	—	—	—	—	—	—	—	—	—	—	—	49
163·75	—	—	—	—	—	—	—	1	—	—	2	1	4	4	4	7	6	5	5	—	2	6	2	1	2	—	—	—	—	—	—	—	—	—	—	—	—	54
164·75	—	—	—	—	—	—	—	—	2	—	2	2	1	6	6	4	5	7	6	7	4	5	1	—	2	—	—	1	—	—	—	—	—	—	—	—	—	60
165·75	—	—	—	—	—	—	—	—	—	1	1	4	2	1	6	5	9	9	4	2	4	5	1	1	2	—	1	—	—	—	—	—	—	—	—	—	—	58
166·75	—	—	—	—	—	—	—	—	1	—	—	1	2	1	1	1	3	9	4	1	6	5	5	—	3	—	—	—	—	—	—	—	—	—	—	—	—	41
167·75	—	—	—	—	—	—	—	—	—	—	1	—	1	2	1	—	3	6	2	3	7	2	—	2	—	1	2	—	1	2	—	—	—	—	—	—	1	39
168·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	4	1	3	3	6	2	4	2	4	2	3	—	—	—	1	—	—	—	—	—	—	38
169·75	—	—	—	—	—	—	—	—	—	1	—	—	—	—	2	3	4	1	5	4	—	4	3	1	2	2	—	—	—	1	—	—	—	—	—	—	—	36
170·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	4	2	2	—	5	6	3	1	—	—	—	1	1	—	—	—	—	—	—	—	32
171·75	—	—	—	—	—	—	—	—	—	—	1	—	—	2	—	1	2	—	1	1	—	1	3	1	—	4	1	—	—	—	—	—	—	—	—	—	—	22
172·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	1	1	3	3	1	2	—	1	1	—	—	—	—	—	—	17
173·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	2	1	5	1	1	—	—	1	1	1	—	—	—	1	—	—	—	16
174·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	1	1	—	1	1	—	—	—	—	—	—	—	—	—	7
175·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1	1	1	—	—	—	—	—	—	—	—	4
176·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	3
177·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
178·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2
179·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	3
180·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1
181·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
182·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
183·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
184·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	1
	3	3	4	10	10	11	10	23	30	34	34	44	49	49	47	50	70	61	54	41	52	49	42	15	25	14	11	9	5	6	3	1	—	1	1	—	2	873

TABLE XI.

Parietal Arc Measurement P_4.

Parietal Bone, P_4. (Central Values.) Right.

Parietal Bone, P_4. (Central Values.) Left.	156·75	157·75	158·75	159·75	160·75	161·75	162·75	163·75	164·75	165·75	166·75	167·75	168·75	169·75	170·75	171·75	172·75	173·75	174·75	175·75	176·75	177·75	178·75	179·75	180·75	181·75	182·75	183·75	184·75	185·75	186·75	187·75	188·75	189·75	190·75	191·75	192·75	193·75	
155·75	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
156·75	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
157·75	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
158·75	—	—	—	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
159·75	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
160·75	—	—	—	—	—	—	1	—	—	—	1	—	—	1	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
161·75	—	—	1	—	—	1	—	—	1	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5
162·75	—	—	—	—	—	—	—	1	1	—	2	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5
163·75	—	—	—	—	1	—	—	—	—	—	—	—	1	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5
164·75	—	—	—	—	—	1	—	—	—	—	1	1	—	1	4	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8
165·75	—	—	—	—	—	—	—	—	—	—	—	1	—	4	—	—	4	2	—	1	—	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	14
166·75	—	—	—	—	—	—	—	—	—	1	—	2	1	4	—	—	—	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	10
167·75	—	—	—	—	—	—	—	1	—	1	1	1	—	2	4	4	—	3	3	—	1	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	23
168·75	—	—	—	—	—	—	1	—	—	1	2	3	2	1	1	2	1	1	2	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	19
169·75	—	—	—	—	—	—	—	—	—	—	—	4	2	4	5	2	5	3	3	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	31
170·75	—	—	—	—	—	—	—	—	1	—	1	1	4	3	4	1	4	3	4	3	1	1	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	33
171·75	—	—	—	—	—	—	—	—	—	—	—	—	—	2	4	7	7	5	6	6	2	—	2	4	1	—	1	—	2	—	—	—	—	—	—	—	—	—	48
172·75	—	—	—	—	—	—	—	—	—	—	—	—	—	2	2	2	4	7	4	4	3	5	3	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	38
173·75	—	—	—	—	—	—	—	—	—	—	1	—	1	3	3	3	1	5	5	3	2	2	2	2	2	—	—	—	—	—	—	—	—	—	—	—	—	—	35
174·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	4	4	4	7	7	3	5	4	3	—	—	1	—	—	—	—	—	—	—	—	—	—	43
175·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	3	3	1	7	5	6	3	4	3	3	1	—	—	—	—	—	—	—	—	—	—	—	42
176·75	—	—	—	—	—	—	—	—	—	—	—	1	—	1	1	1	—	1	9	4	5	10	6	8	6	7	—	4	1	—	—	1	—	—	—	—	—	—	66
177·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	6	4	4	7	8	7	5	6	1	3	3	—	—	1	1	—	—	—	—	—	57
178·75	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	2	3	3	5	3	8	8	2	4	2	5	1	2	—	—	—	—	—	—	—	—	50
179·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	1	2	—	3	4	4	6	5	3	4	3	2	1	—	—	1	—	—	—	—	—	—	41
180·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	3	1	2	6	3	3	6	5	—	5	3	2	1	—	—	—	—	—	—	41
181·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	4	1	1	3	2	1	1	4	1	1	—	—	—	—	—	—	—	20
182·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1	1	—	3	2	1	4	1	—	1	1	1	—	—	—	—	17
183·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	2	1	1	—	2	1	5	1	1	1	—	2	—	—	—	—	1	19
184·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	1	—	3	3	2	2	2	2	1	1	2	—	1	—	—	—	—	23
185·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	1	—	2	1	1	—	2	2	2	—	—	18
186·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	2	1	—	—	—	—	—	—	1	—	—	6
187·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	1	1	2	—	—	—	1	—	—	—	6
188·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	1	1	3
189·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1	1	3
190·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	1	3
	1	—	1	3	3	2	2	2	4	4	9	15	11	30	31	30	35	42	51	51	44	53	57	55	36	42	20	27	26	12	9	10	4	4	3	3	2	4	738

TABLE XII.

Temporal Chord Measurement T_1.

Temporal Bone, T_1. (Central Values.) Right.

Temporal Bone, T_1. (Central Values.) Left.	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	
74	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
75	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
76	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
77	—	—	1	2	2	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6
78	—	—	—	2	2	1	1	1	3	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	11
79	—	—	—	2	4	2	1	4	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	16
80	—	—	—	—	4	2	2	3	4	2	2	—	1	—	—	—	—	—	—	—	—	—	—	—	—	20
81	—	—	—	1	1	—	6	5	11	7	5	1	—	—	—	—	—	—	—	—	—	—	—	—	—	37
82	—	—	—	—	—	3	8	11	8	14	6	4	1	1	—	1	—	—	—	—	—	—	—	—	—	57
83	—	—	—	—	1	1	7	5	10	14	10	11	4	3	—	—	—	—	—	—	—	—	—	—	—	66
84	—	—	—	—	—	—	2	6	9	9	18	15	11	6	8	2	1	—	—	—	—	—	—	—	—	87
85	—	—	—	—	—	—	2	6	5	9	12	11	16	11	5	3	4	—	—	—	—	—	—	—	—	84
86	—	—	—	—	—	—	—	—	2	9	8	22	19	15	14	11	3	2	1	—	—	—	—	—	—	106
87	—	—	—	—	—	—	1	2	1	2	8	11	15	15	9	4	4	3	1	—	—	1	—	—	—	77
88	—	—	—	—	—	—	—	2	—	1	1	7	8	7	16	10	9	5	2	2	—	—	—	—	—	70
89	—	—	—	—	—	—	—	1	1	—	—	1	10	13	7	10	10	7	4	—	—	—	—	—	—	64
90	—	—	—	—	—	—	—	—	—	—	2	2	4	7	4	11	6	10	3	2	—	—	2	—	—	53
91	—	—	—	—	—	—	—	—	—	—	—	3	2	3	4	2	6	4	8	2	4	—	1	—	—	39
92	—	—	—	—	—	—	—	—	—	—	—	1	—	2	3	5	6	4	6	1	1	1	—	—	—	30
93	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	—	3	3	—	2	—	—	1	1	—	13
94	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	1	—	1	1	1	4	—	—	—	—	10
95	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1	1	1	2	1	—	—	—	7
96	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	2	—	—	4
97	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	2
98	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
99	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	2
	1	1	2	8	15	9	31	46	56	68	73	89	91	86	73	60	52	40	27	12	12	3	7	3	1	866

TABLE XIII.

Temporal Chord Measurement T_2.

Temporal Bone, T_1. (Central Values.) Right.

Temporal Bone, T_2. (Central Values.) Left.

	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	
35	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
36	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
37	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
38	1	—	—	2	2	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
39	—	—	2	1	—	2	2	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	9
40	—	—	—	1	1	2	4	8	2	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	21
41	—	1	1	—	3	4	6	5	5	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	27
42	—	—	1	1	3	3	8	9	6	6	7	2	—	—	1	1	—	—	—	—	—	—	—	—	48
43	—	—	—	—	—	1	5	13	9	6	6	2	2	1	—	1	—	—	—	—	—	—	—	—	46
44	—	—	—	—	1	1	4	5	8	19	9	10	5	1	1	1	1	—	—	—	—	—	—	—	66
45	—	—	—	—	—	—	3	2	16	15	8	16	9	4	4	2	1	1	—	—	—	—	—	—	81
46	—	—	—	—	—	—	2	2	8	11	15	16	15	11	9	3	2	—	—	—	—	—	—	—	94
47	—	—	—	—	—	—	—	1	2	11	13	13	16	9	14	8	3	1	—	—	—	—	—	—	91
48	—	—	—	—	—	—	—	1	1	6	8	17	9	14	14	8	5	—	—	—	—	—	—	—	83
49	—	—	—	—	—	—	—	—	—	1	5	7	11	14	8	10	8	4	1	2	—	—	—	—	71
50	—	—	—	—	—	—	—	—	—	2	3	6	4	13	16	13	9	2	3	3	1	1	—	—	76
51	—	—	—	—	—	—	—	—	—	—	1	1	2	6	6	11	12	8	4	2	1	1	—	—	55
52	—	—	—	—	—	—	—	—	—	—	—	2	2	2	4	5	4	6	4	2	1	—	—	—	32
53	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	3	3	2	3	—	3	—	—	16
54	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	1	2	3	2	1	—	—	—	11
55	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	—	2	2	—	—	—	—	—	—	7
56	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	4	—	—	—	7
57	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	2
58	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	2
59	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
	2	1	6	5	10	14	34	46	59	80	77	92	76	76	80	66	51	30	18	16	9	7	—	1	856

TABLE XIV.

Temporal Chord Measurement T_3.

Temporal Bone, T_3. (Central Values.) Right.

Temporal Bone, T_3. (Central Values.) Left.

	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	
54	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
55	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
56	1	2	—	—	1	—	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
57	—	—	1	4	—	1	3	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	11
58	—	—	—	1	4	5	2	2	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	17
59	—	—	—	3	4	4	3	3	2	5	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	25
60	—	—	—	2	—	7	7	6	4	8	4	1	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	41
61	—	—	—	—	2	3	6	5	8	14	7	3	1	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	51
62	—	—	—	—	1	1	2	4	8	9	8	6	2	3	—	—	1	1	—	—	—	—	—	—	—	—	—	—	46
63	—	—	—	—	1	1	3	4	10	12	7	12	8	3	1	—	1	—	—	—	—	—	—	—	—	—	—	—	63
64	—	—	—	—	—	—	2	2	3	5	14	14	11	6	5	2	—	—	—	—	—	—	—	—	—	—	—	—	64
65	—	—	—	—	—	—	2	1	7	4	12	16	12	9	4	4	4	—	1	—	—	—	—	—	—	—	—	—	78
66	—	—	—	—	—	—	—	—	1	3	6	11	18	23	14	4	4	—	—	—	—	—	—	—	—	—	—	—	84
67	—	—	—	—	—	—	1	—	3	2	8	9	16	20	9	16	6	3	5	—	—	—	—	—	—	—	—	—	96
68	—	—	—	—	—	—	—	—	—	2	1	8	9	4	12	18	7	6	5	2	1	—	—	—	—	—	—	—	84
69	—	—	—	—	—	—	—	—	—	1	1	6	6	8	7	10	10	9	6	2	1	1	—	—	—	—	—	—	68
70	—	—	—	—	—	—	—	—	—	—	—	2	3	4	7	1	13	6	6	1	—	3	—	—	—	—	—	—	44
71	—	—	—	—	—	—	—	—	—	—	—	—	—	1	3	8	3	8	7	4	2	2	1	1	1	—	—	—	41
72	—	—	—	—	—	—	—	—	—	—	—	—	1	1	2	2	5	4	3	3	2	—	—	—	—	—	—	—	25
73	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	—	2	2	3	2	2	1	1	—	—	—	—	16
74	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	1	2	1	1	—	1	—	—	—	9
75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	1	1	2	2	—	1	—	—	—	—	10
76	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	2	—	—	—	—	—	[illegible]
77	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	[illegible]
78	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	2
79	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	1
80	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1
81	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	2
	1	3	1	10	14	22	34	28	49	65	69	86	88	84	64	63	55	39	37	17	13	11	7	5	3	2	—	1	871

TABLE XV.

Temporal Arc Measurement T_4.

Temporal Bone, T_4. (Central Values.) Right.

Temporal Bone, T_4. (Central Values.) Left.

	85·75	86·75	87·75	88·75	89·75	90·75	91·75	92·75	93·75	94·75	95·75	96·75	97·75	98·75	99·75	100·75	101·75	102·75	103·75	104·75	105·75	106·75	107·75	108·75	109·75	110·75	111·75	112·75	113·75	114·75	
81·75	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
82·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
83·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
84·75	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
85·75	1	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
86·75	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
87·75	—	—	—	—	1	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
88·75	—	—	—	—	—	—	1	1	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
89·75	—	—	—	1	1	1	4	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8
90·75	—	—	—	—	—	1	2	—	1	2	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
91·75	—	—	—	—	—	4	3	2	—	2	1	1	—	1	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	15
92·75	—	—	—	—	—	—	1	4	—	2	1	3	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	12
93·75	—	—	—	—	—	1	2	1	3	7	7	—	2	—	2	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	27
94·75	—	—	—	—	—	1	—	2	2	5	6	6	4	2	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	30
95·75	—	—	—	—	—	1	2	—	2	7	10	8	14	3	1	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	50
96·75	—	—	—	—	—	2	—	—	4	6	10	9	12	7	5	3	4	—	1	1	—	—	—	—	—	—	—	—	—	—	64
97·75	—	—	—	—	—	—	1	1	1	3	7	8	17	12	8	7	3	3	1	—	—	—	—	—	—	—	—	—	—	—	72
98·75	—	—	—	—	—	—	—	—	1	—	3	7	6	7	10	10	7	3	1	2	—	2	1	—	—	—	—	—	—	—	60
99·75	—	—	—	—	—	—	—	—	—	—	1	2	7	10	13	9	6	7	3	—	1	—	—	—	—	—	—	—	—	—	59
100·75	—	—	—	—	1	—	—	—	—	—	—	3	6	7	8	6	13	7	3	2	2	—	—	—	—	—	—	—	—	—	58
101·75	—	—	—	—	—	—	—	—	1	—	—	3	7	4	8	9	9	6	6	1	—	—	—	—	—	—	—	—	—	—	54
102·75	—	—	—	—	—	—	—	—	—	—	2	1	—	3	2	9	5	10	7	5	2	—	1	—	—	—	—	—	—	—	47
103·75	—	—	—	—	—	—	—	—	—	—	—	—	2	1	2	4	2	10	3	6	7	6	2	—	—	—	—	—	—	—	45
104·75	—	—	—	—	—	—	—	—	—	—	—	1	—	2	1	2	7	5	5	8	2	6	4	—	—	—	—	—	—	—	43
105·75	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	—	2	3	1	4	4	2	3	1	2	—	—	—	—	—	24
106·75	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	1	2	—	2	3	4	1	—	—	—	—	—	—	15
107·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	5	2	3	1	—	—	—	—	—	—	12
108·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	—	—	—	—	—	1	—	—	4
109·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	—	1	—	—	—	—	—	3
110·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	—	—	—	1	—	—	3
111·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
112·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
113·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1
114·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
115·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
	1	1	—	1	4	12	18	12	16	34	40	54	78	62	60	61	61	56	34	31	27	25	19	3	3	—	1	2	—	1	726

TABLE XVI.

Temporal Chord Measurement T_5.

Temporal Bone, T_5. (Central Values.) Right.

Temporal Bone, T_5. (Central Values.) Left.

	38·25	38·75	39·25	39·75	40·25	40·75	41·25	41·75	42·25	42·75	43·25	43·75	44·25	44·75	45·25	45·75	46·25	46·75	47·25	47·75	48·25	48·75	49·25	49·75	50·25	50·75	51·25	51·75	52·25	52·75	53·25	53·75	54·25	54·75	55·25	
37·75	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
38·25	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
38·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
39·25	—	1	—	1	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
39·75	—	—	—	1	—	—	1	—	—	1	1	—	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
40·25	—	—	—	—	—	3	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6
40·75	1	1	—	1	2	—	—	1	1	1	—	2	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	12
41·25	—	—	—	—	—	4	2	2	—	—	—	2	1	1	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	13
41·75	—	—	—	2	1	1	3	2	4	2	1	2	2	2	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	23
42·25	—	—	—	—	—	—	3	4	—	3	3	4	1	3	1	—	1	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	25
42·75	—	—	—	—	—	—	1	4	1	2	6	1	2	6	2	2	1	—	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	30
43·25	—	—	—	1	1	1	—	2	3	1	3	7	1	6	5	3	2	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	38
43·75	—	1	—	—	—	3	2	—	2	3	3	2	8	9	5	5	4	—	2	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	51
44·25	—	—	—	—	1	—	3	2	1	4	2	2	7	6	2	3	4	3	3	1	—	1	3	1	—	—	—	—	—	—	—	—	—	—	—	50
44·75	—	—	—	—	—	—	2	—	3	3	2	6	5	5	6	5	7	6	5	3	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	60
45·25	—	—	—	—	—	—	1	—	1	1	5	7	6	4	9	5	4	3	2	4	3	—	1	—	—	—	1	—	—	—	—	—	—	—	—	57
45·75	—	—	—	—	—	—	—	—	1	2	2	1	7	9	6	5	3	6	7	1	4	—	2	2	—	1	3	—	—	—	—	—	—	—	—	62
46·25	—	—	—	—	—	—	—	1	—	3	5	2	3	3	8	5	11	7	3	9	1	2	—	3	—	1	—	—	—	1	—	—	—	—	—	68
46·75	—	—	—	—	—	—	—	1	1	1	2	6	5	4	4	7	4	11	6	4	8	2	4	—	—	—	—	—	—	—	—	—	—	—	—	66
47·25	—	—	—	—	—	—	—	—	—	—	1	3	—	2	2	2	6	11	6	6	3	3	2	2	1	1	—	—	—	—	—	—	—	—	—	50
47·75	—	—	—	—	1	—	—	—	1	—	1	1	—	2	4	4	2	1	5	9	1	5	2	1	2	1	—	3	—	—	—	—	—	—	—	46
48·25	—	—	—	—	—	—	—	—	—	1	—	—	—	1	4	7	1	5	4	4	3	1	2	4	1	4	—	—	—	—	—	—	—	—	1	43
48·75	—	—	—	—	—	—	—	—	—	—	1	2	—	4	1	3	1	3	2	3	5	3	2	5	1	—	1	1	1	—	1	—	—	—	—	40
49·25	—	—	—	—	—	—	—	1	—	—	—	—	—	1	1	2	2	1	5	2	4	4	3	3	4	2	1	—	—	—	—	—	—	—	—	36
49·75	—	—	—	—	—	—	—	—	—	—	—	1	—	1	1	—	1	2	2	4	1	2	4	3	3	1	—	—	—	—	—	—	1	—	—	27
50·25	—	—	—	—	—	—	—	—	—	1	—	—	1	—	1	—	—	1	1	1	1	—	1	—	2	2	4	—	—	—	—	—	—	—	—	16
50·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	2	1	1	—	—	1	1	2	—	—	—	1	—	—	—	—	—	10
51·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	1	—	1	1	1	2	—	—	—	—	1	—	—	—	—	—	—	9
51·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	3	1	—	1	3	—	—	—	—	—	—	—	—	—	10
52·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	1	—	1	—	—	—	—	—	—	—	4
52·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	1	—	—	—	1	1	1	—	—	—	—	—	6
53·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	1
53·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	1	—	—	—	—	—	1	—	3
54·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
54·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	1	—	2
	1	3	—	6	7	12	19	21	21	30	38	51	51	70	64	60	57	60	56	55	35	32	31	30	18	20	10	7	3	3	1	—	1	2	1	876

TABLE XVII.

Temporal Chord Measurement T_8.

Temporal Bone, T_8. (Central Values.) Right.

Rows: Temporal Bone, T_8. (Central Values.) Left.

	33·5	34·5	35·5	36·5	37·5	38·5	39·5	40·5	41·5	42·5	43·5	44·5	45·5	46·5	47·5	48·5	49·5	50·5	51·5	52·5	53·5	54·5	55·5	56·5	57·5	58·5	
33·5	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
34·5	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
35·5	—	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
36·5	—	1	—	1	—	1	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5
37·5	—	—	—	2	1	2	—	2	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6
38·5	—	—	—	—	—	2	3	1	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8
39·5	—	—	—	—	1	4	5	4	3	3	—	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	23
40·5	—	—	—	1	1	1	5	5	9	2	4	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	30
41·5	—	—	—	—	2	3	7	8	8	5	4	5	1	—	1	—	—	—	—	—	—	—	—	—	—	—	44
42·5	—	—	—	—	2	2	6	1	10	12	8	4	5	3	2	—	1	—	—	—	—	—	—	—	—	—	56
43·5	—	—	—	—	1	1	4	3	10	6	16	17	10	9	9	—	—	—	—	—	—	—	—	—	—	—	86
44·5	—	—	—	—	—	—	—	4	9	8	8	16	9	7	3	3	4	2	—	—	—	—	—	—	—	—	78
45·5	—	—	—	—	—	1	—	2	2	3	8	13	16	9	12	6	3	3	2	—	—	—	—	—	—	—	80
46·5	—	—	—	—	1	—	1	1	4	4	16	9	19	15	10	9	6	3	2	—	—	—	—	—	—	—	100
47·5	—	—	—	—	—	—	—	—	1	4	8	6	11	14	10	11	9	2	2	—	—	—	—	—	—	—	78
48·5	—	—	—	—	—	—	—	—	—	—	3	6	4	8	15	11	6	1	—	2	—	—	—	—	—	—	56
49·5	—	—	—	—	—	—	—	—	—	1	3	3	4	4	8	9	13	8	5	2	1	1	—	1	—	—	63
50·5	—	—	—	—	—	—	—	—	—	—	1	2	2	5	6	4	13	3	7	1	—	—	—	—	—	—	44
51·5	—	—	—	—	—	—	—	—	—	—	—	—	2	—	6	5	2	4	3	4	2	—	—	—	—	—	27
52·5	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	1	1	2	1	3	—	1	1	—	—	—	12
53·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	3	1	3	1	2	—	1	—	—	—	13
54·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	2	1	—	—	—	—	—	4
55·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	2	—	—	—	—	—	4
56·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	1	1	—	2	6
57·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1
	1	2	—	4	9	17	32	32	56	53	80	86	84	74	64	61	62	29	25	17	9	3	3	2	—	2	827

TABLE XVIII.

Temporal Chord Measurement T_7.

Temporal Bone, T_7. (Central Values.) Right.

Rows: Temporal Bone, T_7. (Central Values.) Left.

	27·25	27·75	28·25	28·75	29·25	29·75	30·25	30·75	31·25	31·75	32·25	32·75	33·25	33·75	34·25	34·75	35·25	35·75	36·25	36·75	37·25	37·75	38·25	38·75	39·25	39·75	40·25	40·75	41·25	41·75	42·25	42·75	43·25	43·75	
26·25	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
26·75	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
27·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
27·75	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
28·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
28·75	1	—	—	—	—	—	1	1	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5
29·25	—	—	—	—	—	—	—	1	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
29·75	—	—	—	—	—	1	—	—	—	1	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
30·25	—	—	—	1	2	—	—	—	1	—	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6
30·75	—	—	—	1	1	—	1	1	—	5	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	11
31·25	—	—	1	1	—	1	1	2	1	—	2	2	3	1	1	—	1	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	16
31·75	—	—	—	—	—	4	1	1	—	3	2	—	2	2	2	1	1	1	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	22
32·25	—	—	—	—	—	—	1	—	—	2	7	4	2	3	3	—	6	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	28
32·75	1	—	—	—	—	—	—	1	—	2	1	3	4	5	4	—	—	2	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	25
33·25	—	—	—	—	1	—	—	—	—	3	1	5	2	6	3	6	2	3	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	33
33·75	—	—	—	—	—	—	—	1	3	2	2	2	4	4	2	3	6	1	2	4	—	2	2	—	—	—	—	—	—	—	—	—	—	—	41
34·25	—	—	—	—	—	—	—	—	—	3	4	3	3	2	3	7	5	4	3	1	2	1	—	2	—	—	—	—	—	—	—	—	—	—	46
34·75	—	—	—	—	—	—	—	1	—	1	2	3	4	7	6	6	6	6	3	4	5	—	2	—	1	2	—	—	—	—	—	—	—	—	56
35·25	—	—	—	—	—	—	1	—	—	2	—	1	1	5	3	9	7	8	7	2	1	1	—	4	—	—	—	—	—	—	—	—	—	—	52
35·75	—	—	—	—	—	—	—	—	1	—	—	1	1	3	4	7	12	4	6	4	3	3	3	3	—	—	—	1	—	—	—	—	—	—	56
36·25	—	—	—	—	—	—	—	—	1	1	—	—	3	9	6	6	4	8	11	12	2	4	3	3	—	1	—	—	—	—	—	—	—	—	63
36·75	—	—	—	—	—	—	—	—	—	1	—	—	—	—	3	6	4	9	7	10	7	11	7	2	—	2	—	—	—	1	—	—	—	—	72
37·25	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	1	4	3	5	6	7	6	2	6	1	2	2	2	—	1	—	—	—	—	60
37·75	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	—	2	3	6	6	3	7	7	5	1	1	2	1	1	—	—	—	—	—	47
38·25	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	—	1	—	5	6	2	3	3	2	8	—	—	1	—	1	2	1	—	40
38·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	2	1	3	1	7	2	4	6	4	2	1	—	—	—	—	1	—	36
39·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	4	—	1	1	3	2	5	3	1	2	1	—	2	—	—	—	27
39·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	2	—	2	3	6	6	2	1	1	3	1	—	—	29
40·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	3	1	—	3	3	4	—	—	—	—	1	1	—	17
40·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	3	1	2	2	4	1	1	—	2	—	—	17
41·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	3	1	—	2	1	2	—	—	—	10
41·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	1	—	1	2	—	—	—	—	6
42·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	1	—	1	1	6
42·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1	—	—	—	—	—	1	3
43·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	—	—	—	—	3
43·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	1	—	—	—	—	—	—	2
44·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1
44·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	2
	2	—	3	3	4	8	6	9	8	29	28	25	32	46	42	54	59	56	57	60	42	51	40	41	26	36	21	16	11	8	10	6	5	2	840

TABLE XIX.

Sphenoidal Chord Measurement S_2.

Sphenoid Bone, S_2. (Central Values.) Right.

Sphenoid Bone, S_2. (Central Values.) Left.	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	
65	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
66	—	—	1	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
67	—	—	—	2	2	2	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8
68	—	2	—	1	3	4	3	2	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	17
69	—	—	—	—	1	5	3	3	3	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	17
70	—	—	—	1	—	4	6	6	7	4	1	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	31
71	—	—	—	1	1	2	6	3	8	12	2	5	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	42
72	—	—	—	—	—	—	5	7	20	8	14	3	5	1	2	1	—	—	—	—	—	—	—	—	—	—	—	66
73	—	—	—	—	—	—	4	8	14	21	18	12	9	2	—	—	—	—	—	—	—	—	—	—	—	—	—	88
74	—	—	—	—	—	—	2	2	3	13	20	11	10	8	4	1	—	—	—	—	—	—	—	—	—	—	—	74
75	—	—	—	—	—	—	1	1	5	12	10	18	9	7	4	4	2	—	—	—	—	—	—	—	—	—	—	73
76	—	—	—	—	—	—	—	—	1	5	9	6	24	12	8	4	2	1	1	—	—	—	—	—	—	—	—	73
77	—	—	—	—	—	—	—	—	—	1	8	11	10	19	5	7	3	4	2	—	—	—	—	—	—	—	—	70
78	—	—	—	—	—	—	—	—	—	1	2	6	7	6	15	5	3	3	—	—	—	—	—	—	—	—	—	48
79	—	—	—	—	—	—	—	—	—	—	—	3	3	6	8	19	4	1	1	—	—	—	—	—	—	—	—	45
80	—	—	—	—	—	—	—	—	—	—	1	—	1	3	5	1	6	—	1	2	1	—	—	—	—	—	—	21
81	—	—	—	—	—	—	—	—	—	—	—	—	1	1	2	3	4	6	3	1	—	—	—	—	—	—	—	21
82	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	2	5	—	2	—	—	—	—	—	—	12
83	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	—	—	—	1	2	—	—	—	—	—	6
84	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	—	—	—	—	—	—	—	2
85	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
86	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
87	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
88	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
89	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
90	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
	1	2	1	5	8	17	30	35	62	80	85	77	80	66	54	47	28	17	13	4	4	2	—	—	—	—	1	719

TABLE XX.

Sphenoidal Chord Measurement S_3.

Sphenoid Bone, S_3. (Central Values.) Right.

Sphenoid Bone, S_3. (Central Values.) Left.	30·25	30·85	31·45	32·05	32·65	33·25	33·85	34·45	35·05	35·65	36·25	36·85	37·45	38·05	38·65	39·25	39·85	40·45	41·05	41·65	42·25	42·85	43·45	
30·25	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
30·85	—	1	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
31·45	1	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
32·05	—	—	—	2	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
32·65	—	1	2	3	—	4	4	1	3	1	1	—	—	—	—	—	—	—	—	—	—	—	—	20
33·25	1	—	—	2	4	3	7	4	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	23
33·85	—	1	—	3	3	3	6	14	4	4	3	1	—	1	—	—	—	—	—	—	—	—	—	43
34·45	—	—	1	2	3	5	8	13	7	12	11	—	2	—	—	—	—	—	—	—	—	—	—	64
35·05	—	—	—	1	1	2	7	11	24	5	16	3	1	—	1	—	—	—	—	—	—	—	—	72
35·65	—	—	—	1	—	3	6	11	18	17	19	16	3	4	—	1	—	—	—	—	—	—	—	99
36·25	—	—	—	1	2	2	4	10	9	16	32	15	10	3	1	2	—	—	—	—	—	—	—	107
36·85	—	—	—	—	1	1	2	1	4	16	21	19	13	11	2	2	—	—	—	—	—	—	—	93
37·45	—	—	—	—	—	—	2	—	2	10	20	22	23	15	3	2	2	1	—	—	—	—	—	102
38·05	—	—	—	—	—	—	—	—	2	6	5	14	13	18	8	6	1	—	—	—	—	—	—	73
38·65	—	—	—	—	—	—	—	—	—	3	3	8	8	6	5	4	1	—	1	—	—	—	—	39
39·25	—	—	—	—	—	—	—	—	2	4	3	8	3	12	5	8	3	3	—	—	—	—	—	51
39·85	—	—	—	—	—	—	—	—	—	—	—	2	6	4	7	10	—	2	3	1	—	—	—	35
40·45	—	—	—	—	—	—	—	—	—	—	—	1	2	1	3	3	1	7	1	—	—	—	—	19
41·05	—	—	—	—	—	—	—	—	—	—	—	1	1	1	2	4	—	—	1	—	—	—	1	11
41·65	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1	—	—	—	2
42·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1
42·85	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1
43·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1
	2	4	5	15	15	25	46	65	76	94	135	110	85	76	38	42	8	13	8	3	—	—	1	866

TABLE XXI.

Sphenoidal Chord Measurement S_5.

Sphenoid Bone, S_5. (Central Values.) Right.

Sphenoid Bone, S_5. (Central Values.) Left.	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	
46	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
47	—	1	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
48	—	—	2	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
49	—	—	1	1	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
50	—	—	2	3	6	1	2	—	—	—	—	2	1	—	—	—	—	—	—	—	—	—	17
51	1	—	2	1	6	2	2	4	—	—	2	1	—	—	—	—	—	—	—	—	—	—	21
52	—	—	1	2	6	9	14	8	6	2	1	—	1	—	—	—	—	—	—	—	—	—	50
53	—	—	—	4	3	7	9	8	7	6	4	2	1	1	—	—	—	—	—	—	—	—	52
54	—	—	—	—	3	5	14	16	13	13	6	2	2	2	—	—	—	—	—	—	—	—	76
55	—	—	—	—	1	3	4	14	13	16	9	6	2	4	—	2	—	—	—	—	—	—	74
56	—	—	—	2	—	3	4	7	12	16	15	11	10	2	1	—	—	—	—	—	—	—	83
57	—	—	—	—	—	1	2	3	5	14	11	18	12	6	4	—	—	—	—	—	—	—	76
58	—	—	—	—	—	—	—	3	4	9	13	10	10	13	2	3	—	—	—	—	—	—	67
59	—	—	—	—	—	—	1	3	1	3	8	10	14	9	9	—	1	—	—	—	—	—	59
60	—	—	—	—	—	—	—	1	—	3	4	5	10	10	14	7	—	2	—	—	—	—	56
61	—	—	—	—	—	—	—	1	—	1	1	4	6	3	10	1	6	—	—	—	—	—	33
62	—	—	—	—	—	—	—	—	—	1	—	3	1	6	3	2	3	1	2	—	—	—	22
63	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	3	1	—	—	—	—	7
64	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	2	3	1	—	—	—	8
65	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	1	1	1	—	—	5
66	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	—	2
67	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
68	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1	2
	2	1	8	14	26	34	52	69	61	84	74	74	70	57	46	17	16	10	4	1	1	1	722

TABLE XXII.

Sphenoidal Chord Measurement S_6.

Sphenoid Bone, S_6. (Central Values.) Right.

Sphenoid Bone, S_6. (Central Values.) Left.	28·45	29·05	29·65	30·25	30·85	31·45	32·05	32·65	33·25	33·85	34·45	35·05	35·65	36·25	36·85	37·45	38·05	38·65	39·25	39·85	40·45	41·05	41·65	42·25	
28·45	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
29·05	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
29·65	—	—	1	—	—	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
30·25	—	—	1	2	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5
30·85	1	1	1	—	3	4	2	1	1	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	16
31·45	—	—	—	4	—	5	6	2	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	18
32·05	—	—	—	—	2	1	8	5	4	6	3	1	1	—	—	—	—	—	—	—	—	—	—	—	31
32·65	—	—	—	—	2	4	4	8	6	6	1	1	—	—	—	—	—	—	—	—	—	—	—	—	32
33·25	—	—	—	—	—	—	2	11	23	11	5	4	4	1	1	—	—	—	—	—	—	—	—	—	62
33·85	—	—	—	—	—	1	4	9	10	17	10	10	4	2	1	—	—	—	—	—	—	—	—	—	68
34·45	—	—	—	—	—	—	1	—	6	11	15	10	15	4	1	1	1	—	—	—	—	—	—	—	65
35·05	—	—	—	—	—	—	1	4	7	7	16	18	15	11	6	2	4	1	2	—	—	—	—	—	94
35·65	—	—	—	—	—	—	—	—	—	5	8	13	21	16	6	3	1	—	—	—	—	—	—	—	73
36·25	—	—	—	—	—	—	—	—	—	3	8	13	13	22	9	3	2	1	1	—	—	—	—	—	75
36·85	—	—	—	—	—	—	—	—	—	—	2	9	10	13	13	8	7	6	2	1	—	—	—	—	71
37·45	—	—	—	—	—	—	—	—	—	—	—	—	2	8	10	7	10	7	3	2	—	—	—	—	49
38·05	—	—	—	—	—	—	—	—	—	—	—	—	4	3	5	6	14	2	1	1	1	—	—	—	37
38·65	—	—	—	—	—	—	—	—	—	—	—	—	—	3	1	4	8	13	2	—	1	1	—	—	33
39·25	—	—	—	—	—	—	—	—	—	—	—	—	—	2	1	4	2	3	7	3	4	1	—	—	27
39·85	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	1	3	5	1	4	—	1	17
40·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	2	1	3	—	2	1	—	11
41·05	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	4	1	—	—	—	6
41·65	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	1	—	1	—	1	5
42·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	1	3
42·85	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	2
	1	2	4	7	9	18	29	40	57	69	68	79	89	85	54	41	52	36	23	21	9	9	2	4	808

TABLE XXIII.

Malar Arc Measurement Ml_1.

Malar Bone, Ml_1. (Central Values.) Right.

Malar Bone, Ml_1. (Central Values.) Left.

	37·75	38·75	39·75	40·75	41·75	42·75	43·75	44·75	45·75	46·75	47·75	48·75	49·75	50·75	51·75	52·75	53·75	54·75	55·75	56·75	57·75	58·75	59·75	
37·75	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
38·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
39·75	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
40·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
41·75	—	—	—	1	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
42·75	—	—	—	2	3	4	3	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	14
43·75	—	—	—	—	—	5	3	3	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	12
44·75	—	—	—	—	—	—	4	5	6	1	2	—	—	—	—	—	—	—	—	—	—	—	—	18
45·75	—	—	—	—	—	1	4	13	15	9	8	1	—	—	—	—	—	—	—	—	—	—	—	46
46·75	—	—	—	—	—	—	2	8	23	18	13	2	1	—	—	—	—	—	—	—	—	—	—	67
47·75	—	—	—	—	—	—	—	3	14	17	20	15	4	2	—	—	—	—	—	—	—	—	—	84
48·75	—	—	—	—	—	—	—	—	5	10	20	29	15	2	1	—	—	—	—	—	—	—	—	82
49·75	—	—	—	—	—	—	—	—	1	6	16	20	34	14	1	—	—	—	—	—	—	—	—	101
50·75	—	—	—	—	—	—	—	—	—	2	1	8	40	41	13	3	—	—	—	—	—	—	—	108
51·75	—	—	—	—	—	—	—	—	—	—	1	5	17	27	23	15	2	—	—	—	—	—	—	90
52·75	—	—	—	—	—	—	—	—	—	—	1	3	4	16	17	22	6	2	1	—	—	—	—	72
53·75	—	—	—	—	—	—	—	—	—	—	—	1	5	5	8	11	13	1	—	—	—	—	—	44
54·75	—	—	—	—	—	—	—	—	—	—	—	—	—	2	6	6	9	7	3	—	—	—	—	33
55·75	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	3	2	8	7	1	—	—	—	22
56·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	2	2	—	1	1	9
57·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	1	—	1	—	4
58·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	2
59·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
60·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	2
61·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1
	1	—	—	4	5	11	16	34	65	63	86	93	120	110	69	60	33	20	15	5	2	4	1	817

TABLE XXIV.

Malar Arc Measurement Ml_2.

Malar Bone, Ml_2. (Central Values.) Right.

Malar Bone, Ml_2. (Central Values.) Left.

	37·75	38·75	39·75	40·75	41·75	42·75	43·75	44·75	45·75	46·75	47·75	48·75	49·75	50·75	51·75	52·75	53·75	54·75	55·75	56·75	57·75	58·75	59·75	60·75	61·75	62·75	63·75	64·75	65·75	66·75	67·75	68·75	69·75	70·75	71·75	72·75	73·75	
37·75	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
38·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
39·75	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
40·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
41·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
42·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
43·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
44·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
45·75	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
46·75	—	—	—	—	—	—	—	—	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
47·75	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
48·75	—	—	—	—	—	—	—	—	—	—	2	2	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6
49·75	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
50·75	—	—	—	—	—	—	—	—	—	—	—	—	1	3	1	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
51·75	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	8	3	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	14
52·75	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	7	4	2	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	17
53·75	—	—	—	—	—	—	—	—	—	—	—	—	1	—	3	3	6	7	4	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	24
54·75	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	5	9	7	2	4	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	29
55·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	5	10	10	9	5	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	43
56·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	4	7	12	17	4	2	1	—	1	—	—	—	—	—	—	—	—	—	—	—	50
57·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	5	7	17	23	16	7	5	—	—	—	—	—	—	—	—	—	—	—	—	—	81
58·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	8	9	20	12	7	5	—	—	—	—	—	—	—	—	—	—	—	—	62
59·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	9	14	15	13	2	2	—	—	—	—	—	—	—	—	—	—	—	58
60·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8	8	10	17	13	4	2	—	—	—	—	—	—	—	—	—	—	57
61·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	3	8	12	19	9	6	—	—	—	—	—	—	—	—	—	—	58
62·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4	3	6	14	17	7	3	—	—	—	—	—	—	—	—	—	54
63·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	5	5	13	9	6	3	—	—	—	—	—	—	—	—	44
64·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	2	6	7	6	2	3	—	—	—	—	—	—	—	28
65·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	4	6	7	6	2	—	—	—	—	—	—	28
66·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	4	2	6	—	—	—	—	—	—	—	14
67·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	2	5	1	2	—	—	—	—	—	12
68·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	2	1	2	3	—	—	—	—	—	10
69·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	1	3	1	—	—	7
70·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	—	2
71·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
72·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	2
	1	—	—	1	—	—	1	—	—	2	3	3	4	8	7	23	24	38	38	58	71	70	61	69	61	55	37	28	18	21	7	5	2	4	2	—	1	718

TABLE XXV.

Maxillary Chord Measurement Mx_1.

Maxillary Bone, Mx_1. (Central Values.) Right.

Maxillary Bone, Mx_1. (Central Values.) Left.

	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	
52	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
53	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
54	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
55	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
56	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
57	—	—	—	—	1	1	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5
58	—	—	—	—	—	5	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
59	—	—	—	—	1	2	3	4	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	10
60	—	—	—	—	—	—	5	9	4	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	21
61	—	—	—	—	—	—	—	2	12	7	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	21
62	—	—	—	—	—	—	—	—	3	19	5	3	2	—	—	—	—	—	—	—	—	—	—	—	—	—	32
63	—	—	—	—	—	—	—	—	1	9	26	10	1	1	—	—	—	—	—	—	—	—	—	—	—	—	48
64	—	—	—	—	—	—	—	—	—	1	12	28	19	3	2	—	—	—	—	—	—	—	—	—	—	—	65
65	—	—	—	—	—	—	—	—	—	—	1	10	22	24	5	1	—	—	—	—	—	—	—	—	—	—	63
66	—	—	—	—	—	—	—	—	—	—	—	2	9	29	25	3	—	—	—	—	—	—	—	—	—	—	68
67	—	—	—	—	—	—	—	—	—	—	—	—	2	6	36	21	6	2	—	—	—	—	—	—	—	—	73
68	—	—	—	—	—	—	—	—	—	—	—	—	2	1	11	28	17	3	1	—	—	—	—	—	—	—	63
69	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5	12	26	22	—	2	—	—	—	—	—	—	67
70	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8	20	12	2	—	—	—	—	—	—	42
71	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	9	20	7	3	—	—	—	—	—	41
72	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	16	6	2	—	—	—	—	27
73	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	2	5	1	—	—	—	11
74	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	6	1	—	—	—	10
75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	2	—	—	—	4
76	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	—	—	3
77	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	1	4
	2	—	3	—	4	8	12	16	20	39	44	53	57	64	84	65	59	56	36	30	14	15	5	3	2	1	692

TABLE XXVI.

Maxillary Chord Measurement Mx_2.

Maxillary Bone, Mx_2. (Central Values.) Right.

Maxillary Bone, Mx_2. (Central Values.) Left.

	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	
47	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
48	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
49	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
50	—	—	—	1	7	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	10
51	—	—	—	1	5	6	5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	17
52	—	—	—	—	3	9	8	10	1	1	—	—	—	—	—	—	—	—	—	—	—	—	32
53	—	—	—	—	1	6	8	7	11	1	5	—	—	—	—	—	—	—	—	—	—	—	39
54	—	—	—	—	—	—	6	7	20	11	13	—	—	—	—	—	—	—	—	—	—	—	57
55	—	—	—	—	—	—	1	4	10	34	15	1	—	—	—	—	—	—	—	—	—	—	65
56	—	—	—	—	—	—	—	1	12	18	37	10	—	—	—	—	—	—	—	—	—	—	78
57	—	—	—	—	—	—	—	—	1	8	13	26	7	2	—	—	—	—	—	—	—	—	57
58	—	—	—	—	—	—	—	—	—	2	9	17	23	3	7	—	—	—	—	—	—	—	61
59	—	—	—	—	—	—	—	—	—	—	1	8	9	15	12	—	—	—	—	—	—	—	45
60	—	—	—	—	—	—	—	—	—	—	—	1	5	11	5	1	—	—	—	—	—	—	23
61	—	—	—	—	—	—	—	—	—	—	—	—	1	1	5	5	3	—	—	—	—	—	15
62	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	3	2	—	1	—	—	—	8
63	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	2
64	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	2
65	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
66	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
67	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
	2	—	1	3	17	22	29	29	55	75	93	63	45	33	30	10	7	1	1	—	—	1	517

TABLE XXVII.

Maxillary Chord Measurement Mx_3.

Maxillary Bone, Mx_3. (Central Values.) Right.

Maxillary Bone, Mx_3. (Central Values.) Left.

	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	
31	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1
32	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
33	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
34	—	—	—	—	4	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6
35	—	—	—	—	1	3	1	3	1	—	—	—	—	—	—	—	—	—	—	—	9
36	—	—	—	—	1	3	2	8	3	—	—	—	—	—	—	—	—	—	—	—	17
37	—	—	—	—	1	—	5	13	8	2	—	1	—	—	—	—	—	—	—	—	30
38	—	—	—	—	—	1	1	8	13	9	7	—	—	—	—	—	—	—	—	—	39
39	—	—	—	—	—	—	—	4	5	21	8	1	3	1	—	—	—	—	—	—	43
40	—	—	—	—	—	—	1	2	7	6	25	13	5	—	—	—	—	—	—	—	59
41	—	—	—	—	—	—	—	—	2	4	14	15	17	6	2	—	—	—	—	—	60
42	—	—	—	—	—	—	—	—	—	1	8	11	16	7	2	—	—	—	—	—	45
43	—	—	—	—	—	—	—	—	—	1	3	3	5	12	13	2	—	—	—	—	39
44	—	—	—	—	—	—	—	—	—	—	1	—	3	11	14	4	3	—	—	—	36
45	—	—	—	—	—	—	—	—	—	—	—	1	3	4	8	9	5	—	—	—	30
46	—	—	—	—	—	—	—	—	—	—	—	—	—	2	1	1	3	4	3	1	15
47	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	3	2	—	—	8
48	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	1	—	3
49	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	1	3	6
50	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
51	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
	1	—	—	1	8	10	11	38	39	44	66	45	52	43	41	21	15	6	5	5	451

TABLE XXVIII.

Maxillary Chord Measurement Mx_4.

Maxillary Bone, Mx_4. (Central Values.) Right.

Maxillary Bone, Mx_4. (Central Values.) Left.	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	
30	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
31	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
32	—	1	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
33	—	—	1	7	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	9
34	—	—	6	6	6	4	1	1	—	—	—	—	—	—	—	—	—	—	—	—	24
35	—	2	1	5	5	14	4	2	—	—	—	—	—	—	—	—	—	—	—	—	33
36	—	—	—	4	6	16	12	6	2	3	—	—	—	—	—	—	—	—	—	—	49
37	—	—	—	1	6	17	32	27	7	6	1	2	—	—	—	—	—	—	—	—	99
38	—	—	—	—	2	7	16	29	14	10	—	—	—	—	—	—	—	—	—	—	78
39	—	—	—	—	1	2	6	15	25	15	7	2	—	—	—	—	—	—	—	—	73
40	—	—	—	—	—	—	4	6	15	20	10	4	1	—	—	—	—	—	—	—	60
41	—	—	—	—	—	—	—	—	6	12	18	9	—	—	—	—	—	—	—	—	45
42	—	—	—	—	—	—	—	2	6	6	8	13	3	1	1	—	—	—	—	—	40
43	—	—	—	—	—	—	—	—	1	2	1	1	4	2	—	—	—	—	—	—	11
44	—	—	—	—	—	—	—	—	—	—	2	4	—	1	2	—	—	—	—	—	9
45	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	1	—	—	—	—	4
46	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	—	—	—	—	2
47	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	—	2
48	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
49	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
	2	3	10	23	27	61	75	88	76	74	47	35	10	5	5	1	2	—	—	1	545

TABLE XXIX.

Occipital Arc Measurement O_7.

Occipital Bone, O_7. (Central Values.) Right.

Occipital Bone, O_7. (Central Values.) Left.	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100	101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	
85	1	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
86	1	1	—	—	2	2	—	—	2	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	9
87	—	—	2	—	—	—	2	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6
88	—	—	—	—	3	—	—	—	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6
89	1	—	—	—	1	2	1	1	3	1	2	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	14
90	—	—	—	—	3	2	1	1	4	2	1	—	2	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	18
91	—	—	1	1	3	2	3	2	3	1	4	2	1	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	25
92	—	1	1	—	1	2	3	3	4	4	2	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	23
93	—	—	—	1	4	6	2	3	7	3	6	2	3	2	2	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	43
94	—	—	1	2	—	1	5	4	4	3	5	6	3	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	36
95	—	—	—	—	—	3	3	4	9	7	13	6	6	2	4	6	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	64
96	—	—	—	—	—	2	1	4	3	—	11	4	7	7	6	3	1	1	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	52
97	—	—	—	—	—	—	1	—	1	3	7	4	5	3	6	2	4	1	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	39
98	—	—	—	1	—	1	—	1	3	5	8	11	5	7	10	7	3	2	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	65
99	—	—	—	—	—	1	—	—	2	1	5	5	6	9	8	2	6	1	2	2	3	—	—	—	—	—	—	—	—	—	—	—	—	—	53
100	—	—	—	—	—	—	—	2	1	1	2	5	7	9	8	7	4	10	6	8	1	2	—	—	—	—	—	—	—	—	—	—	—	—	73
101	—	—	—	—	—	—	—	—	1	1	—	4	3	3	6	4	13	4	5	4	2	2	2	1	—	—	—	—	—	—	—	—	—	—	55
102	—	—	—	—	—	—	—	—	—	—	1	1	5	8	3	8	6	7	7	8	4	6	2	1	—	—	—	1	—	—	—	—	—	—	68
103	—	—	—	—	—	—	—	—	—	2	1	1	3	7	4	3	4	6	3	2	2	1	—	1	—	—	—	—	—	—	—	—	—	—	40
104	—	—	—	—	—	—	—	—	—	—	—	1	3	4	2	2	3	2	6	3	1	1	4	2	—	—	—	—	—	—	—	—	—	—	34
105	—	—	—	—	—	—	—	—	—	—	—	—	3	—	3	1	3	7	7	4	6	4	3	2	—	—	1	—	—	—	—	—	—	—	44
106	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	1	3	3	6	2	2	1	—	1	2	—	—	1	—	—	—	—	—	—	24
107	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	2	3	2	3	2	2	2	1	1	—	—	—	—	—	—	—	20
108	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	1	—	3	1	2	2	3	—	—	—	—	2	—	—	—	—	—	—	16
109	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	2	—	—	—	3	2	—	2	3	—	—	—	—	—	—	—	—	13
110	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	1	1	—	—	—	—	—	—	—	—	—	—	—	5
111	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	1	—	—	—	—	—	—	—	—	3
112	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1	—	—	—	—	1	—	—	—	1	—	4
113	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	2
114	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	—	—	—	—	2
115	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	2
116	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	2
117	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
118	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
119	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
120	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1
	3	2	5	6	18	24	22	26	49	36	68	54	65	61	69	49	54	48	51	39	28	27	21	13	7	6	3	4	1	1	—	—	3	1	864

TABLE XXX.

Occipital Arc Measurement O_3.

Occipital Bone, O_2. (Central Values.) Right.

Occipital Bone, O_2. (Central Values.) Left.	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	
54	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
55	—	—	—	1	—	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
56	—	—	—	—	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
57	—	—	—	—	1	1	1	1	1	1	1	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	9
58	—	—	—	—	—	—	1	2	6	6	6	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	23
59	1	—	—	—	—	1	3	4	5	4	7	2	6	3	1	—	—	—	—	—	—	—	—	—	—	—	—	—	37
60	—	—	—	—	1	1	7	7	11	15	11	8	8	6	3	4	2	—	—	—	—	1	—	—	—	—	—	—	85
61	—	—	—	—	2	4	2	7	5	13	7	8	14	5	4	4	—	1	—	2	—	—	—	—	—	—	—	—	78
62	—	—	—	—	—	1	4	8	8	11	17	15	12	11	7	8	2	1	1	—	—	—	2	—	1	—	—	—	102
63	—	—	—	1	—	—	3	1	3	10	21	17	19	12	11	6	3	1	1	—	—	2	1	1	—	—	—	—	118
64	—	—	—	—	—	1	—	4	5	9	11	9	11	7	10	4	1	1	1	2	—	—	—	1	—	—	—	1	78
65	—	—	—	1	—	—	1	1	4	3	5	11	5	14	12	6	7	2	1	2	1	1	—	1	1	—	—	—	79
66	—	—	—	—	—	—	—	1	1	2	9	8	10	14	8	11	4	5	1	—	—	1	—	1	—	—	—	—	78
67	—	—	—	—	—	—	1	—	—	1	1	2	14	7	6	7	11	6	4	4	2	—	—	—	—	—	—	—	66
68	—	—	—	—	—	—	—	1	—	—	1	—	2	5	6	5	5	5	3	1	3	—	—	—	—	—	—	—	37
69	—	1	—	—	—	—	—	—	1	—	2	2	3	3	3	2	2	8	2	2	—	1	—	—	—	—	—	—	32
70	—	—	—	—	1	—	—	—	—	—	—	1	1	2	1	1	5	1	4	1	—	—	—	—	—	—	—	—	18
71	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	1	1	2	1	1	—	1	—	—	—	—	—	—	9
72	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	1	2	—	—	—	—	—	—	—	5
73	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1	1	—	—	—	—	—	—	—	—	3
74	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
76	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
77	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1
	1	1	—	3	5	13	23	37	40	75	100	83	106	91	75	55	44	33	21	17	8	7	3	4	2	—	1	1	858

TABLE XXXI.

Occipital Chord Measurement O_9.

Occipital Bone, O_2. (Central Values.) Right.

Occipital Bone, O_2. (Central Values.) Left.	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	
64	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
65	1	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
66	—	1	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
67	3	1	1	1	—	—	—	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	9
68	—	2	1	2	4	2	2	2	2	—	—	—	—	—	—	—	—	—	—	—	—	—	17
69	—	1	1	3	2	1	4	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	15
70	—	—	—	2	6	7	7	6	3	5	2	2	1	—	—	—	—	—	—	—	—	—	41
71	—	—	1	—	1	13	10	11	10	8	6	2	1	—	1	—	—	—	—	—	—	—	64
72	—	—	1	1	2	4	14	24	12	8	14	3	—	—	—	—	—	—	—	—	—	—	83
73	—	—	—	1	3	9	11	21	22	8	14	11	6	—	—	—	—	—	—	—	—	—	106
74	—	—	1	—	2	2	7	7	14	20	20	21	8	4	1	2	—	—	—	—	—	—	109
75	—	—	—	—	—	—	3	5	9	16	16	20	16	11	3	3	—	—	—	—	—	—	103
76	—	—	—	—	—	—	—	1	6	7	17	18	16	4	7	2	3	—	1	—	—	—	82
77	—	—	—	—	—	—	—	3	—	8	10	13	17	15	6	4	—	—	—	—	—	—	76
78	—	—	—	—	—	—	—	1	2	—	7	13	5	11	6	8	8	—	—	—	—	—	59
79	—	—	—	—	—	—	—	—	1	—	1	3	9	3	4	9	8	—	—	—	—	—	38
80	—	—	—	—	—	—	—	—	—	—	—	1	—	2	3	4	4	2	—	—	—	—	16
81	—	—	—	—	—	—	—	—	—	—	—	2	1	1	1	4	4	2	1	—	—	—	16
82	—	—	—	—	—	—	—	—	—	—	1	1	—	—	1	3	1	—	—	1	1	—	9
83	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	—	—	1	1	—	4
84	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	2
85	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
86	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	2
87	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1
	4	5	8	10	22	38	59	87	81	80	108	110	80	52	33	40	26	4	3	2	3	3	858

THE MEAN AND SECOND MOMENT COEFFICIENT OF THE MULTIPLE CORRELATION COEFFICIENT, IN SAMPLES FROM A NORMAL POPULATION.

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Our knowledge regarding the sampling distribution of the multiple correlation coefficient has been very greatly increased in recent years. It has been known since 1924 that, for the special case of zero correlation in the universe, the distribution of R for samples from a normal population is given by*

$$df = \frac{(a+b-1)!}{(a-1)!\,(b-1)!}(R^2)^{a-1}(1-R^2)^{b-1}\,d(R^2) \quad \text{..............(1)},$$

where a is put, for convenience, for one-half the number of degrees of freedom due to the regression function (i.e. the number of independent variates), and b for one-half the number of degrees of freedom due to deviations from the regression function (i.e. the total number in the sample less the total number of variates). Tables exist for determining the probability of occurrence of a given R from this distribution, and extend to six independent variates and for a size of sample of about 100†. More recently the general distribution of R has been reached by Dr R. A. Fisher‡, and a table, appropriate for large samples, has been furnished whereby the experimenter may, by suitable transformations, determine approximately the significance of an observed R in relation to a given multiple correlation in the universe, exact account being taken of the positive bias of small observed multiple correlations. It is an interesting mathematical exercise, not altogether devoid of practical interest, to use Fisher's distribution to determine the exact nature of this bias, i.e. the amount by which the mean value of R (or R^2, which as we shall see is the more amenable to analysis) is in excess of the true correlation ρ (or ρ^2) existing in the universe. The purpose of the first section of this paper is to determine the mean value of R^2. Later, the analysis is extended to the derivation of the second moment coefficient, or variance, of R^2, although the utility of this quantity, for a distribution which is far from normal, is not so great as would at first sight appear. In both cases the results are compared with Hall's large sample approximations§.

* R. A. Fisher, *Phil. Trans.* B, Vol. 213, 1924, pp. 89—142.

† J. Wishart, *Quart. Journ. Roy. Met. Soc.* Vol. LIV. 1928, pp. 258—259.

‡ R. A. Fisher, *Proc. Roy. Soc.* A, Vol. 121, 1928, pp. 654—673.

§ P. Hall, *Biometrika*, Vol. XIX. 1927, pp. 100—109.

Fisher's general distribution is

$$df = \frac{(a+b-1)!}{(a-1)!(b-1)!}(1-\rho^2)^{a+b} \cdot F(a+b, a+b, a, \rho^2 R^2) \cdot (R^2)^{a-1}(1-R^2)^{b-1} d(R^2) \quad \text{.........(2)},$$

and he notes that, when $2b$ is even, we may use the Euler transformation of the hypergeometric function to obtain the distribution in the form

$$df = \frac{(a+b-1)!}{(a-1)!(b-1)!} \frac{(1-\rho^2)^{a+b}}{(1-\rho^2 R^2)^{a+2b}} F(-b, -b, a, \rho^2 R^2) \cdot (R^2)^{a-1}(1-R^2)^{b-1} d(R^2) \quad \text{.........(3)},$$

giving a terminating series.

The fact that, for $2b$ even, he has given the probability integral enables us without a great deal of difficulty to determine for the special cases $b=1$, 2 and 3 the first and second moments of the distribution (3), and thence to infer the general result for any b, which is probably true without any restrictions as to whether $2b$ is even or odd.

A. *Determination of Mean Value of R^2.*

We may conveniently put $R^2\rho^2 = x$.

Case 1. $b=1$. The distribution is

$$df = \frac{(1-\rho^2)^{a+1}}{(\rho^2)^a} \frac{(a+x)x^{a-1}}{(1-x)^{a+2}} dx \quad \text{.........................(4)}.$$

We now multiply by R^2, i.e. by x/ρ^2, and integrate with respect to x from 0 to ρ^2. Noting that the indefinite integral of (4) is

$$\frac{(1-\rho^2)^{a+1}}{(\rho^2)^a} \frac{x^a}{(1-x)^{a+1}},$$

we have

$$\bar{R}^2 = \left(\frac{1-\rho^2}{\rho^2}\right)^{a+1} \int_0^{\rho^2} x\, d\left(\frac{x^a}{(1-x)^{a+1}}\right) = 1 - \left(\frac{1-\rho^2}{\rho^2}\right)^{a+1} \int_0^{\rho^2} \frac{x^a dx}{(1-x)^{a+1}} \quad \text{...(5)},$$

on integrating by parts. Leaving the result in this form meantime we shall consider other cases.

Case 2. $b=2$. The distribution is

$$df = \frac{(1-\rho^2)^{a+2}}{(\rho^2)^a} \cdot \frac{\{a(a+1)+4(a+1)x+2x^2\}\, x^{a-1}(1-x/\rho^2)}{(1-x)^{a+4}} dx \text{......(6)},$$

and the indefinite integral

$$\frac{(1-\rho^2)^{a+2}}{(\rho^2)^a} \left\{\frac{(a+2)(1-x/\rho^2)}{(1-x)^{a+3}} - \frac{1-2x/\rho^2}{(1-x)^{a+2}}\right\} x^a,$$

which may be written

$$\frac{(1-\rho^2)^{a+2}}{(\rho^2)^a} \left\{\frac{(a+1+x)(1-x/\rho^2)}{(1-x)^{a+3}} + \frac{x/\rho^2}{(1-x)^{a+2}}\right\} x^a.$$

We have, then,

$$\bar{R}^2 = \frac{(1-\rho^2)^{a+2}}{(\rho^2)^{a+1}} \int_0^{\rho^2} x\, d\left\{\frac{(a+1+x)(1-x/\rho^2)\,x^a}{(1-x)^{a+3}} + \frac{x^{a+1}/\rho^2}{(1-x)^{a+2}}\right\}$$

$$= 1 - \frac{(1-\rho^2)^{a+2}}{(\rho^2)^{a+1}} \int_0^{\rho^2} \frac{(a+1+x)(1-x/\rho^2)\,x^a}{(1-x)^{a+3}}\,dx - \left(\frac{1-\rho^2}{\rho^2}\right)^{a+2} \int_0^{\rho^2} \frac{x^{a+1}}{(1-x)^{a+2}}\,dx,$$

on integrating by parts,

$$= 1 - \frac{(1-\rho^2)^{a+2}}{(\rho^2)^{a+1}} \int_0^{\rho^2} (1-x/\rho^2)\, d\left\{\frac{x^{a+1}}{(1-x)^{a+2}}\right\} - \left(\frac{1-\rho^2}{\rho^2}\right)^{a+2} \int_0^{\rho^2} \frac{x^{a+1}}{(1-x)^{a+2}}\,dx,$$

from the integral of (4) on replacing a by $a+1$,

$$= 1 - 2\left(\frac{1-\rho^2}{\rho^2}\right)^{a+2} \int_0^{\rho^2} \frac{x^{a+1}}{(1-x)^{a+2}}\,dx \quad \text{.....................(7)},$$

on further integrating by parts.

CASE 3. $b=3$. The distribution is

$$df = \frac{(1-\rho^2)^{a+3}}{2(\rho^2)^a}$$
$$\times \frac{\{a(a+1)(a+2)+9(a+1)(a+2)x+18(a+2)x^2+6x^3\}}{(1-x)^{a+6}}\,x^{a-1}(1-x/\rho^2)^2\,dx \quad (8),$$

and the indefinite integral

$$\frac{(1-\rho^2)^{a+3}}{2(\rho^2)^a}\left\{\frac{(a+3)(a+4)(1-x/\rho^2)^2}{(1-x)^{a+5}} - \frac{2(a+3)(2-3x/\rho^2)(1-x/\rho^2)}{(1-x)^{a+4}} + \frac{2(1-3x/\rho^2+3x^2/\rho^4)}{(1-x)^{a+3}}\right\}x^a.$$

This may be written

$$\frac{(1-\rho^2)^{a+3}}{2(\rho^2)^a}\left[\frac{\{(a+1)(a+2)+4(a+2)x+2x^2\}(1-x/\rho^2)^2}{(1-x)^{a+5}} + \frac{2x/\rho^2(a+2+x)(1-x/\rho^2)}{(1-x)^{a+4}} + \frac{2x^2/\rho^4}{(1-x)^{a+3}}\right]x^a.$$

For the mean value of R^2 we have, on integrating by parts,

$$\bar{R}^2 = 1 - \frac{(1-\rho^2)^{a+3}}{2(\rho^2)^{a+1}} \int_0^{\rho^2} \frac{\{(a+1)(a+2)+4(a+2)x+2x^2\}\,x^a(1-x/\rho^2)^2}{(1-x)^{a+5}}\,dx$$
$$- \frac{(1-\rho^2)^{a+3}}{(\rho^2)^{a+2}} \int_0^{\rho^2} \frac{(a+2+x)\,x^{a+1}(1-x/\rho^2)}{(1-x)^{a+4}}\,dx - \left(\frac{1-\rho^2}{\rho^2}\right)^{a+3} \int_0^{\rho^2} \frac{x^{a+2}}{(1-x)^{a+3}}\,dx.$$

Now utilising the integral of (6) and replacing a by $a+1$, we may write the first of these integrals in the form

$$\frac{(1-\rho^2)^{a+3}}{2(\rho^2)^{a+1}} \int_0^{\rho^2} (1-x/\rho^2)\, d\left\{\frac{(a+2+x)(1-x/\rho^2)\,x^{a+1}}{(1-x)^{a+4}} + \frac{x^{a+2}/\rho^2}{(1-x)^{a+3}}\right\}$$
$$= \frac{(1-\rho^2)^{a+3}}{2(\rho^2)^{a+2}} \int_0^{\rho^2} \frac{(a+2+x)(1-x/\rho^2)\,x^{a+1}}{(1-x)^{a+4}}\,dx + \frac{1}{2}\left(\frac{1-\rho^2}{\rho^2}\right)^{a+3} \int_0^{\rho^2} \frac{x^{a+2}}{(1-x)^{a+3}}\,dx,$$

on integrating by parts. It follows that

$$\bar{R}^2 = 1 - \frac{3}{2}\,\frac{(1-\rho^2)^{a+3}}{(\rho^2)^{a+2}} \int_0^{\rho^2} \frac{(a+2+x)\,x^{a+1}(1-x/\rho^2)}{(1-x)^{a+4}}\,dx - \frac{3}{2}\left(\frac{1-\rho^2}{\rho^2}\right)^{a+3} \int_0^{\rho^2} \frac{x^{a+2}}{(1-x)^{a+3}}\,dx.$$

The reduced integral may be simplified by using the integral of (4), in which a is now replaced by $a+2$, and is

$$\frac{3}{2}\frac{(1-\rho^2)^{a+3}}{(\rho^2)^{a+2}}\int_0^{\rho^2}(1-x/\rho^2)\,d\left\{\frac{x^{a+2}}{(1-x)^{a+3}}\right\}=\frac{3}{2}\left(\frac{1-\rho^2}{\rho^2}\right)^{a+3}\int_0^{\rho^2}\frac{x^{a+2}}{(1-x)^{a+3}}\,dx,$$

on integrating by parts. Finally we have

$$\bar{R}^2=1-3\left(\frac{1-\rho^2}{\rho^2}\right)^{a+3}\int_0^{\rho^2}\frac{x^{a+2}}{(1-x)^{a+3}}\,dx \quad\text{......................(9).}$$

It is evident from (5), (7) and (9) that the general result for any integral b is

$$\bar{R}^2=1-b\left(\frac{1-\rho^2}{\rho^2}\right)^{a+b}\int_0^{\rho^2}\frac{x^{a+b-1}}{(1-x)^{a+b}}\,dx \quad\text{..................(10).}$$

Now when x is less than unity the denominator may be expanded in a convergent series, which when integrated term by term yields the result

$$\int_0^{\rho^2}\frac{x^{a+b-1}}{(1-x)^{a+b}}\,dx=\left[\frac{x^{a+b}}{a+b}\left\{1+\frac{(a+b)^2}{a+b+1}x+\frac{(a+b)^2(a+b+1)^2}{2!\,(a+b+1)(a+b+2)}x^2+\dots\right\}\right]_0^{\rho^2}$$

$$=\frac{(\rho^2)^{a+b}}{a+b}F(a+b,\,a+b,\,a+b+1,\,\rho^2)$$

$$=\frac{1}{a+b}\frac{(\rho^2)^{a+b}}{(1-\rho^2)^{a+b-1}}F(1,\,1,\,a+b+1,\,\rho^2) \quad\text{......................(11),}$$

using the Euler transformation of the hypergeometric series. The series in (11) is absolutely convergent even for $\rho^2=1$, since $a+b-1$ is always positive*. We therefore have

$$\bar{R}^2=1-\frac{b}{a+b}(1-\rho^2)F(1,\,1,\,a+b+1,\,\rho^2) \quad\text{..............(12)}$$

as our final form. Since x (or ρ^2) may take the value unity in the limiting case, some consideration is necessary as to the validity of the solution we have reached for the integral in (10), where the integrand may become infinite. In this case the important part of the integral is the denominator, and we have

$$b\left(\frac{1-\rho^2}{\rho^2}\right)^{a+b}\int_0^{\rho^2}\frac{\text{const.}\,dx}{(1-x)^{a+b}}=\text{const.}\left(\frac{1-\rho^2}{\rho^2}\right)^{a+b}\left[\frac{1}{(1-x)^{a+b-1}}\right]_0^{\rho^2}\to 0 \text{ as } \rho^2\to 1.$$

Equation (12) therefore gives the mean value of R^2 valid over the whole range of ρ^2 from 0 to 1. We note that for $\rho=0$ we have

$$\bar{R}^2=\frac{a}{a+b},$$

agreeing with Fisher's result† from the simplified distribution (1). Also for $\rho^2=1$ we have $\bar{R}^2=1$.

For comparison with our exact result (12) we have Hall's approximate value‡, which in our notation is

$$\bar{R}^2=\rho^2+\frac{(1-\rho^2)(a-\rho^2)}{a+b+\frac{1}{2}}=\frac{a+(b-\frac{1}{2})\rho^2+\rho^4}{a+b+\frac{1}{2}} \quad\text{..............(13).}$$

* See Whittaker and Watson, *Modern Analysis*, p. 25.

† R. A. Fisher, *Phil. Trans.* B, Vol. 213, 1924, p. 92.

‡ P. Hall, *loc. cit.*

Now (12) can be written

$$\frac{a}{a+b}+\frac{b}{a+b+1}\rho^2 F(1, 1, a+b+2, \rho^2)$$
$$=\frac{a}{a+b}+\frac{b}{a+b+1}\rho^2\left\{1+\frac{1}{a+b+2}\rho^2+\frac{1.2}{(a+b+2)(a+b+3)}\rho^4+\ldots\right\} \quad \ldots(14).$$

It is evident that the approximation in (13) consists in supposing that for large N (which is equal to $2(a+b+\frac{1}{2})$ in our notation) $a+b$ and $a+b+1$ may be safely replaced by $a+b+\frac{1}{2}$, while b is replaced by $b-\frac{1}{2}$ and $b/(a+b+2)$ is replaced by unity, and terms of higher order are neglected. To give a numerical example, suppose there are 6 independent variates and the sample is of size 101. Then $a=3$, $b=47$. If we take $\rho^2=0$, 0·5 and 1, we find $\bar{R}^2=0{\cdot}0594$, 0·5248 and 1 respectively from (13), and 0·06 (the correct result), 0·5252 (correct result 0·5253) and 0·9993 from the first three terms only of (14). (We would naturally, however, use (12) for preference for ρ^2 nearly equal to 1; for $\rho^2=1$ exactly we get the correct result from (12), or by using the well-known formula for the sum of the hypergeometric in (14).) It would appear, therefore, to be desirable to improve the approximate formula (13), as it gives an underestimate of the correct mean value, and we would suggest the use of the first three terms of (14), except when ρ^2 is large, when formula (12), using the first two or three terms of the hypergeometric series, should be used, e.g. for $\rho^2=0{\cdot}9$ formula (13) gives $\bar{R}^2=0{\cdot}90416$, while two terms of the hypergeometric in (12) give 0·90434 and three terms 0·90428, which is correct to the last place shown.

It may be mentioned that the value derived for $\bar{R}^2$ by Fisher in 1924* by averaging the numerator and denominator of the expression for R^2, and which may be written

$$\bar{R}^2=1-\frac{b}{a+b}(1-\rho^2),$$

differs from the exact value (12) by a term involving $(F-1)$, of the order of $1/N$.

B. *Determination of Second Moment of R^2.*

This involves a repetition of the procedure we have gone through for determining the mean value. The second moment about zero is obtained by multiplying the distribution by R^4, i.e. by x^2/ρ^4, and integrating for x from 0 to ρ^2.

CASE 1. $b=1$.

We have

$$\mu_2'(R^2)=\frac{(1-\rho^2)^{a+1}}{(\rho^2)^{a+2}}\int_0^{\rho^2} x^2 d\left(\frac{x^a}{(1-x)^{a+1}}\right)=1-\frac{2(1-\rho^2)^{a+1}}{(\rho^2)^{a+2}}\int_0^{\rho^2}\frac{x^{a+1}}{(1-x)^{a+1}}dx,$$

on integrating by parts. For comparison with the other cases we shall leave this result meantime in the form

$$1+\left(\frac{1-\rho^2}{\rho^2}\right)^{a+1}\int_0^{\rho^2}\frac{x^a(0-2x/\rho^2)}{(1-x)^{a+1}}dx \quad \ldots\ldots\ldots\ldots(15).$$

* R. A. Fisher, *loc. cit.* p. 92.

CASE 2. $b=2$.

From A, Case 2,

$$\mu_2'(R^2)=\left(\frac{1-\rho^2}{\rho^2}\right)^{a+2}\int_0^{\rho^2} x^2 d\left\{\frac{(a+1+x)(1-x/\rho^2)x^a}{(1-x)^{a+3}}+\frac{x^{a+1}/\rho^2}{(1-x)^{a+2}}\right\}$$

$$=1-2\left(\frac{1-\rho^2}{\rho^2}\right)^{a+2}\int_0^{\rho^2}\frac{(a+1+x)x^{a+1}(1-x/\rho^2)}{(1-x)^{a+3}}dx-2\frac{(1-\rho^2)^{a+2}}{(\rho^2)^{a+3}}\int_0^{\rho^2}\frac{x^{a+2}}{(1-x)^{a+2}}dx$$

$$=1-2\left(\frac{1-\rho^2}{\rho^2}\right)^{a+2}\int_0^{\rho^2} x(1-x/\rho^2)\,d\left\{\frac{x^{a+1}}{(1-x)^{a+2}}\right\}-2\frac{(1-\rho^2)^{a+2}}{(\rho^2)^{a+3}}\int_0^{\rho^2}\frac{x^{a+2}}{(1-x)^{a+2}}dx$$

$$=1+2\left(\frac{1-\rho^2}{\rho^2}\right)^{a+2}\int_0^{\rho^2}\frac{x^{a+1}(1-3x/\rho^2)}{(1-x)^{a+2}}dx \quad\text{......................................(16).}$$

CASE 3. $b=3$.

From A, Case 3, we have

$$\mu_2'(R^2)=\frac{(1-\rho^2)^{a+3}}{2(\rho^2)^{a+2}}\int_0^{\rho^2} x^2 d\left[\frac{\{(a+1)(a+2)+4(a+2)x+2x^2\}(1-x/\rho^2)^2x^a}{(1-x)^{a+5}}\right.$$
$$\left.+\frac{2x^{a+1}(a+2+x)(1-x/\rho^2)}{\rho^2(1-x)^{a+4}}+\frac{2x^{a+2}}{\rho^4(1-x)^{a+3}}\right]$$

$$=1-\frac{(1-\rho^2)^{a+3}}{(\rho^2)^{a+2}}\int_0^{\rho^2}\frac{\{(a+1)(a+2)+4(a+2)x+2x^2\}x^{a+1}(1-x/\rho^2)^2}{(1-x)^{a+5}}dx$$
$$-2\left(\frac{1-\rho^2}{\rho^2}\right)^{a+3}\int_0^{\rho^2}\frac{(a+2+x)x^{a+2}(1-x/\rho^2)}{(1-x)^{a+4}}dx-2\frac{(1-\rho^2)^{a+3}}{(\rho^2)^{a+4}}\int_0^{\rho^2}\frac{x^{a+3}}{(1-x)^{a+3}}dx$$

$$=1-\frac{(1-\rho^2)^{a+3}}{(\rho^2)^{a+2}}\int_0^{\rho^2} x(1-x/\rho^2)\,d\left\{\frac{(a+2+x)(1-x/\rho^2)x^{a+1}}{(1-x)^{a+4}}+\frac{x^{a+2}/\rho^2}{(1-x)^{a+3}}\right\}$$
$$-2\left(\frac{1-\rho^2}{\rho^2}\right)^{a+3}\int_0^{\rho^2}\frac{(a+2+x)x^{a+2}(1-x/\rho^2)}{(1-x)^{a+4}}dx-2\frac{(1-\rho^2)^{a+3}}{(\rho^2)^{a+4}}\int_0^{\rho^2}\frac{x^{a+3}}{(1-x)^{a+3}}dx$$

$$=1+\frac{(1-\rho^2)^{a+3}}{(\rho^2)^{a+2}}\int_0^{\rho^2}\frac{(a+2+x)x^{a+1}(1-x/\rho^2)(1-4x/\rho^2)}{(1-x)^{a+4}}dx$$
$$+\left(\frac{1-\rho^2}{\rho^2}\right)^{a+3}\int_0^{\rho^2}\frac{x^{a+2}(1-4x/\rho^2)}{(1-x)^{a+3}}dx$$

$$=1+\frac{(1-\rho^2)^{a+3}}{(\rho^2)^{a+2}}\int_0^{\rho^2}(1-x/\rho^2)(1-4x/\rho^2)\,d\left\{\frac{x^{a+2}}{(1-x)^{a+3}}\right\}$$
$$+\left(\frac{1-\rho^2}{\rho^2}\right)^{a+3}\int_0^{\rho^2}\frac{x^{a+2}(1-4x/\rho^2)}{(1-x)^{a+3}}dx$$

$$=1+3\left(\frac{1-\rho^2}{\rho^2}\right)^{a+3}\int_0^{\rho^2}\frac{x^{a+2}(2-4x/\rho^2)}{(1-x)^{a+3}}dx \quad\text{......................................(17).}$$

From (15), (16) and (17) it appears that the general result for any integral b is

$$\mu_2'(R^2)=1+b\left(\frac{1-\rho^2}{\rho^2}\right)^{a+b}\int_0^{\rho^2}\frac{x^{a+b-1}\{b-1-(b+1)x/\rho^2\}}{(1-x)^{a+b}}dx.$$

Now we already have

$$\left(\frac{1-\rho^2}{\rho^2}\right)^{a+b}\int_0^{\rho^2}\frac{x^{a+b-1}}{(1-x)^{a+b}}dx=\frac{(1-\rho^2)}{a+b}F(1,1,a+b+1,\rho^2),$$

while by a similar expansion of the denominator and integration term by term we have

$$\frac{(1-\rho^2)^{a+b}}{(\rho^2)^{a+b+1}}\int_0^{\rho^2}\frac{x^{a+b}}{(1-x)^{a+b}}\,dx=\frac{(1-\rho^2)}{a+b+1}F(2,1,a+b+2,\rho^2)\quad\text{......(18).}$$

The same considerations as were examined in the deduction of (11) show that this integration is valid even when $\rho^2 \to 1$, and the hypergeometric series in (18) is also absolutely convergent even for $\rho^2=1$. We then have, for the second moment of R^2 about zero,

$$\mu_2'(R^2)=1+\frac{b(b-1)}{a+b}(1-\rho^2)F(1,1,a+b+1,\rho^2)$$
$$-\frac{b(b+1)}{a+b+1}(1-\rho^2)F(2,1,a+b+2,\rho^2)\quad\text{.........(19).}$$

For $\rho^2=0$ this becomes

$$\mu_2'(R^2)=\frac{a(a+1)}{(a+b)(a+b+1)},$$

which may be derived directly from the distribution (1), while for $\rho^2=1$ we have

$$\mu_2'(R^2)=1.$$

The second moment about the mean, or variance, of R^2, may now be obtained, for, since

$$\bar{R}^2=1-\frac{b}{a+b}(1-\rho^2)F(1,1,a+b+1,\rho^2),$$

we have

$$\sigma^2_{R^2}=\mu_2'(R^2)-(\bar{R}^2)^2$$
$$=b(b+1)(1-\rho^2)\left\{\frac{1}{a+b}F(1,1,a+b+1,\rho^2)-\frac{1}{a+b+1}F(2,1,a+b+2,\rho^2)\right\}$$
$$-\frac{b^2}{(a+b)^2}(1-\rho^2)^2F^2(1,1;a+b+1,\rho^2),$$

and this, on reduction, is equal to

$$\frac{b(b+1)(1-\rho^2)^2}{(a+b)(a+b+1)}F(2,2,a+b+2,\rho^2)-\frac{b^2(1-\rho^2)^2}{(a+b)^2}F^2(1,1,a+b+1,\rho^2)\quad\text{...(20).}$$

An alternative form for this expression is

$$\frac{b(b+1)}{a+b}(1-\rho^2)^2\frac{dF}{d(\rho^2)}-(1-\bar{R}^2)^2\quad\text{.....................(21),}$$

where F stands for the hypergeometric series $F(1,1,a+b+1,\rho^2)$, i.e. the same series which occurs in the expression (12) for the mean value of R^2. This series is the only part of our results (12) and (21) which is at all difficult to calculate, although for values of ρ^2 up to 0·5 and a reasonably large N a very few terms of the series should suffice. A table of the series F for values of $a+b+1$ proceeding by half-integers, and for a number of values of ρ^2, with, possibly, a table of its derivative, would be useful in this connection.

The only result known hitherto for the variance of R^2 is the approximate one of P. Hall (*loc. cit.*):

$$\sigma^2_{R^2}=4\rho^2(1-\rho^2)^2/N=2\rho^2(1-\rho^2)^2/(a+b+\tfrac{1}{2})\quad\text{............(22).}$$

This result is correct to terms of the order of $1/N$, but its weakness lies in the fact that for $\rho^2 \to 0$ it gives $\sigma^2_{R^2} \to 0$, whereas in fact we know from the distribution (1) directly*, or from (20) on putting $\rho^2 = 0$, that

$$\text{For } \rho^2 = 0, \qquad \sigma^2_{R^2} = \frac{ab}{(a+b)^2(a+b+1)}.$$

This result is of the order of $1/N^2$, which explains wherein the approximation (22) is insufficient†. The terms in $1/N$ involve ρ^2, and when this is equal to zero the terms all vanish, while the part that does not vanish with ρ^2 is not given, being of the order of $1/N^2$. An exact formula is always to be preferred to an approximate one, proceeding in powers of $1/N$. If N is not really large the first term or two will not be adequate to give precision enough; while a more serious objection, illustrated in the case before us, is that in particular cases the early terms of a series may vanish, and the first term of importance may be a term neglected.

The nature of the approximation in (22) may be seen from (20) on expanding the hypergeometric series as far as the terms in ρ^2. The parts outside involving a and b are nearly unity for large N. If we count them as unity and replace the $a+b+2$ and $a+b+1$ of the hypergeometric series by $\frac{1}{2}N$ we find, approximately,

$$\sigma^2_{R^2} = (1-\rho^2)^2\left[1 + \frac{4}{\frac{1}{2}N}\rho^2 - 1 - \frac{2}{\frac{1}{2}N}\rho^2\right] = 4\rho^2(1-\rho^2)^2/N,$$

as in (22).

As a numerical example let $a = 3$, $b = 47$, and $\rho^2 = 0{\cdot}5$. From (22) we have $\sigma^2_{R^2} = 0{\cdot}00495$. The correct result, from (20), is $0{\cdot}0047241$. A much better approximation is obtained by retaining the exact values of the parts outside the hypergeometric series in (20) and calculating the series up to terms in ρ^4. This yields $0{\cdot}00470$. We have chosen the case of $N = 101$ for the purposes of illustration, and even here the approximate forms (13) and (22) are not good enough. For smaller samples the discrepancy will be even wider, and it is obvious that the exact forms (12) and (20) must be used in such cases to secure reliable results.

C. *Mean and Second Moment of R.*

We have dealt so far with R^2, as having a rather simpler sampling distribution than R. We know that the mean value of R, for the special case of no correlation in the population, is of the form‡

$$\bar{R} = \frac{(a-\frac{1}{2})!\,(a+b-1)!}{(a-1)!\,(a+b-\frac{1}{2})!} \quad \text{.......................(23)},$$

where $x!$ is written for the factorial function, or $\Gamma(x+1)$, even when x is not an integer. It is hardly to be expected, therefore, that the more general form for any ρ should be simple. A similar method of attack to that in A does, in fact, lead to a solution for the special cases $b = 1$, 2 and 3, and it is seen that the mean value will in the general case involve a number of hypergeometric series equal in number to b,

* J. Wishart, *Mem. Roy. Met. Soc.* Vol. XI. No. 13, 1928, p. 84.

† Indicated on other grounds by P. Hall, *loc. cit.* p. 108.

‡ J. Wishart, *loc. cit.* p. 84; P. Hall, *loc. cit.* p. 109.

but it does not appear that the expressions are capable of any very great degree of simplification. The first three results are

$$b=1.\quad \bar{R}=1-\frac{1-\rho^2}{2a+1}F(\tfrac{1}{2},1,a+\tfrac{3}{2},\rho^2);$$

$$b=2.\quad \bar{R}=1-\frac{1}{2}\frac{1-\rho^2}{2a+1}F(-\tfrac{1}{2},1,a+\tfrac{3}{2},\rho^2)-\frac{3}{2}\frac{1-\rho^2}{2a+3}F(\tfrac{1}{2},1,a+\tfrac{5}{2},\rho^2);$$

$$b=3.\quad \bar{R}=1-\frac{3}{8}\frac{1-\rho^2}{2a+1}F(-\tfrac{3}{2},1,a+\tfrac{3}{2},\rho^2)-\frac{3}{4}\frac{1-\rho^2}{2a+3}F(-\tfrac{1}{2},1,a+\tfrac{5}{2},\rho^2)$$
$$-\frac{15}{8}\frac{1-\rho^2}{2a+5}F(\tfrac{1}{2},1,a+\tfrac{7}{2},\rho^2).$$

In view of this difficulty, and also bearing in mind that R^2 is calculated first before extracting the square root, it would seem desirable to apply the usual tests of significance to R^2, and not to R. The second moment of R about zero is, of course, identical with the mean value of R^2, i.e. our formula (12), and we therefore have

$$\sigma^2_R=1-\frac{b}{a+b}(1-\rho^2)F(1,1,a+b+1,\rho^2)-(\bar{R})^2 \quad \text{.........(24).}$$

Further than this it is hardly practicable to proceed. What we have done in this section will illustrate the difficulties experienced by other authors* in obtaining approximate expressions for the mean value of R.

* L. Isserlis, *Phil. Mag.* Vol. xxxiv. 1917, pp. 205—220; P. Hall, *loc. cit.* pp. 108—109.

APPENDIX TO A PAPER BY DR WISHART.

Tables of the Mean Value and Squared Standard Deviation of the Square of a Multiple Correlation Coefficient.

EDITORIAL.

Dr Wishart has provided in his paper the formulae giving the Mean Value, $\bar{R}^2$, and the Squared Standard Deviation $\sigma^2_{R^2}$ of the square of a multiple correlation coefficient R. Let us suppose N = size of sample and n = *total* number of variates*, then Dr Wishart's formulae may be expressed as follows:

$$\bar{R}^2 = 1 - \frac{N-n}{N-1}(1-\rho^2) F(1, 1, \tfrac{1}{2}(N+1), \rho^2) \quad\text{...............................(i)},$$

$$\sigma^2_{R^2} = \frac{(N-n)(N-n+2)}{(N-1)(N+1)}(1-\rho^2)^2 F(2, 2, \tfrac{1}{2}(N+3), \rho^2) - (1-\bar{R}^2)^2 \quad\text{...(ii)},$$

where F is the hypergeometrical function. We may write these as follows:

$$\bar{R}^2 = 1 - \frac{N-n}{N-2}\gamma_1 \quad\text{...(i) } bis,$$

$$\sigma^2_{R^2} = \frac{(N-n)(N-n+2)}{(N-2)N}\gamma_2 - (1-\bar{R}^2)^2 \quad\text{..............(ii) } bis.$$

The only parts of these formulae which involve n, the total number of variates, are the coefficients of γ_1 and γ_2; these change with the order $n-1$ of the multiple correlation coefficient.

Now $\gamma_1 = \mu_2' - 1$ and $\gamma_2 = \mu_4' - 2\mu_2' + 1$, of the paper in *Biometrika*, Vol. XI. pp. 384—385, and although the numerical values of μ_2' and μ_4' are not given in the Tables attached to it, they exist in the Archives of the Laboratory on the working sheets from which the μ_2 and μ_4 of the frequency distributions of r were obtained. It is therefore only a matter of picking out of those sheets the values of μ_2' and μ_4' and so finding γ_1 and γ_2. This has been done and Tables I and II below provide their values.

For samples of 8† to 25, there is no need of interpolation for N; we require only to interpolate for ρ. For most practical purposes central difference interpolation to δ^2 will suffice. Beyond 25, the two adjacent values of γ_1 and γ_2 are so close that linear interpolation for N will as a rule be adequate.

Illustration (i). Let us take Dr Wishart's example $N = 101$, $n = 7$, and $\rho^2 = \cdot 5$, or $\rho = \cdot 7071$ nearly. This lies between $\rho = \cdot 7$ and $\cdot 8$.

* Wishart puts $a = \frac{1}{2}$ (number of *independent* variates) $= \frac{1}{2}$ (Fisher's n_1) $= \frac{1}{2}$ (our $(n-1)$) and $b = \frac{1}{2}$ (size of sample − *total* number of variates) $= \frac{1}{2}$ (Fisher's n_2) $= \frac{1}{2}$ (our $N - n$).

† For $N = 2$, we can only take $n = 2$, i.e. ordinary correlation and then $\bar{R}^2 = 1$ and $\sigma^2_{R^2} = 0$.

First to find γ_1 using Everett's Central Difference formula we have for $N = 100$, $\theta = \cdot 071$, $\phi = \cdot 929$,

$$z_0 = \cdot 509{,}8429,\quad z_1 = \cdot 360{,}9965,\quad \text{and}\quad \delta^2 z_0 = -\cdot 020{,}5732,\quad \delta^2 z_1 = -\cdot 020{,}9530.$$

Hence:

$$\begin{aligned} z_\theta &= \cdot 929 \times \cdot 509{,}8429 + \cdot 071 \times \cdot 360{,}9965 \\ &\qquad - \tfrac{1}{6}(\cdot 929 \times \cdot 071)\{1\cdot 929(-\cdot 020{,}5732) + 1\cdot 071(-\cdot 020{,}9530)\} \\ &= \cdot 499{,}2748 + \cdot 010{,}993 \times \cdot 062{,}1264 \\ &= \cdot 499{,}9578 = \gamma_1 \text{ for } N = 100. \end{aligned}$$

Similarly for $N = 200$,

$$\begin{aligned} z_\theta &= \cdot 929 \times \cdot 509{,}9355 + \cdot 071 \times \cdot 360{,}5013 \\ &\qquad - \cdot 010{,}993\{1\cdot 929 \times (-\cdot 020{,}2884) + 1\cdot 071 \times (-\cdot 020{,}4733)\} \\ &= \cdot 499{,}9970 = \gamma_1 \text{ for } N = 200. \end{aligned}$$

Clearly linear interpolation for $N = 101$ will suffice and we have

$$\gamma_1 = \cdot 499{,}9582 \text{ for } N = 101.$$

Thus
$$\begin{aligned} \bar{R}^2 &= 1 - \frac{N-n}{N-2}\gamma_1 = 1 - \frac{94}{99} \times \cdot 499{,}9582 \\ &= \cdot 525{,}2922, \end{aligned}$$

agreeing completely with Dr Wishart's

$$\bar{R}^2 = \cdot 5253.$$

We turn now to Table II to find γ_2. We have for $N = 100$,

$$z_0 = \cdot 265{,}0540, \qquad z_1 = \cdot 133{,}7050,$$
$$\delta^2 z_0 = +\cdot 016{,}6294, \quad \delta^2 z_1 = +\cdot 035{,}4198.$$

The values of θ and ϕ are as before. Hence

$$\begin{aligned} z_\theta &= \cdot 929 \times \cdot 265{,}0540 + \cdot 071 \times \cdot 133{,}7050 \\ &\qquad - \cdot 010{,}993\{1\cdot 929 \times \cdot 016{,}6294 + 1\cdot 071 \times \cdot 035{,}4198\} \\ &= \cdot 254{,}9586 = \gamma_2 \text{ for } N = 100. \end{aligned}$$

Similarly

$$\begin{aligned} z_\theta &= \cdot 929 \times \cdot 262{,}5877 + \cdot 071 \times \cdot 131{,}6374 \\ &\qquad - \cdot 010{,}993\{1\cdot 929 \times \cdot 017{,}8219 + 1\cdot 071 \times \cdot 036{,}2360\} \\ &= \cdot 252{,}4857 = \gamma_2 \text{ for } N = 200. \end{aligned}$$

Interpolating linearly for $N = 101$

$$\begin{aligned} \gamma_2 &= \cdot 254{,}9586 - \tfrac{1}{100}(\cdot 254{,}9586 - \cdot 252{,}4857) \\ &= \cdot 254{,}9586 - \cdot 000{,}0247 \\ &= \cdot 254{,}9339. \end{aligned}$$

Thus
$$\begin{aligned} \sigma^2_{R^2} &= \tfrac{94}{99} \times \tfrac{96}{101} \times \cdot 254{,}9339 - (\cdot 474{,}7078)^2 \\ &= \cdot 2300{,}7556 - \cdot 2253{,}4750 \\ &= \cdot 0047{,}2806, \end{aligned}$$

and accordingly $\qquad \sigma_{R^2} = \cdot 0688.$

TABLE I. *Values of* γ_1.

ρ = Multiple Correlation Coefficient in Parent Population.

N	$\rho=0$	$\rho=\cdot1$	$\rho=\cdot2$	$\rho=\cdot3$	$\rho=\cdot4$	$\rho=\cdot5$	$\rho=\cdot6$	$\rho=\cdot7$	$\rho=\cdot8$	$\rho=\cdot9$	$\rho=1\cdot0$
3	·500 0000*	·497 4916	·489 8639	·476 7929	·457 6776	·431 5231	·396 6996	·350 4140	·287 3394	·194 7771	$\gamma_1=0$ throughout
4	·666 6667	·662 6552	·650 4805−	·629 7020	·599 5127	·558 6019	·504 8836	·434 9287	·342 5947	·214 7098	
5	·750 0000*	·744 9874	·729 7967	·703 9493	·666 5772	·616 2920	·550 9352	·467 0746	·358 8647	·214 7865−	
6	·800 0000	·794 2730	·776 9367	·747 5077	·705 1162	·648 3886	·575 2286	·482 3934	·364 5810	·212 0292	
7	·833 3333	·827 0708	·808 1306	·776 0392	·729 9499	·668 5399	·589 8216	·490 7889	·366 6893	·209 2399	
8	·857 1429	·850 4640	·830 2798	·796 1328	·747 2098	·682 2515−	·599 3905−	·495 8553	·367 3848	·206 6786	
9	·875 0000	·867 9883	·846 8112	·811 0298	·759 8680	·692 1325−	·606 0737	·499 1362	·367 4822	·204 9492	
10	·888 8889	·881 6049	·859 6170	·822 5056	·769 5313	·699 5654	·610 9686	·501 3787	·367 3186	·203 3748	
11	·900 0000	·892 4893	·869 8868	·831 6120	·777 1412	·705 3462	·614 6887	·502 9781	·367 0441	·202 0788	
12	·909 0909	·901 3883	·878 1560	·839 0113	·783 2842	·709 9629	·617 6007	·504 1586	·366 7291	·200 9995−	
13	·916 6667	·908 7997	·885 0795+	·845 1405−	·788 3441	·713 7305+	·619 9357	·505 0551	·366 4072	·200 0899	
14	·923 0769	·915 0675−	·890 9251	·850 2997	·792 5824	·716 8607	·621 8458	·505 7521	·366 0942	·199 3146	
15	·928 5714	·920 4374	·895 9258	·854 7017	·796 1830	·719 5009	·623 4348	·506 3051	·365 7975+	·198 6470	
16	·933 3333	·925 0894	·900 2524	·858 5012	·799 2788	·721 7567	·624 7757	·506 7515+	·365 5199	·198 0666	
17	·937 5000*	·929 1583	·904 0324	·861 8137	·801 9687	·723 7055+	·625 9214	·507 1173	·365 2619	·197 5579	
18	·941 1765−	·932 7474	·907 3630	·864 7270	·804 3271	·725 4056	·626 9108	·507 4209	·365 0229	·197 1085+	
19	·944 4444	·935 9367	·910 3200	·867 3089	·806 4115−	·726 9012	·627 7733	·507 6759	·364 8018	·196 7039	
20	·947 3684	·938 7895	·912 9627	·869 6130	·808 2668	·728 2289	·628 5313	·507 8923	·364 5972	·196 3513	
21	·950 0000*	·941 3564	·915 3387	·871 6815+	·809 9288	·729 4099	·629 2029	·508 0776	·364 4077	·196 0294	
22	·952 3810	·943 6783	·917 4864	·873 5489	·811 4260	·730 4719	·629 8016	·508 2377	·364 2320	·195 7384	
23	·954 5455−	·945 7887	·919 4371	·875 2431	·812 7817	·731 4305−	·630 3382	·508 3768	·364 0689	·195 4738	
24	·956 5217	·947 7152	·921 2169	·876 7871	·814 0150−	·732 2999	·630 8217	·508 4988	·363 9171	·195 2325−	
25	·958 3333	·949 4809	·922 8471	·878 1999	·815 1417	·733 0921	·631 2607	·508 6062	·363 7757	·195 0113	
50	·979 5918	·970 1765+	·941 8878	·894 5964	·828 0836	·742 0365+	·636 0401	·509 5673	·361 9643	·192 4322	
100	·989 8990	·980 1941	·951 0569	·902 4191	·834 1662	·746 1358	·638 1161	·509 8429	·360 9965−	·191 1971	
200	·994 9748	·985 1232	·955 5563	·906 2394	·837 1137	·748 0967	·639 0813	·509 9355−	·360 5013	·190 5938	
400	·997 4937	·987 5681	·957 7851	·908 1271	·838 5644	·749 0554	·639 5464	·509 9712	·360 2513	·190 2957	

TABLE II. *Values of* γ_2.

ρ=multiple correlation coefficient in parent population.

N	$\rho=0$	$\rho=\cdot1$	$\rho=\cdot2$	$\rho=\cdot3$	$\rho=\cdot4$	$\rho=\cdot5$	$\rho=\cdot6$	$\rho=\cdot7$	$\rho=\cdot8$	$\rho=\cdot9$	$\rho=1\cdot0$
3	·375 0000	·372 4937	·364 8983	·351 9747	·333 2885	·308 1460	·275 4675	·233 5373	·179 4323	·107 3932	$\gamma_2=0$ throughout
4	·533 3333	·528 7543	·514 9245	·491 5596	·458 1615	·413 9809	·357 9592	·288 6614	·204 2769	·103 3340	
5	·625 0000	·618 7437	·599 9000	·568 2444	·523 4070	·464 8896	·392 1267	·304 6799	·202 9060	·090 7534	
6	·685 7143	·678 0919	·655 1842	·616 8793	·563 0261	·493 5091	·408 4240	·308 5062	·196 3093	·080 2621	
7	·729 1667	·720 4167	·694 1694	·650 4492	·589 3602	·511 2142	·416 8017	·307 9994	·189 2911	·072 4402	
8	·761 9048	·752 2115	·723 1804	·674 9803	·607 9805	·522 9222	·421 2652	·305 9241	·182 9972	·066 6492	
9	·787 5000	·777 0076	·745 6250	·693 6651	·621 7588	·531 0590	·423 6466	·303 3544	·177 6088	·062 2889	
10	·808 0808	·796 9032	·763 5103	·708 3551	·632 3174	·536 9375	·424 8626	·300 7280	·173 0497	·058 9315	
11	·825 0000	·813 2292	·778 0992	·720 1968	·640 6374	·541 3193	·425 4021	·298 2218	·169 1896	·056 2887	
12	·839 1608	·826 8713	·790 2266	·729 9390	·647 3439	·544 6706	·425 5401	·295 9003	·165 9037	·054 1657	
13	·851 1905	·838 4441	·800 4668	·738 0896	·652 8528	·547 2896	·425 4371	·293 7793	·163 0858	·052 4291	
14	·861 5384	·848 3864	·809 2288	·745 0063	·657 4508	·549 3742	·425 1897	·291 8522	·160 6501	·050 9860	
15	·870 5357	·857 0213	·816 8104	·750 9476	·661 3413	·551 0599	·424 8575	·290 1046	·158 5286	·049 7700	
16	·878 4314	·864 5913	·823 4353	·756 1045	·664 6719	·552 4422	·424 4778	·288 5192	·156 6673	·048 7326	
17	·885 4167	·871 2823	·829 2736	·760 6217	·667 5529	·553 5893	·424 0747	·287 0785	·155 0230	·047 8384	
18	·891 6409	·877 2395	·834 4572	·764 6106	·670 0676	·554 5517	·423 6630	·285 7662	·153 5612	·047 0598	
19	·897 2221	·882 5772	·839 0908	·768 1579	·672 2801	·555 3668	·423 2526	·284 5679	·152 2541	·046 3765	
20	·902 2556	·887 3877	·843 2571	·771 3331	·674 2408	·556 0630	·422 8491	·283 4706	·151 0788	·045 7721	
21	·906 8182	·891 7453	·847 0233	·774 1910	·675 9898	·556 6624	·422 4575	·282 4630	·150 0169	·045 2338	
22	·910 9731	·895 7098	·850 4445	·776 7769	·677 5588	·557 1820	·422 0789	·281 5353	·149 0532	·044 7517	
23	·914 7728	·899 3363	·853 5659	·779 1278	·678 9738	·557 6355	·421 7147	·280 6785	·148 1749	·044 3171	
24	·918 2608	·902 6624	·856 4254	·781 2741	·680 2560	·558 0334	·421 3656	·279 8856	·147 3711	·043 9240	
25	·921 4743	·905 7253	·859 0544	·783 2410	·681 4230	·558 3846	·421 0318	·279 1496	·146 6331	·043 5661	
50	·960 3841	·942 6956	·890 4692	·806 2600	·694 4546	·561 4768	·416 1002	·269 9046	·137 9239	·039 5739	
100	·980 0980	·961 3408	·906 0742	·817 3368	·700 2710	·562 2454	·413 0324	·265 0540	·133 7050−	·037 7758	
200	·990 0248	·970 7058	·913 8476	·822 7588	·702 9974	·562 4365+	·411 3599	·262 5877	·131 6374	·036 9231	
400	·995 0062	·975 3992	·917 7265+	·825 4397	·704 3143	·562 4842	·410 4906	·261 3463	·130 6148	·036 5079	

Illustration (ii). Let $\rho = \cdot 3$, $N = 315$, $n = 5$. We require to find $\bar{R}^2$ and σ_{R^2}.

This is about as unfavourable an example as we can take for the Tables. It is easier to expand Dr Wishart's Hypergeometrical

$$\bar{R}^2 = 1 - \frac{N-n}{N-1}(1-\rho^2) F(1, 1, \tfrac{1}{2}(N+1), \rho^2),$$

which gives $\bar{R}^2 = \cdot 10108$, in the present case. We will, however, compute $\bar{R}^2$ by aid of Table I. The value $N = 315$ occurs in a part of that table, where the argument does not run by equal intervals but logarithmically, i.e.

$$\begin{aligned}
\log\ \ 25 &= \log 25 + 0 \times \log 2,\\
\log\ \ 50 &= \log 25 + 1 \times \log 2,\\
\log 100 &= \log 25 + 2 \times \log 2,\\
\log 200 &= \log 25 + 3 \times \log 2,\\
\log 400 &= \log 25 + 4 \times \log 2;
\end{aligned}$$

and we need

$$\begin{aligned}
\log 315 &= \log 25 + \left(\frac{\log 12 \cdot 6}{\log 2}\right) \log 2\\
&= \log 25 + 3 \cdot 655{,}3516 \log 2.
\end{aligned}$$

Hence for equal argument intervals we shall need to interpolate at a distance

$$4 - 3 \cdot 655{,}3516 = \cdot 344{,}6484 = \theta$$

from the 400 value. Write down the terms in reverse order and difference them:

	Δ	Δ^2	Δ^3	Δ^4
·908,1271,				
·906,2394,	− ·001,8877,			
·902,4191,	− ·003,8203,	− ·001,9326,		
·894,5964,	− ·007,8227,	− ·004,0024,	− ·002,0698,	
·878,1999,	− ·016,3965,	− ·008,5738,	− ·004,5714,	− ·002,5016.

The differences are thus slightly diverging, but the forward difference formula will suffice. Accordingly:

$$\begin{aligned}
z_\theta = &\cdot 908{,}1271 - \cdot 344{,}6484 \times \cdot 001{,}8877 + \cdot 112{,}9329 \times \cdot 001{,}9326\\
&- \cdot 062{,}3146 \times \cdot 002{,}0698 + \cdot 041{,}3668 \times \cdot 002{,}5016\\
= &\cdot 908{,}1271 - \cdot 000{,}6506 + \cdot 000{,}2183 - \cdot 000{,}1290 + \cdot 000{,}1035.
\end{aligned}$$

Clearly the required value is greater than ·907,5658 and less than ·907,6693. Taking it as the mean of these we have

$$\gamma_1 = \cdot 907{,}6175,$$

$$\bar{R}^2 = 1 - \tfrac{310}{314} \times \cdot 907{,}6175 = 1 - \cdot 898{,}9183$$

$$= \cdot 101{,}0817,$$

in excellent agreement with Dr Wishart's result ·10108.

Calculated from the formula

$$\sigma^2_{R^2} = \frac{(N-n)(N-n+2)}{(N-1)(N+1)}(1-\rho^2)^2 F(2, 2, \tfrac{1}{2}(N+3), \rho^2) - (1 - \bar{R}^2)^2,$$

we find

$$\sigma_{R^2} = \cdot 03124,$$

again the quicker method.

But it is of interest to see how closely by aid of a logarithmic formula we can get comparable results from a table with apparently absurd stretches of argument. Our differences from Table II are:

	Δ	Δ^2	Δ^3	Δ^4
·825,4397,				
·822,7588,	− ·002,6809,			
·817,3368,	− ·005,4220,	− ·002,7411,		
·806,2600,	− ·011,0768,	− ·005,6548,	− ·002,9137,	
·783,2410,	− ·023,0190,	− ·011,9422,	− ·006,2874,	− ·003,3737.

Accordingly:

$$\begin{aligned} x_\theta &= \cdot825{,}4397 - \cdot344{,}6484 \times \cdot002{,}6809 + \cdot112{,}9329 \times \cdot002{,}7411 \\ &\qquad - \cdot062{,}3146 \times \cdot002{,}9137 + \cdot041{,}3668 \times \cdot003{,}3737 \\ &= \cdot825{,}4397 - \cdot000{,}9240 + \cdot000{,}3096 - \cdot000{,}1816 + \cdot000{,}1396. \end{aligned}$$

Thus γ_2 lies between ·824,7833 and ·824,6437.

Taking as before the mean of these values we have

$$\gamma_2 = \cdot824{,}7135.$$

Using the Equation (ii) *bis*, we have

$$\begin{aligned} \sigma^2_{R^2} &= \tfrac{310}{318} \times \tfrac{312}{316} \times \cdot824{,}7135 - (\cdot898{,}9183)^2 \\ &= \cdot000{,}97566, \end{aligned}$$

or $$\sigma_{R^2} = \cdot03124,$$

agreeing with the directly computed value.

These results are interesting as showing that by the use of a logarithmic interpolation we may cover by three properly chosen intermediate values the range from 25 to 400, with sufficient accuracy for most statistical purposes.

Table of Normal Curve Functions to each Permille of Frequency.
Computed by T. Kondo, Ph.D., Lond., and revised by Ethel M. Elderton.

$\frac{1}{2}(1+\alpha_x)$	x	z	$\frac{\frac{1}{2}(1+\alpha_x)}{z}$	$\frac{\frac{1}{2}(1-\alpha_x)}{z}$	$\frac{z}{\frac{1}{2}(1+\alpha_x)}$	$\frac{z}{\frac{1}{2}(1-\alpha_x)}$	$\frac{1}{2}(1-\alpha_x)$
·500	·00000 00000	·39894 22804	1 25331 41373	1·25331 41373	·79788 45608	·79788 45608	·500
·501	·00250 66309	·39894 10271	1·25582 47109	1·25081 14380	·79628 94752	·79948 10162	·499
·502	·00501 32775*	·39893 72071	1·25834 32067	1·24831 65676	·79469 57513	·80107 88497	·498
·503	·00751 99557	·39893 10005+	1·26086 96726	1·24582 94777	·79310 33808	·80267 80695−	·497
·504	·01002 66811	·39892 22272	1·26340 41566	1·24335 01223	·79151 23556	·80427 86839	·496
·505	·01253 34696	·39891 09471	1·26594 67072	1·24087 84556	·78992 26676	·80588 07013	·495
·506	·01504 03367	·39889 71602	1·26849 73733	1·23841 44317	·78833 43088	·80748 41300*	·494
·507	·01754 72984	·39888 08664	1·27105 62042	1·23595 80052	·78674 72711	·80908 89786	·493
·508	·02005 43703*	·39886 20656	1·27362 32492	1·23350 91311*	·78516 15465−	·81069 52553	·492
·509	·02256 15684	·39884 07576*	1·27619 85585+	1·23106 77647	·78357 71270	·81230 29687	·491
·510	·02506 89083	·39881 69424	1·27878 21824	1·22863 38615+	·78199 40047*	·81391 21274	·490
·511	·02757 64057*	·39879 06198	1·28137 41714	1·22620 73774	·78041 21718	·81552 27398	·489
·512	·03008 40766	·39876 17895+	1·28397 45769	1·22378 82686	·77883 16202	·81713 48146	·488
·513	·03259 19367	·39873 04516	1·28658 34500	1·22137 64915+	·77725 23422	·81874 83605+	·487
·514	·03510 00018	·39869 66056	1·28920 08429	1·21897 20032	·77567 43300	·82036 33860	·486
·515	·03760 82877	·39866 02515−	1·29182 68076	1·21657 47606	·77409 75757	·82197 99000	·485
·516	·04011 68102	·39862 13890	1·29446 13969	1·21418 47212	·77252 20716	·82359 79111	·484
·517	·04262 55852	·39858 00178	1·29710 46639	1·21180 18427	·77094 78100	·82521 74281	·483
·518	·04513 46285+	·39853 61377	1·29975 66619	1·20942 60831	·76937 47832	·82683 84600	·482
·519	·04764 39560	·39848 97484	1·30241 74450−	1·20705 74008	·76780 29835−	·82846 10154*	·481
·520	·05015 35835−	·39844 08497	1·30508 70673	1·20469 57544	·76623 24032	·83008 51035+	·480
·521	·05266 35269	·39838 94412	1·30776 55836	1·20234 11027	·76466 30348*	·83171 07331	·479
·522	·05517 38022	·39833 55225+	1·31045 30490	1·19999 34050	·76309 48707	·83333 79132	·478
·523	·05768 44251	·39827 90934	1·31314 95191	1·19765 26206	·76152 79033	·83496 66529	·477
·524	·06019 54118	·39822 01535*	1·31585 50498*	1·19531 87094	·75996 21251	·83659 69612	·476
·525	·06270 67780	·39815 87025−	1·31856 96978	1·19299 16313	·75839 75285*	·83822 88473	·475
·526	·06521 85397*	·39809 47399	1·32129 35197	1·19067 13466	·75683 41062	·83986 23204	·474
·527	·06773 07130*	·39802 82653	1·32402 65729	1·18835 78159	·75527 18506	·84149 73896	·473
·528	·07024 33138*	·39795 92783	1·32676 89153	1·18605 10000*	·75371 07543	·84313 40642	·472
·529	·07275 63582	·39788 77785−	1·32952 06050	1·18375 08600*	·75215 08100	·84477 23535−	·471
·530	·07526 98622	·39781 37654	1·33228 17008	1·18145 73573	·75059 20102	·84641 22669	·470
·531	·07778 38417	·39773 72386	1·33505 22618	1·17917 04534	·74903 43477	·84805 38137	·469
·532	·08029 83130	·39765 81976	1·33783 23476	1·17689 01103	·74747 78150−	·84969 70034	·468
·533	·08281 32920	·39757 66418	1·34062 20183*	1·17461 62900	·74592 24049	·85134 18454	·467
·534	·08532 87949*	·39749 25708	1·34342 13346*	1·17234 89549*	·74436 81101*	·85298 83494	·466
·535	·08784 48380	·39740 59840*	1·34623 03576	1·17008 80678	·74281 49235−	·85463 65248	·465
·536	·09036 14372	·39731 68810	1·34904 91487	1·16783 35914	·74126 28376	·85628 63814	·464
·537	·09287 86088*	·39722 52610	1·35187 77700	1·16558 54888	·73971 18454	·85793 79287	·463
·538	·09539 63691	·39713 11235+	1·35471 62841	1·16334 37235*	·73816 19397	·85959 11765*	·462
·539	·09791 47343	·39703 44680	1·35756 47542	1·16110 82591	·73661 31132*	·86124 61346	·461
·540	·10043 37206	·39693 52939	1·36042 32436*	1·15887 90594	·73506 53590	·86290 28127*	·460
·541	·10295 33443	·39683 36004	1·36329 18167	1·15665 60885−	·73351 86698*	·86456 12209	·459
·542	·10547 36218	·39672 93870	1·36617 05379	1·15443 93106*	·73197 30387	·86622 13689	·458
·543	·10799 45695−	·39662 26529	1·36905 94725+	1·15222 86905+	·73042 84584	·86788 32668	·457
·544	·11051 62036	·39651 33976	1·37195 86861*	1·15002 41928	·72888 49220	·86954 69245+	·456
·545	·11303 85407	·39640 16203	1·37486 82450	1·14782 57825+	·72734 24225+	·87121 23523	·455
·546	·11556 15972	·39628 73203	1·37778 82159*	1·14563 34250−	·72580 09529	·87287 95601	·454
·547	·11808 53895−	·39617 04968	1·38071 86662	1·14344 70836	·72426 05061	·87454 85581*	·453
·548	·12060 99342	·39605 11492	1·38365 96698	1·14126 67300	·72272 10753	·87621 93567	·452
·549	·12313 52478	·39592 92767	1·38661 12770	1·13909 23241	·72118 26534	·87789 19661	·451
·550	·12566 13469	·39580 48785−	1·38957 35750+	1·13692 38341	·71964 52336	·87956 63966	·450
·551	·12818 82481*	·39567 79538	1·39254 66273	1·13476 12262	·71810 88090	·88124 26587	·449
·552	·13071 59682	·39554 85017	1·39553 05041	1·13260 44671	·71657 33727	·88292 07628	·448
·553	·13324 45236*	·39541 65215+	1·39852 52762	1·13045 35234	·71503 89178	·88460 07193	·447
·554	·13577 39313	·39528 20124	1·40153 10149	1·12830 83622	·71350 54375+	·88628 25390	·446
·555	·13830 42080	·39514 49734	1·40454 77923	1·12616 89506	·71197 29250+	·88796 62323	·445
·556	·14083 53704	·39500 54037	1·40757 56808	1·12403 52559	·71044 13735+	·88965 18101	·444
·557	·14336 74355−	·39486 33023*	1·41061 47537	1·12190 72458	·70891 07762	·89133 92829*	·443
·558	·14590 04200*	·39471 86685−	1·41366 50849	1·11978 48880	·70738 11263	·89302 86618	·442
·559	·14843 43411	·39457 15012	1·41672 67486*	1·11766 81505*	·70585 24172	·89471 99574	·441
·560	·15096 92155+	·39442 17995−	1·41979 98202	1·11555 70016	·70432 46420	·89641 31807	·440

$\frac{1}{2}(1+\alpha_x)$	x	z	$\frac{\frac{1}{2}(1+\alpha_x)}{z}$	$\frac{\frac{1}{2}(1-\alpha_x)}{z}$	$\frac{z}{\frac{1}{2}(1+\alpha_x)}$	$\frac{z}{\frac{1}{2}(1-\alpha_x)}$	$\frac{1}{2}(1-\alpha_x)$
·560	·15096 92155+	·39442 17995−	1·41979 98202	1·11555 70016	·70432 46420	·89641 31807	·440
·561	·15350 50604	·39426 95624	1·42288 43752	1·11345 14094	·70279 77940	·89810 83427	·439
·562	·15604 18928	·39411 47890^{5}	1·42598 04900	1·11135 13427	·70127 18666	·89980 54544^{5}	·438
·563	·15857 97298	·39395 74783	1·42908 82417	1·10925 67702	·69974 68531	·90150 45270	·437
·564	·16111 85885+	·39379 76292^{5}	1·43220 77080	1·10716 76608	·69822 27469	·90320 55717	·436
·565	·16365 84862^{5}	·39363 52408	1·43533 89672^{5}	1·10508 39836	·69669 95412	·90490 85995	·435
·566	·16619 94402	·39347 03119	1·43848 20985+	1·10300 57080^{5}	·69517 72295+	·90661 36219	·434
·567	·16874 14676	·39330 28416	1·44163 71815^{5}	1·10093 28035	·69365 58052	·90832 06502^{5}	·433
·568	·17128 45859	·39313 28286	1·44480 42968	1·09886 52398	·69213 52617	·91002 96959	·432
·569	·17382 88125+	·39296 02720	1·44798 35253	1·09680 29867	·69061 55923	·91174 07703^{5}	·431
·570	·17637 41648	·39278 51706	1·45117 49491	1·09474 60143	·68909 67906	·91345 38852	·430
·571	·17892 06603	·39260 75233	1·45437 86507	1·09269 42927	·68757 88499	·91516 90520	·429
·572	·18146 83166	·39242 73289	1·45759 47133	1·09064 77921	·68606 17638	·91688 62825+	·428
·573	·18401 71512	·39224 45863	1·46082 32211	1·08860 64842	·68454 55258	·91860 55885−	·427
·574	·18656 71819	·39205 92942	1·46406 42588	1·08657 03384	·68303 01293	·92032 69817	·426
·575	·18911 84263	·39187 14515	1·46731 79120	1·08453 93263	·68151 55678^{5}	·92205 04741^{5}	·425
·576	·19167 09023	·39168 10569^{5}	1·47058 42669	1·08251 34187	·68000 18350−	·92377 60777	·424
·577	·19422 46276	·39148 81093	1·47386 34107	1·08049 25870	·67848 89242^{5}	·92550 38045−	·423
·578	·19677 96203	·39129 26073	1·47715 54311^{5}	1·07847 68027	·67697 68292	·92723 36665+	·422
·579	·19933 58981	·39109 45496	1·48046 04169^{5}	1·07646 60372	·67546 55434	·92896 56761	·421
·580	·20189 34792	·39089 39350	1·48377 84575^{5}	1·07446 02624	·67395 50604	·93069 98453	·420
·581	·20445 23816	·39069 07622	1·48710 96432	1·07245 94501	·67244 53739	·93243 61867	·419
·582	·20701 26234	·39048 50298	1·49045 40649	1·07046 35724	·67093 64774	·93417 47125	·418
·583	·20957 42230	·39027 67365	1·49381 18147	1·06847 26016	·66942 83645+	·93591 54353	·417
·584	·21213 71984	·39006 58809	1·49718 29851	1·06648 65099^{5}	·66792 10290	·93765 83676	·416
·585	·21470 15680^{5}	·38985 24617	1·50056 76699	1·06450 52701	·66641 44644	·93940 35221	·415
·586	·21726 73504	·38963 64773	1·50396 59634	1·06252 88547	·66490 86644	·94115 09114	·414
·587	·21983 45638^{5}	·38941 79265	1·50737 79610	1·06055 72366	·66340 36226^{5}	·94290 05484	·413
·588	·22240 32270	·38919 68077	1·51080 37588	1·05859 03888	·66189 93328	·94465 24459	·412
·589	·22497 33584	·38897 31195	1·51424 34539	1·05662 82845−	·66039 57887	·94640 66169	·411
·590	·22754 49767	·38874 68605−	1·51769 71442	1·05467 08968	·65889 29839	·94816 30744	·410
·591	·23011 81007	·38851 80291	1·52116 49287	1·05271 81994	·65739 09122	·94992 18315	·409
·592	·23269 27492	·38828 66238	1·52464 69070	1·05077 01657	·65588 95672	·95168 29014	·408
·593	·23526 89411^{5}	·38805 26431	1·52814 31800	1·04882 67694	·65438 89428	·95344 62975−	·407
·594	·23784 66954	·38781 60854	1·53165 38493	1·04688 79845+	·65288 90326	·95521 20330	·406
·595	·24042 60312	·38757 69492	1·53517 90175−	1·04495 37850^{5}	·65138 98305+	·95698 01214	·405
·596	·24300 69674	·38733 52328	1·53871 87881	1·04302 41450−	·64989 13302	·95875 05762	·404
·597	·24558 95234	·38709 09347	1·54227 32658	1·04109 90387	·64839 35254	·96052 34111	·403
·598	·24817 37185−	·38684 40532	1·54584 25561	1·03917 84407	·64689 64100	·96229 86398	·402
·599	·25075 95719	·38659 45867	1·54942 67655−	1·03726 23254^{5}	·64539 99778	·96407 62760	·401
·600	·25334 71031	·38634 25335−	1·55302 60015−	1·03535 06677	·64390 42225−	·96585 63337	·400
·601	·25593 63317^{5}	·38608 78919	1·55664 03728	1·03344 34421^{5}	·64240 91380	·96763 88269	·399
·602	·25852 72773	·38583 06603	1·56026 99889	1·03154 06239	·64091 47180	·96942 37695−	·398
·603	·26111 99595^{5}	·38557 08368	1·56391 49605+	1·02964 21879	·63942 09565	·97121 11758	·397
·604	·26371 43982	·38530 84198	1·56757 53994	1·02774 81096	·63792 78473	·97300 10600	·396
·605	·26631 06132	·38504 34074	1·57125 14183^{5}	1·02585 83640	·63643 53841	·97479 34364^{5}	·395
·606	·26890 86244	·38477 57979	1·57494 31312	1·02397 29269	·63494 35609^{5}	·97658 83196	·394
·607	·27150 84520	·38450 55895+	1·57865 06529^{5}	1·02209 17737	·63345 23716	·97838 57240	·393
·608	·27411 01160^{5}	·38423 27804	1·58237 40997^{5}	1·02021 48801	·63196 18099	·98018 56643	·392
·609	·27671 36367^{5}	·38395 73687	1·58611 35888	1·01834 22220	·63047 18698	·98198 81552	·391
·610	·27931 90345−	·38367 93525+	1·58986 92385−	1·01647 37754	·62898 25451	·98379 32116	·390
·611	·28192 63296	·38339 87300	1·59364 11683	1·01460 95163	·62749 38298	·98560 08483	·389
·612	·28453 55427	·38311 54992	1·59742 94991	1·01274 94210	·62600 57177	·98741 10804^{5}	·388
·613	·28714 66943	·38282 96583	1·60123 43526	1·01089 34657	·62451 82027	·98922 39231^{5}	·387
·614	·28975 98052^{5}	·38254 12052	1·60505 58520	1·00904 16268	·62303 12788	·99103 93916	·386
·615	·29237 48962^{5}	·38225 01380	1·60889 41216	1·00719 38810	·62154 49398	·99285 75013	·385
·616	·29499 19882	·38195 64547	1·61274 92870	1·00535 02049	·62005 91797	·99467 82675	·384
·617	·29761 11022^{5}	·38166 01533	1·61662 14749	1·00351 05752	·61857 39924	·99650 17058	·383
·618	·30023 22594	·38136 12318	1·62051 08135	1·00167 49688	·61708 93719	·99832 78320	·382
·619	·30285 54809	·38105 96881	1·62441 74319	0·99984 33628	·61560 53120	1·00015 66618	·381
·620	·30548 07881	·38075 55202	1·62834 14610	0·99801 57342	·61412 18067	1·00198 82110	·380

$\frac{1}{2}(1+\alpha_x)$	x	z	$\frac{\frac{1}{2}(1+\alpha_x)}{z}$	$\frac{\frac{1}{2}(1-\alpha_x)}{z}$	$\frac{z}{\frac{1}{2}(1+\alpha_x)}$	$\frac{z}{\frac{1}{2}(1-\alpha_x)}$	$\frac{1}{2}(1-\alpha_x)$
·620	·30548 07881	·38075 55202	1·62834 14610	0·99801 57342	·61412 18067	1·00198 82110	·380
·621	·30810 82025−	·38044 87258^5	1·63228 30326^5	·99619 20602	·61263 88500	1·00382 24956^5	·379
·622	·31073 77455	·38013 93031	1·63624 22801	·99437 23182	·61115 64358	1·00565 95319	·378
·623	·31336 94389	·37982 72496	1·64021 93381	·99255 64855−	·60967 45580	1·00749 93359	·377
·624	·31600 33044	·37951 25634^5	1·64421 43425	·99074 45397	·60819 32107	1·00934 19241	·376
·625	·31863 93640	·37919 52423	1·64822 74308	·98893 64585−	·60671 23877	1·01118 73128	·375
·626	·32127 76396	·37887 52840	1·65225 87417	·98713 22195	·60523 20830^5	1·01303 55187	·374
·627	·32391 81533	·37855 26863	1·65630 84155	·98533 18006	·60375 22907	1·01488 65584	·373
·628	·32656 09274	·37822 74469	1·66037 65938	·98353 51798	·60227 30046	1·01674 04487	·372
·629	·32920 59843	·37789 95637	1·66446 34196	·98174 23349	·60079 42189	1·01859 72066^5	·371
·630	·33185 33464	·37756 90342	1·66856 90374	·97995 32443	·59931 59273	1·02045 68492	·370
·631	·33450 30364	·37723 58562	1·67269 35937	·97816 78860	·59783 81239	1·02231 93935	·369
·632	·33715 50769^5	·37690 00273	1·67683 72357	·97638 62385+	·59636 08028	1·02418 48569	·368
·633	·33980 94910	·37656 15453	1·68100 01126^5	·97460 82802	·59488 39578	1·02605 32568	·367
·634	·34246 63014	·37622 04076−	1·68518 23752	·97283 39894	·59340 75829	1·02792 46108	·366
·635	·34512 55314	·37587 66118^5	1·68938 41755	·97106 33450−	·59193 16722	1·02979 89366	·365
·636	·34778 72042	·37553 01557	1·69360 56675	·96929 63254	·59045 62196	1·03167 62519	·364
·637	·35045 13432	·37518 10366	1·69784 70067	·96753 29096	·58898 12192	1·03355 65747	·363
·638	·35311 79719	·37482 92522	1·70210 83501^5	·96577 30764	·58750 66648	1·03543 99231	·362
·639	·35578 71140	·37447 47998	1·70638 98366	·96401 68047	·58603 25506	1·03732 63153^5	·361
·640	·35845 87932	·37411 76771	1·71069 16865	·96226 40737	·58455 88705−	1·03921 57697	·360
·641	·36113 30335−	·37375 78814	1·71501 40021	·96051 48623	·58308 56184	1·04110 83047	·359
·642	·36380 98589	·37339 54102	1·71935 69672	·95876 91499	·58161 27884	1·04300 39390	·358
·643	·36648 92938	·37303 02608	1·72372 07475−	·95702 69158	·58014 03745	1·04490 26914	·357
·644	·36917 13624	·37266 24307	1·72810 55103	·95528 81392^5	·57866 83707	1·04680 45806	·356
·645	·37185 60893	·37229 19172	1·73251 14250	·95355 27998	·57719 67709	1·04870 96259	·355
·646	·37454 34991	·37191 87176	1·73693 86626	·95182 08770	·57572 55691	1·05061 78464	·354
·647	·37723 36166	·37154 28203	1·74138 73959	·95009 23505−	·57425 47594	1·05252 92615−	·353
·648	·37992 64609	·37116 42495−	1·74585 77998	·94836 71999	·57278 43356	1·05444 38906	·352
·649	·38262 20750	·37078 29755−	1·75035 00510	·94664 54051	·57131 42919	1·05636 17534	·351
·650	·38532 04663	·37039 90044	1·75486 43280^5	·94492 69459	·56984 46222	1·05828 28698	·350
·651	·38802 16661	·37001 23336	1·75940 08115+	·94321 18022	·56837 53204	1·06020 72596	·349
·652	·39072 57000	·36962 29601^5	1·76395 96841	·94149 99541	·56690 63806	1·06213 49429^5	·348
·653	·39343 25939^5	·36923 08812	1·76854 11302	·93979 13816	·56543 77967	1·06406 59402	·347
·654	·39614 23737	·36883 60940	1·77314 53365^5	·93808 60649	·56396 95627	1·06600 02717	·346
·655	·39885 50655−	·36843 85955+	1·77777 24917	·93638 39842	·56250 16725^5	1·06793 79580	·345
·656	·40157 06954^5	·36803 83829	1·78242 27866	·93468 51198	·56103 41203	1·06987 90200	·344
·657	·40428 92901	·36763 54531	1·78709 64141	·93298 94521	·55956 68998	1·07182 34785^5	·343
·658	·40701 08761	·36722 98033	1·79179 35692	·93129 69615+	·55810 00050+	1·07377 13547	·342
·659	·40973 54801	·36682 14304	1·79651 44493	·92960 76285^5	·55663 34300	1·07572 26697^5	·341
·660	·41246 31293	·36641 03313	1·80125 92538	·92792 14338	·55516 71687	1·07767 74451	·340
·661	·41519 38506	·36599 65031	1·80602 81844	·92623 83578	·55370 12150−	1·07963 57024	·339
·662	·41792 76715−	·36557 99426	1·81082 14452	·92455 83814	·55223 55628	1·08159 74633	·338
·663	·42066 46195−	·36516 06467	1·81563 92425+	·92288 14853	·55077 02062	1·08356 27499	·337
·664	·42340 47222	·36473 86123	1·82048 17850+	·92120 76502^5	·54930 51390	1·08553 15842^5	·336
·665	·42614 80077	·36431 38362	1·82534 92838	·91953 68572^5	·54784 03552	1·08750 39887	·335
·666	·42889 45039	·36388 63152	1·83024 19523	·91786 90872	·54637 58487	1·08947 99857	·334
·667	·43164 42392	·36345 60461	1·83516 00066	·91620 43211^5	·54491 16134	1·09145 95979^5	·333
·668	·43439 72421^5	·36302 30256^5	1·84010 36651	·91454 25401	·54344 76432	1·09344 28483	·332
·669	·43715 35413	·36258 72505+	1·84507 31486^5	·91288 37253	·54198 39320	1·09542 97599	·331
·670	·43991 31653^5	·36214 87175−	1·85006 86809	·91122 78578	·54052 04738	1·09742 03559	·330
·671	·44267 61441	·36170 74231	1·85509 04880	·90957 49180	·53905 72624	1·09941 46598	·329
·672	·44544 25062	·36126 33640	1·86013 87987	·90792 48898^5	·53759 42917	1·10141 26952	·328
·673	·44821 22813^5	·36081 65369	1·86521 38445−	·90627 77521	·53613 15556	1·10341 44860	·327
·674	·45098 54993	·36036 69383	1·87031 58595+	·90463 34869^5	·53466 90480	1·10542 00562	·326
·675	·45376 21901	·35991 45648	1·87544 50808	·90299 20759	·53320 67626	1·10742 94301	·325
·676	·45654 23838	·35945 94128	1·88060 17480	·90135 35005+	·53174 46934^5	1·10944 26320	·324
·677	·45932 61108	·35900 14788	1·88578 61038	·89971 77423	·53028 28343	1·11145 96868	·323
·678	·46211 34017	·35854 07594	1·89099 83936	·89808 47828	·52882 11790	1·11348 06191	·322
·679	·46490 42874^5	·35807 72508	1·89623 88659	·89645 46038	·52735 97214	1·11550 54543	·321
·680	·46769 87991	·35761 09496	1·90150 77721	·89482 71869	·52589 84553	1·11753 42174	·320

$\frac{1}{2}(1+\alpha_x)$	x	z	$\frac{\frac{1}{2}(1+\alpha_x)}{z}$	$\frac{\frac{1}{2}(1-\alpha_x)}{z}$	$\frac{z}{\frac{1}{2}(1+\alpha_x)}$	$\frac{z}{\frac{1}{2}(1-\alpha_x)}$	$\frac{1}{2}(1-\alpha_x)$
·680	·46769 87991	·35761 09496	1·90150 77721	·89482 71869	·52589 84553	1·11753 42174	·320
·681	·47049 69679	·35714 18520	1·90680 53665^{+}	·89320 25138	·52443 73744^5	1·11956 69342	·319
·682	·47329 88254	·35666 99544	1·91213 19067	·89158 05665^{-}	·52297 64727	1·12160 36302	·318
·683	·47610 44034^5	·35619 52531	1·91748 76533	·88996 13266	·52151 57439	1·12364 43316	·317
·684	·47891 37341	·35571 77443^5	1·92287 28700	·88834 47762	·52005 51818	1·12568 90644	·316
·685	·48172 68495^5	·35523 74244	1·92828 78239	·88673 08971	·51859 47801	1·12773 78552	·315
·686	·48454 37824	·35475 42894	1·93373 27850^{+}	·88511 96713	·51713 45326	1·12979 07305^5	·314
·687	·48736 45654^5	·35426 83355^{+}	1·93920 80271	·88351 10808	·51567 44331	1·13184 77173	·313
·688	·49018 92317	·35377 95589^5	1·94471 38270	·88190 51076	·51421 44752	1·13390 88428	·312
·689	·49301 78145	·35328 79558	1·95025 04650^{+}	·88030 17338^5	·51275 46528	1·13597 41343	·311
·690	·49585 03473	·35279 35220	1·95581 82250^{-}	·87870 09417	·51129 49594	1·13804 36194	·310
·691	·49868 68641	·35229 62538	1·96141 73938	·87710 27130	·50983 53890	1·14011 73262^5	·309
·692	·50152 73993	·35179 61469	1·96704 82640	·87550 70308	·50837 59348	1·14219 52821	·308
·693	·50437 19864	·35129 31975^5	1·97271 11280	·87391 38763	·50691 65910	1·14427 75164^5	·307
·694	·50722 06606	·35078 74016	1·97840 62849	·87232 32323	·50545 73510	1·14636 40575^{+}	·306
·695	·51007 34570	·35027 87549	1·98413 40370	·87073 50810	·50399 82085^{-}	1·14845 49341	·305
·696	·51293 04106	·34976 72533	1·98989 46898	·86914 94047	·50253 91570	1·15055 01753	·304
·697	·51579 15570	·34925 28927	1·99568 85530	·86756 61859	·50108 01903	1·15264 98108	·303
·698	·51865 69321	·34873 56688	2·00151 59402^5	·86598 54068	·49962 13020	1·15475 38701	·302
·699	·52152 65718	·34821 55774	2·00737 71692	·86440 70500^{-}	·49816 24855^{+}	1·15686 23833	·301
·700	·52440 05127	·34769 26142	2·01327 25614^5	·86283 10978	·49670 37346	1·15897 53807	·300
·701	·52727 87914	·34716 67749	2·01920 24429	·86125 75327	·49524 50427	1·16109 28927	·299
·702	·53016 14450^{+}	·34663 80552	2·02516 71435^{-}	·85968 63373	·49378 64034	1·16321 49502	·298
·703	·53304 85109	·34610 64506	2·03116 69975^{+}	·85811 74940	·49232 78102	1·16534 15844	·297
·704	·53594 00266	·34557 19567	2·03720 23436	·85655 09854	·49086 92566	1·16747 28265^{+}	·296
·705	·53883 60303	·34503 45690	2·04327 35248	·85498 67941	·48941 07362	1·16960 87085^{-}	·295
·706	·54173 65601	·34449 42831	2·04938 08886	·85342 49026	·48795 22423	1·17174 92622	·294
·707	·54464 16548	·34395 10944	2·05552 47871	·85186 52936^5	·48649 37685^{+}	1·17389 45200	·293
·708	·54755 13533	·34340 49982	2·06170 55769	·85030 79498	·48503 53082^5	1·17604 45145^{+}	·292
·709	·55046 56950^{+}	·34285 59901	2·06792 36194	·84875 28537	·48357 68549	1·17819 92787	·291
·710	·55338 47196	·34230 40653	2·07417 92809	·84719 99880	·48211 84018	1·18035 88458	·290
·711	·55630 84670	·34174 92191	2·08047 29324^5	·84564 93354	·48065 99425^{-}	1·18252 32494	·289
·712	·55923 69776	·34119 14468	2·08680 49500	·84410 08786	·47920 14702	1·18469 25235^5	·288
·713	·56217 02922^5	·34063 07435^{+}	2·09317 57146	·84255 46004	·47774 29783	1·18686 67022	·287
·714	·56510 84520	·34006 71046	2·09958 56124	·84101 04834	·47628 44602	1·18904 58202	·286
·715	·56805 14983	·33950 05250^5	2·10603 50348^5	·83946 85104	·47482 59091	1·19122 99123	·285
·716	·57099 94731	·33893 09999	2·11252 43786	·83792 86641	·47336 73183	1·19341 90138	·284
·717	·57395 24186	·33835 85244	2·11905 40456	·83639 09273^5	·47190 86812	1·19561 31604	·283
·718	·57691 03773	·33778 30934	2·12562 44436	·83485 52829	·47044 99908	1·19781 23880	·282
·719	·57987 33924	·33720 47020	2·13223 59855^{+}	·83332 17134	·46899 12406	1·20001 67329	·281
·720	·58284 15073	·33662 33449	2·13888 90902^5	·83179 02018	·46753 24235^{+}	1·20222 62319	·280
·721	·58581 47657	·33603 90172	2·14558 41823	·83026 07307	·46607 35329	1·20444 09220	·279
·722	·58879 32119	·33545 17137	2·15232 16921	·82873 32831	·46461 45619	1·20666 08406	·278
·723	·59177 68906	·33486 14291	2·15910 20561	·82720 78417	·46315 55035^{+}	1·20888 60255^{-}	·277
·724	·59476 58468	·33426 81581	2·16592 57168	·82568 43893	·46169 63510	1·21111 65150	·276
·725	·59776 01260	·33367 18956	2·17279 31227	·82416 29086	·46023 70974	1·21335 23476	·275
·726	·60075 97742^5	·33307 26361	2·17970 47290	·82264 33826	·45877 77357	1·21559 35624	·274
·727	·60376 48378	·33247 03742	2·18666 09970	·82112 57939	·45731 82589	1·21784 01987	·273
·728	·60677 53635^{+}	·33186 51046	2·19366 23945^{-}	·81961 01254	·45585 86602	1·22009 22963	·272
·729	·60979 13987	·33125 68217	2·20070 93961	·81809 63599	·45439 89323	1·22234 98955^{-}	·271
·730	·61281 29910	·33064 55199^5	2·20780 24832	·81658 44801	·45293 90684	1·22461 30368	·270
·731	·61584 01887	·33003 11938	2·21494 21439	·81507 44688	·45147 90613	1·22688 17614	·269
·732	·61887 30405^{+}	·32941 38377	2·22212 88733	·81356 63088	·45001 89039	1·22915 61108	·268
·733	·62191 15956	·32879 34458^5	2·22936 31739	·81205 99829	·44855 85891^5	1·23143 61268	·267
·734	·62495 59035^{-}	·32817 00125^{+}	2·23664 55558	·81055 54739	·44709 81097	1·23372 18515^{+}	·266
·735	·62800 60144	·32754 35321	2·24397 65343	·80905 27641	·44563 74586^5	1·23601 33287	·265
·736	·63106 19790^5	·32691 39986	2·25135 66356	·80755 18367	·44417 66285^{+}	1·23831 06007^5	·264
·737	·63412 38485^{+}	·32628 14062	2·25878 63913	·80605 26742	·44271 56122	1·24061 37117	·263
·738	·63719 16745^5	·32564 57489	2·26626 63414^5	·80455 52594	·44125 44023	1·24292 27058	·262
·739	·64026 55092	·32500 70208	2·27379 70340	·80305 95749^5	·43979 29916^5	1·24523 76277	·261
·740	·64334 54054	·32436 52159	2·28137 90252	·80156 56034^5	·43833 13728	1·24755 85226	·260

$\frac{1}{2}(1+\alpha_x)$	x	z	$\frac{\frac{1}{2}(1+\alpha_x)}{z}$	$\frac{\frac{1}{2}(1-\alpha_x)}{z}$	$\frac{z}{\frac{1}{2}(1+\alpha_x)}$	$\frac{z}{\frac{1}{2}(1-\alpha_x)}$	$\frac{1}{2}(1-\alpha_x)$
·740	·64334 54054	·32436 52159	2·28137 90252	·80156 56034^{s}	·43833 13728	1·24755 85226	·260
·741	·64643 14163	·32372 03280	2·28901 28792	·80007 33275^{s}	·43686 95384	1·24988 54362	·259
·742	·64952 35958	·32307 23510	2·29660 91689	·79858 27299	·43540 74811	1·25221 84147	·258
·743	·65262 19983^{s}	·32242 12787	2·30443 84756	·79709 37910	·43394 51934	1·25455 75047	·257
·744	·65572 66788	·32176 71049	2·31223 13893^{s}	·79560 64996	·43248 26679	1·25690 27535^{s}	·256
·745	·65883 76927	·32110 98232	2·32007 85092	·79412 08320	·43101 98970	1·25925 42088	·255
·746	·66195 50963	·32044 94274	2·32798 04433	·79263 67729	·42955 68732	1·26161 19188	·254
·747	·66507 89462	·31978 59109	2·33593 78089	·79115 43047	·42809 35889	1·26397 59324	·253
·748	·66820 92997	·31911 92673	2·34395 12327	·78967 34099	·42663 00365^{+}	1·26634 62989	·252
·749	·67134 62149	·31844 94901	2·35202 13511^{s}	·78819 40709	·42516 62084	1·26872 30682	·251
·750	·67448 97502	·31777 65727	2·36014 88104	·78671 62701	·42370 20969	1·27110 62907	·250
·751	·67763 99649	·31710 05084	2·36833 42667	·78523 99899	·42223 76942^{s}	1·27349 60176	·249
·752	·68079 69188	·31642 12905^{+}	2·37657 83864	·78376 52125^{+}	·42077 29927	1·27589 23004	·248
·753	·68396 06724	·31573 89123	2·38488 18461	·78229 19203	·41930 79844	1·27829 51914	·247
·754	·68713 12868	·31505 33669	2·39324 53332	·78082 00954	·41784 26616^{s}	1·28070 47434	·246
·755	·69030 88240	·31436 46474	2·40166 95461	·77934 97202	·41637 70164	1·28312 10098	·245
·756	·69349 33463	·31367 27469	2·41015 51936	·77788 07768	·41491 10409	1·28554 40446	·244
·757	·69668 49171	·31297 76583^{s}	2·41870 29962	·77641 32471	·41344 47270	1·28797 39027	·243
·758	·69988 36002	·31227 93747	2·42731 36859	·77494 71134	·41197 80669	1·29041 06392	·242
·759	·70308 94604	·31157 78888	2·43598 80062	·77348 23577	·41051 10524	1·29285 43102	·241
·760	·70630 25629	·31087 31933^{s}	2·44472 67125^{+}	·77201 89619	·40904 36755^{-}	1·29530 49723	·240
·761	·70952 29739	·31016 52812	2·45353 05727	·77055 69078^{s}	·40757 59280	1·29776 26828	·239
·762	·71275 07602^{s}	·30945 41449^{s}	2·46240 03667	·76909 61775^{+}	·40610 78018	1·30022 74998	·238
·763	·71598 59896	·30873 97772	2·47133 68874	·76763 67527	·40463 92886	1·30269 94818	·237
·764	·71922 87305^{-}	·30802 21705^{-}	2·48034 09405^{+}	·76617 86151	·40317 03802	1·30517 86884	·236
·765	·72247 90519	·30730 13172	2·48941 33450^{-}	·76472 17465	·40170 10682^{s}	1·30766 51796	·235
·766	·72573 70241	·30657 72098	2·49855 49332	·76326 61284	·40023 13444	1·31015 90163	·234
·767	·72900 27178	·30584 98406	2·50776 65515^{-}	·76181 17425^{-}	·39876 12002^{s}	1·31266 02600	·233
·768	·73227 62048	·30511 92018	2·51704 90599^{s}	·76035 85702	·39729 06273	1·31516 89732	·232
·769	·73555 75574	·30438 52859	2·52640 33306	·75890 65921	·39581 96175^{+}	1·31768 52202	·231
·770	·73884 68492	·30364 80841	2·53583 02605^{s}	·75745 57921	·39434 81611	1·32020 90609^{s}	·230
·771	·74214 41544	·30290 75892	2·54533 07461^{s}	·75600 61490	·39287 62506	1·32274 05641	·229
·772	·74544 95482	·30216 37930	2·55490 57096	·75455 76448	·39140 38770	1·32527 97939	·228
·773	·74876 31066	·30141 66874	2·56455 60860	·75311 02607	·38993 10315^{+}	1·32782 68166	·227
·774	·75208 49067	·30066 62640^{s}	2·57428 28263	·75166 39777	·38845 77055^{-}	1·33038 16993	·226
·775	·75541 50264	·29991 25148	2·58408 68980	·75021 87769^{s}	·38698 38900	1·33294 45101	·225
·776	·75875 35445^{+}	·29915 54312	2·59396 92891	·74877 46390	·38550 95763	1·33551 53178^{s}	·224
·777	·76210 05410	·29839 50049	2·60393 09886^{s}	·74733 15450^{-}	·38403 47553	1·33809 41922	·223
·778	·76545 60967	·29763 12273	2·61397 30268	·74588 94755^{+}	·38255 94181	1·34068 12039	·222
·779	·76882 02935^{-}	·29686 40898	2·62409 64360	·74444 84112	·38108 35556	1·34327 64244	·221
·780	·77219 32142	·29609 35838	2·63430 22705^{-}	·74300 83327	·37960 71587	1·34587 99262	·220
·781	·77557 49428	·29531 97004	2·64459 16031^{s}	·74156 92203^{s}	·37813 02182	1·34849 17828	·219
·782	·77896 55644	·29454 24309	2·65496 55259	·74013 10545^{+}	·37665 27250^{-}	1·35111 20684	·218
·783	·78236 51649	·29376 17663	2·66542 51499	·73869 38155^{-}	·37517 46696	1·35374 08586	·217
·784	·78577 38315^{+}	·29297 76976	2·67597 16064	·73725 74834	·37369 60428	1·35637 82295^{s}	·216
·785	·78919 16527	·29219 02156	2·68660 60467	·73582 20383	·37221 68352	1·35902 42586	·215
·786	79261 87177	·29139 93112	2·69732 96430	·73438 74600	·37073 70371	1·36167 90242	·214
·787	·79605 51173	·29060 49750^{s}	2·70814 35886	·73295 37286	·36925 66392	1·36434 26059	·213
·788	·79950 09431	·28980 71978	2·71904 90987	·73152 08235^{s}	·36777 56317	1·36701 50840	·212
·789	·80295 62883	·28900 59700	2·73004 74105^{+}	·73008 87245^{-}	·36629 40051	1·36969 65402^{s}	·211
·790	·80642 12470	·28820 12820	2·74113 97841	·72865 74109^{s}	·36481 17494	1·37238 70573	·210
·791	·80989 59177	·28739 31243	2·75232 75027	·72722 68623	·36332 88549^{s}	1·37508 67190	·209
·792	·81338 03882	·28658 14869	2·76361 18733^{s}	·72579 70576^{s}	·36184 53118	1·37779 56103	·208
·793	·81687 47655^{-}	·28576 63602	2·77499 42277	·72436 79762	·36036 11099^{s}	1·38051 38173	·207
·794	·82037 91459^{s}	·28494 77341	2·78647 59220	·72293 95969	·35887 62394	1·38324 14275^{+}	·206
·795	·82389 36303	·28412 55985^{+}	2·79805 83380	·72151 18985^{-}	·35739 06900	1·38597 85294	·205
·796	·82741 83207	·28329 99434	2·80974 28839	·72008 48597	·35590 44515^{+}	1·38872 52128	·204
·797	·83095 33205^{s}	·28247 07584^{s}	2·82153 90045^{s}	·71865 84591	·35441 75137	1·39148 15687	·203
·798	·83449 87348	·28163 80333	2·83342 41319	·71723 26750^{+}	·35292 98663	1·39424 76896	·202
·799	·83805 46698^{s}	·28080 17575^{-}	2·84542 37865^{-}	·71580 74857	·35144 14987	1·39702 36690	·201
·800	·84162 12335^{s}	·27996 19204	2·85753 14772	·71438 28693	·34995 24005^{+}	1·39980 96021	·200

$\frac{1}{2}(1+a_x)$	x	z	$\frac{\frac{1}{2}(1+a_x)}{z}$	$\frac{\frac{1}{2}(1-a_x)}{z}$	$\frac{z}{\frac{1}{2}(1+a_x)}$	$\frac{z}{\frac{1}{2}(1-a_x)}$	$\frac{1}{2}(1-a_x)$
·800	·84162 12335^5	·27996 19204	2·85753 14772	·71438 28693	·34995 24005+	1·39980 96021	·200
·801	·84519 85353	·27911 85114	2·86974 87526	·71295 88037	·34846 25611	1·40260 55856^5	·199
·802	·84878 66859	·27827 15197	2·88207 71913	·71153 52667	·34697 19697	1·40541 17158	·198
·803	·85238 57979	·27742 09344	2·89451 84029	·71011 22358	·34548 06157	1·40822 80934	·197
·804	·85599 59855−	·27656 67444	2·90707 40288	·70868 96886	·34398 84881	1·41105 48186	·196
·805	·85961 73642	·27570 89387	2·91974 57427	·70726 76023	·34249 55760	1·41389 19933^5	·195
·806	·86325 00516	·27484 75059	2·93253 52517	·70584 59539	·34100 18684	1·41673 97213	·194
·807	·86689 41666	·27398 24348	2·94544 42969	·70442 47203	·33950 73541	1·41959 81077	·193
·808	·87054 98302	·27311 37138	2·95847 46547	·70300 38783^5	·33801 20220	1·42246 72591^5	·192
·809	·87421 71648^5	·27224 13312	2·97162 81372	·70158 34044^5	·33651 58606	1·42534 72840	·191
·810	·87789 62950	·27136 52755−	2·98490 65933	·70016 32750	·33501 88586	1·42823 82927	·190
·811	·88158 73470	·27048 55347	2·99831 19097	·69874 34660	·33352 10045+	1·43114 03951	·189
·812	·88529 04488	·26960 20968	3·01184 60119	·69732 39535^5	·33202 22867	1·43405 37063	·188
·813	·88900 57306	·26871 49497	3·02551 08652	·69590 47131^5	·33052 26934	1·43697 83407	·187
·814	·89273 33243	·26782 40812	3·03930 84755^5	·69448 57205−	·32902 22128	1·43991 44151	·186
·815	·89647 33640	·26692 94789	3·05324 08909	·69306 69507	·32752 08330	1·44286 20482	·185
·816	·90022 59857	·26603 11303	3·06731 02020	·69164 83789	·32601 85420	1·44582 13603	·184
·817	·90399 13276	·26512 90227	3·08151 85439^5	·69022 99799	·32451 53277	1·44879 24738	·183
·818	·90776 95299	·26422 31433^5	3·09586 80970^5	·68881 17282	·32301 11777	1·45177 55129	·182
·819	·91156 07351	·26331 34793	3·11036 10881	·68739 35982	·32150 60797	1·45477 06039	·181
·820	·91536 50879	·26240 00175−	3·12499 97917	·68597 55640	·32000 00213	1·45777 78750−	·180
·821	·91918 27352	·26148 27447	3·13978 65314^5	·68455 75994	·31849 29899	1·46079 74564	·179
·822	·92301 38263	·26056 16475+	3·15472 36815+	·68313 96780	·31698 49727	1·46382 94806	·178
·823	·92685 85128	·25963 67125+	3·16981 36678	·68172 17730	·31547 59569	1·46687 40821	·177
·824	·93071 69489	·25870 79259	3·18505 89695+	·68030 38576	·31396 59295+	1·46993 13974	·176
·825	·93458 92911	·25777 52740	3·20046 21205+	·67888 59043^5	·31245 48776	1·47300 15656	·175
·826	·93847 56984	·25683 87427	3·21602 57109^5	·67746 78858	·31094 27877	1·47608 47280	·174
·827	·94237 63326	·25589 83178^5	3·23175 23888	·67604 97742	·30942 96467	1·47918 10280	·173
·828	·94629 13579^5	·25495 39852	3·24764 48616	·67463 15413	·30791 54411	1·48229 06117	·172
·829	·95022 09415+	·25400 57303	3·26370 58979	·67321 31587	·30640 01571^5	1·48541 36274	·171
·830	·95416 52531	·25305 35384	3·27993 83292^5	·67179 45976	·30488 37812	1·48855 02260	·170
·831	·95812 44654	·25209 73948	3·29634 50520	·67037 58289	·30336 62994	1·49170 05610	·169
·832	·96209 87539	·25113 72844^5	3·31292 90293	·66895 68232	·30184 76977	1·49486 47884	·168
·833	·96608 82971	·25017 31922	3·32969 32922	·66753 75508	·30032 79618	1·49804 30671^5	·167
·834	·97009 32766	·24920 51027	3·34664 09431	·66611 79815−	·29880 70776	1·50123 55585^5	·166
·835	·97411 38770	·24823 30005−	3·36377 51567	·66469 80848^5	·29728 50305−	1·50444 24270	·165
·836	·97815 02862	·24725 68697	3·38109 91825−	·66327 78300^5	·29576 18059	1·50766 38396	·164
·837	·98220 26953	·24627 66945^5	3·39861 63471	·66185 71859	·29423 73889	1·51089 99665^5	·163
·838	·98627 12987	·24529 24589	3·41633 00565^5	·66043 61207	·29271 17648	1·51415 09809	·162
·839	·99035 62942	·24430 41465−	3·43424 37985−	·65901 46026	·29118 49183	1·51741 70589	·161
·840	·99445 78832	·24331 17408	3·45236 11450−	·65759 25990	·28965 68343	1·52069 83799	·160
·841	·99857 62706	·24231 52251	3·47068 57548^5	·65617 00773	·28812 74972	1·52399 51265+	·159
·842	1·00271 16650+	·24131 45826	3·48922 13765+	·65474 70041	·28659 68914	1·52730 74847	·158
·843	1·00686 42788	·24030 97961	3·50797 18507	·65332 33459	·28506 50012	1·53063 56436	·157
·844	1·01103 43281	·23930 08482	3·52694 11133	·65189 90683	·28353 18107	1·53397 97962	·156
·845	1·01522 20332	·23828 77215+	3·54613 31983	·65047 41370	·28199 73036	1·53734 01388	·155
·846	1·01942 76182	·23727 03982	3·56555 22410	·64904 85167	·28046 14636	1·54071 68714	·154
·847	1·02365 13115+	·23624 88602	3·58520 24816	·64762 21720	·27892 42742	1·54411 01976	·153
·848	1·02789 33458	·23522 30895−	3·60508 82671	·64619 50667	·27738 57187	1·54752 03255−	·152
·849	1·03215 39579	·23419 30674	3·62521 40578	·64476 71646	·27584 57802	1·55094 74659	·151
·850	1·03643 33895−	·23315 87753	3·64558 44265+	·64333 84282	·27430 44415−	1·55439 18351	·150
·851	1·04073 18864	·23212 01943	3·66620 40662	·64190 88200	·27276 16854	1·55785 36527	·149
·852	1·04504 96998	·23107 73050−	3·68707 77942	·64047 83023	·27121 74941	1·56133 31419	·148
·853	1·04938 70848	·23003 00883	3·70821 05495+	·63904 68356	·26967 18502^5	1·56483 05324	·147
·854	1·05374 43022	·22897 85242	3·72960 74067	·63761 43810	·26812 47356^5	1·56834 60565−	·146
·855	1·05812 16178	·22792 25929^5	3·75127 35746	·63618 08986	·26657 61321	1·57187 99514	·145
·856	1·06251 93023	·22686 22742	3·77321 44003	·63474 63477	·26502 60213	1·57543 24599	·144
·857	1·06693 76321	·22579 75475	3·79543 53775+	·63331 06873	·26347 43845−	1·57900 38288	·143
·858	1·07137 68892	·22472 83920	3·81794 21493	·63187 38755+	·26192 12028	1·58259 43099	·142
·859	1·07583 73609	·22365 47867	3·84074 05129	·63043 58699	·26036 64571	1·58620 41608	·141
·860	1·08031 93408	·22257 67101	3·86383 64255−	·62899 66274	·25881 01280	1·58983 36437	·140

$\frac{1}{2}(1+\alpha_x)$	x	z	$\frac{\frac{1}{2}(1+\alpha_x)}{z}$	$\frac{\frac{1}{2}(1-\alpha_x)}{z}$	$\frac{z}{\frac{1}{2}(1+\alpha_x)}$	$\frac{z}{\frac{1}{2}(1-\alpha_x)}$	$\frac{1}{2}(1-\alpha_x)$
·860	1·08031 93408	·22257 67101	3·86383 64255−	·62899 66274	·25881 01280	1·58983 36437	·140
·861	1·08482 31279	·22149 41407	3·88723 60109	·62755 61040	·25725 21960	1·59348 30267*	·139
·862	1·08934 90270	·22040 70565−	3·91094 55647	·62611 42551	·25569 26409	1·59715 25832	·138
·863	1·09389 73525−	·21931 54352	3·93497 15591	·62467 10355−	·25413 14429	1·60084 25927	·137
·864	1·09846 84202	·21821 92542	3·95932 06524	·62322 63990	·25256 85813	1·60455 33399	·136
·865	1·10306 25561	·21711 84907	3·98399 96923	·62178 02988	·25100 40354*	1·60828 51160	·135
·866	1·10768 00920	·21601 31213	4·00901 57241	·62033 26871	·24943 77844	1·61203 82188	·134
·867	1·11232 13671	·21490 31226	4·03437 59995+	·61888 35155	·24786 98069	1·61581 29518	·133
·868	1·11698 67277	·21378 84706	4·06008 79819	·61743 27346	·24630 00813	1·61960 96255−	·132
·869	1·12167 65277	·21266 91410	4·08615 93548	·61598 02940	·24472 85857	1·62342 85573	·131
·870	1·12639 11289	·21154 51092	4·11259 80324	·61452 61428	·24315 52980	1·62727 00710	·130
·871	1·13113 09007	·21041 63503	4·13941 21638	·61307 02286	·24158 01955*	1·63113 44986	·129
·872	1·13589 62211	·20928 28389	4·16661 01464	·61161 24985*	·24000 32556	1·63502 21789	·128
·873	1·14068 74762	·20814 45492*	4·19420 06320	·61015 28984	·23842 44550	1·63893 34586	·127
·874	1·14550 50613	·20700 14552	4·22219 25411	·60869 13732	·23684 37702	1·64286 86918	·126
·875	1·15034 93802	·20585 35302	4·25059 50669*	·60722 78667	·23526 11774	1·64682 82416	·125
·876	1·15522 08464	·20470 07474	4·27941 76924	·60576 23218	·23367 66523	1·65081 24789	·124
·877	1·16011 98829*	·20354 30793	4·30867 01991	·60429 46801	·23209 01702	1·65482 17829	·123
·878	1·16504 69221	·20238 04983	4·33836 26756	·60282 48820	·23050 17065−	1·65885 65433	·122
·879	1·17000 24074	·20121 29760	4·36850 55383*	·60135 28670*	·22891 12355−	1·66291 71569	·121
·880	1·17498 67920	·20004 04838	4·39910 95366*	·59987 85732	·22731 87316	1·66700 40317	·120
·881	1·18000 05403	·19886 29927	4·43018 57685+	·59840 19370	·22572 41687	1·67111 75854	·119
·882	1·18504 41279	·19768 04728	4·46174 57036	·59692 28946	·22412 75201	1·67525 82437	·118
·883	1·19011 80420	·19649 28942	4·49380 11811	·59544 13796	·22252 87590	1·67942 64461	·117
·884	1·19522 27816	·19530 02264	4·52636 44411*	·59395 73249	·22092 78579	1·68362 26411	·116
·885	1·20035 88581	·19410 24382	4·55944 81355+	·59247 06617	·21932 47889	1·68784 72888	·115
·886	1·20552 67961	·19289 94980	4·59306 53476	·59098 13201	·21771 95237	1·69210 08600	·114
·887	1·21072 71329	·19169 13738	4·62722 96054	·58948 92282	·21611 20336	1·69638 38390	·113
·888	1·21596 04197	·19047 80328	4·66195 49084	·58799 43128	·21450 22892	1·70069 67215+	·112
·889	1·22122 72221	·18925 94418	4·69725 57433	·58649 64989	·21289 02607	1·70504 00163	·111
·890	1·22652 81200	·18803 55670	4·73314 71070	·58499 57099	·21127 59180	1·70941 42455	·110
·891	1·23186 37089	·18680 63740	4·76964 45309	·58349 18674	·20965 92301	1·71381 99448	·109
·892	1·23723 45993	·18557 18278	4·80676 41009	·58198 48911	·20804 01657	1·71825 76648	·108
·893	1·24264 14187	·18433 18928	4·84452 24882	·58047 46990	·20641 86929*	1·72272 79702	·107
·894	1·24808 48112	·18308 65328	4·88293 69721	·57896 12070	·20479 47794	1·72723 14413	·106
·895	1·25356 54386	·18183 57108	4·92202 54712	·57744 43290	·20316 83919	1·73176 86740	·105
·896	1·25908 39805+	·18057 93893	4·96180 65699	·57592 39769	·20153 94969	1·73634 02813	·104
·897	1·26464 11358	·17931 75299	5·00229 95540	·57440 00603	·19990 80601	1·74094 68925*	·103
·898	1·27023 76225+	·17805 00939	5·04352 44402	·57287 24865	·19827 40466	1·74558 91556	·102
·899	1·27587 41794	·17677 70413	5·08550 20158*	·57134 11608*	·19663 74208	1·75026 77359	·101
·900	1·28155 15658	·17549 83319	5·12825 38719	·56980 59858	·19499 81465+	1·75498 33188	·100
·901	1·28727 05633	·17421 39243	5·17180 24462*	·56826 68615−	·19335 61868	1·75973 66093*	·099
·902	1·29303 19763	·17292 37766	5·21617 10643*	·56672 36855−	·19171 15040	1·76452 83330	·098
·903	1·29883 66327	·17162 78460	5·26138 39837	·56517 63526	·19006 40598	1·76935 92369	·097
·904	1·30468 53854*	·17032 60887	5·30746 64436	·56362 47551	·18841 38149	1·77423 00902	·096
·905	1·31057 91123	·16901 84602	5·35444 47090	·56206 87816	·18676 07295	1·77914 16864	·095
·906	1·31651 87185*	·16770 49152	5·40234 61337	·56050 83185*	·18510 47629	1·78409 48421	·094
·907	1·32250 51368	·16638 54072	5·45119 92086	·55894 32485+	·18344 58734	1·78909 03999	·093
·908	1·32853 93290	·16505 98890	5·50103 36292	·55737 34514	·18178 40187	1·79412 92278	·092
·909	1·33462 22868	·16372 83123	5·55188 03522	·55579 88031	·18011 91554	1·79921 22228	·091
·910	1·34075 50338	·16239 06278	5·60377 16730*	·55421 91764*	·17845 12393	1·80434 03088	·090
·911	1·34693 86262	·16104 67852*	5·65674 12916*	·55263 44401	·17678 02253	1·80951 44410	·089
·912	1·35317 41546	·15969 67332*	5·71082 43946	·55104 44591	·17510 60672	1·81473 56051*	·088
·913	1·35946 27455−	·15834 04193	5·76605 77398	·54944 90946	·17342 87177	1·82000 48191	·087
·914	1·36580 55627	·15697 77897	5·82247 97405*	·54784 82032	·17174 81287	1·82532 31356	·086
·915	1·37220 38091	·15560 87897	5·88013 05635	·54624 16370*	·17006 42510	1·83069 16430	·085
·916	1·37865 87286	·15423 33632	5·93905 22329	·54462 92441	·16837 70340	1·83611 14665−	·084
·917	1·38517 16082	·15285 14529	5·99928 87383*	·54301 08673	·16668 64263	1·84158 37698	·083
·918	1·39174 37794	·15146 30002	6·06088 61488	·54138 63444*	·16499 23749*	1·84710 97586	·082
·919	1·39837 66208	·15006 79451	6·12389 27423	·53975 55083	·16329 48260	1·85269 06805+	·081
·920	1·40507 15603	·14866 62263	6·18835 91389	·53811 81860	·16159 37242	1·85832 78284	·080

$\frac{1}{2}(1+a_x)$	x	z	$\frac{\frac{1}{2}(1+a_x)}{z}$	$\frac{\frac{1}{2}(1-a_x)}{z}$	$\frac{z}{\frac{1}{2}(1+a_x)}$	$\frac{z}{\frac{1}{2}(1-a_x)}$	$\frac{1}{2}(1-a_x)$
·920	1·40507 15603	·14866 62263	6·18835 91389	·53811 81860	·16159 37242	1·85832 78284	·080
·921	1·41183 00774	·14725 77808	6·25433 84457	·53647 41989	·15988 90128	1·86402 25420	·079
·922	1·41865 37061	·14584 25444	6·32188 64133	·53482 33625+	·15818 06339	1·86977 62104	·078
·923	1·42554 40371	·14442 04512	6·39106 16025+	·53316 54858	·15646 85278	1·87559 02747^5	·077
·924	1·43250 27208	·14299 14335^5	6·46192 55649^5	·53150 03711	·15475 26337	1·88146 62309	·076
·925	1·43953 14708	·14155 54224	6·53454 30392	·52982 78140	·15303 28891	1·88740 56318	·075
·926	1·44663 20671	·14011 23467	6·60898 21635−	·52814 76027	·15130 92297	1·89341 00901	·074
·927	1·45380 63589	·13866 21337	6·68531 46959	·52645 95176	·14958 15897	1·89948 12832	·073
·928	1·46105 62691	·13720 47087	6·76361 62673	·52476 33311	·14784 99017	1·90562 09547^5	·072
·929	1·46838 37982	·13573 99953	6·84396 66452^5	·52305 88071	·14611 40961	1·91183 09192	·071
·930	1·47579 10282	·13426 79144	6·92645 00320	·52134 57013	·14437 41015+	1·91811 30629	·070
·931	1·48328 01274	·13278 83859	7·01115 53337^5	·51962 37573	·14262 98452	1·92446 93607	·069
·932	1·49085 33552	·13130 13263	7·09817 65869^5	·51789 27123^5	·14088 12514	1·93090 18570	·068
·933	1·49851 30679	·12980 66504	7·18761 32490^5	·51615 22912	·13912 82426	1·93741 26921	·067
·934	1·50626 17234	·12830 42705−	7·27957 06377	·51440 22078	·13737 07393	1·94400 40980	·066
·935	1·51410 18877^5	·12679 40964	7·37416 03637^5	·51264 21643	·13560 86592^5	1·95067 84061	·065
·936	1·52203 62418	·12527 60353	7·47150 08155+	·51087 18506	·13384 19181	1·95743 80517^5	·064
·937	1·53006 75881	·12374 99916	7·57171 76848	·50909 09436	·13207 04286	1·96428 55812	·063
·938	1·53819 88586	·12221 58668	7·67494 45411	·50729 91061	·13029 41011	1·97122 36586	·062
·939	1·54643 31223	·12067 35595^5	7·78132 34603	·50549 59862	·12851 28430	1·97825 50747	·061
·940	1·55477 35946	·11912 29652	7·89100 57227	·50368 12163	·12672 65587	1·98538 27531	·060
·941	1·56322 36470	·11756 39758	8·00415 25763	·50185 44123	·12493 51497	1·99260 97598	·059
·942	1·57178 68165+	·11599 64802	8·12093 60702	·50001 51720^5	·12313 85140	1·99993 93136	·058
·943	1·58046 68184	·11442 03633	8·24153 99938	·49816 30749	·12133 65464	2·00737 47942	·057
·944	1·58926 75570^5	·11283 55063	8·36616 08906	·49629 76799^5	·11952 91380	2·01491 97556	·056
·945	1·59819 31399	·11124 17865^5	8·49500 01988	·49441 85248	·11771 61762	2·02257 79372	·055
·946	1·60724 78919	·10963 90770	8·62831 05085+	·49252 51242	·11589 75444	2·03035 32773	·054
·947	1·61643 63711	·10802 72462	8·76630 69580	·49061 69681	·11407 31216	2·03824 99283	·053
·948	1·62576 33863	·10640 61581	8·90925 87998	·48869 35207	·11224 27828	2·04627 22703	·052
·949	1·63523 40154	·10477 56715^5	9·05744 61226	·48675 42173	·11040 63978	2·05442 49323	·051
·950	1·64485 36270	·10313 56404	9·21117 08093	·48479 84636^5	·10856 38320	2·06271 28074	·050
·951	1·65462 79023	·10148 59128	9·37075 87012	·48282 56323	·10671 49451	2·07114 10766	·049
·952	1·66456 28611	·09982 63310	9·53656 20484	·48083 50613	·10485 95914	2·07971 52299	·048
·953	1·67466 48890	·09815 67313	9·70896 22581	·47882 60505+	·10299 76195	2·08844 10924	·047
·954	1·68494 07677	·09647 69433	9·88837 29856	·47679 78588^5	·10112 88714	2·09732 48546^5	·046
·955	1·69539 77100	·09478 67895−	10·07524 36616	·47474 97013	·09925 31827	2·10637 30998	·045
·956	1·70604 33967	·09308 60850−	10·27006 34581	·47268 07449	·09737 03818	2·11559 28409	·044
·957	1·71688 60181^5	·09137 46371	10·47336 58131	·47059 01044^5	·09548 02895+	2·12499 15596	·043
·958	1·72793 43222	·08965 22444	10·68573 35964	·46847 68383^5	·09358 27186	2·13457 72476	·042
·959	1·73919 76650^5	·08791 86967	10·90780 50013	·46633 99426	·09167 74731	2·14435 84571	·041
·960	1·75068 60710	·08617 37741	11·14028 03288	·46417 83470	·08976 43480	2·15434 43514^5	·040
·961	1·76241 02977	·08441 72460	11·38392 97729	·46199 09065−	·08784 31280	2·16454 47690	·039
·962	1·77438 19102^5	·08264 88710	11·63960 24344	·45977 63955+	·08591 35873	2·17497 02893	·038
·963	1·78661 33654	·08086 83956	11·90823 67514	·45753 34993	·08397 54886	2·18563 23123^5	·037
·964	1·79911 81067	·07907 55532	12·19087 26685^5	·45526 08050^5	·08202 85821	2·19654 31438	·036
·965	1·81191 06729	·07727 00634	12·48866 58227	·45295 67915	·08007 26046	2·20771 60974	·035
·966	1·82500 68211	·07545 16306^5	12·80290 42138	·45061 98170^5	·07810 72781	2·21916 56074	·034
·967	1·83842 36691	·07361 99429	13·13502 78467	·44824 81064^5	·07613 23091	2·23090 73605−	·033
·968	1·85217 98586^5	·07177 46702	13·48665 20044	·44583 97357	·07414 73866	2·24295 84444	·032
·969	1·86629 57434	·06991 54633	13·85959 49232	·44339 26136	·07215 21809	2·25533 75253	·031
·970	1·88079 36081	·06804 19514	14·25591 09446	·44090 44622	·07014 63417	2·26806 50475−	·030
·971	1·89569 79240	·06615 37406	14·67793 03948	·43837 27924	·06812 94960	2·28116 34692	·029
·972	1·91103 56476^5	·06425 04111	15·12830 78674	·43579 48768	·06610 12460	2·29465 75399	·028
·973	1·92683 65733	·06233 15149	15·61008 10466	·43316 77166	·06406 11664	2·30857 46275−	·027
·974	1·94313 37511^5	·06039 65726	16·12674 29263	·43048 80042	·06200 88014	2·32294 51001	·026
·975	1·95996 39846	·05844 50698	16·68233 10053	·42775 20771	·05994 36613	2·33780 27919	·025
·976	1·97736 84283	·05647 64533	17·28153 84696	·42495 58640	·05786 52185	2·35318 55534	·024
·977	1·99539 33102	·05449 01262	17·92985 38697	·42209 48199	·05577 29030	2·36913 59215−	·023
·978	2·01409 08121	·05248 54425+	18·63373 82856	·41916 38469	·05366 60967	2·38570 19334	·022
·979	2·03352 01492^5	·05046 17007	19·40085 22490	·41615 71984	·05154 41274	2·40293 81298	·021
·980	2·05374 89105−	·04841 81359	20·24034 96517	·41306 83602	·04940 62611	2·42090 67947^5	·020

$\frac{1}{2}(1+\alpha_x)$	x	z	$\frac{\frac{1}{2}(1+\alpha_x)}{z}$	$\frac{\frac{1}{2}(1-\alpha_x)}{z}$	$\frac{z}{\frac{1}{2}(1+\alpha_x)}$	$\frac{z}{\frac{1}{2}(1-\alpha_x)}$	$\frac{1}{2}(1-\alpha_x)$
·980	2·05374 89105^{-}	·04841 81359	20·24034 96517	·41306 83602	·04940 62611	2·42090 67947^{5}	·020
·981	2·07485 47343	·04635 39107	21·16326 29328	·40988 99039	·04725 16929	2·43967 95103	·019
·982	2·09692 74292	·04426 81043	22·18301 45166	·40661 34007	·04507 95359	2·45933 91270	·018
·983	2·12007 16897	·04215 96988	23·31610 58315^{-}	·40322 86868	·04288 88086	2·47998 22848	·017
·984	2·14441 06210	·04002 75629	24·58306 04688	·39972 45605^{-}	·04067 84176	2·50172 26833	·016
·985	2·17009 03776^{5}	·03787 04310	26·00073 83286	·39608 73857	·03844 71380	2·52469 53982	·015
·986	2·19728 63766	·03568 68772	27·62920 37219^{5}	·39230 10670^{5}	·03619 35874	2·54906 26562	·014
·987	2·22621 17693	·03347 52823	29·48444 14167	·38834 62395	·03391 61928	2·57502 17157	·013
·988	2·25712 92445^{e}	·03123 37993	31·63240 80738	·38419 92877	·03161 31481	2·60281 58617	·012
·989	2·29036 78779	·02896 02511	34·15025 63952	·37983 09609	·02928 23570	2·63275 00991	·011
·990	2·32634 78740	·02665 21422	37·14523 17976	·37520 43616	·02692 13558	2·66521 42202	·010
·991	2·36561 81268	·02430 64606	40·77105 32107	·37027 19262	·02452 72055^{-}	2·70071 78486	·009
·992	2·40891 55459	·02191 95666	45·25636 91596	·36497 07190	·02209 63573	2·73994 58309	·008
·993	2·45726 33903	·01948 69510	50·95717 64741	·35921 47385^{-}	·01962 43212	2·78385 01399	·007
·994	2·51214 43279	·01700 28705^{+}	58·46071 69201	·35288 15911	·01710 55035^{+}	2·83381 17525^{-}	·006
·995	2·57582 93035^{+}	·01445 97430	68·81173 46310	·34578 76112	·01453 24051	2·89194 86054	·005
·996	2·65206 98079	·01184 70585^{+}	84·07150 14744	·33763 65521	·01189 46371	2·96176 46364^{5}	·004
·997	2·74778 13854	·00914 91911	108·07138 1780	·32789 78380	·00917 07213	3·04973 03779	·003
·998	2·87816 17391	·00634 01932	157·40845 1465^{+}	·31544 77985^{+}	·00635 28990	3·17009 66203	·002
·999	3·09023 23062	·00336 70901	296·60535 9238	·29699 23516	·00337 04005^{+}	3·36709 00771	·001

Note. We believe that x and z may be taken as correct to the figures tabled. They were worked of course to more figures than are shown. The possibility of error in the ratio $\frac{1}{2}(1+\alpha_x)/z$ is greater, and may amount to five units in the tenth decimal. It seemed better to leave the last two figures standing with this warning rather than destroy the symmetry of the table by cutting them out. We feel compelled however to show only twelve figures in the last three entries of this ratio.

A more extended system of symbols than heretofore has been adopted in this table to indicate the nature of the last figure. 5^{+} and 5^{-} signify as usual that the real number exceeds 5 and falls short of 5. The symbol 5^{e} denotes that the number is exactly 5 *to the extent of the calculations*, i.e. $\cdot 63719,16745^{e}$ denotes that x for $\frac{1}{2}(1+\alpha_x) = \cdot 738$ was found to be ·63719,16745,00. It does not necessarily indicate that the value terminated at the tenth or twelfth decimal. Another innovation has been made. Consider $\cdot 60075,97742^{5}$; the usual interpretation of this would be that the number as actually worked was terminated by 5, 50 or 500 as the case might be, and the computer was unable to settle whether to enter it as ·60075,97742 or ·60075,97743. In the present table there may be doubt as to the correctness of the twelfth figure and the affixed 5 has been used when the final figures are 48, 49, 50, 51 or 52. Thus ·60075,97742,48 or ·60075,97742,51 would not be printed as usual ·60075,97742 and ·60075,97743, but as $\cdot 60075,97742^{5}$, precisely as ·60075,97742,50 is written $\cdot 60075,97742^{5}$. This seems safer when we cannot be sure of one or two units in the twelfth decimal place, and is more accurate when the 5 is actually put on the machine in computing.

We have to thank most heartily Dr W. F. Sheppard for the original loan to the Laboratory of his twelve figure tables of $\frac{1}{2}(1+\alpha_x)$ to argument x, and more recently for extracts ($x = 2\cdot 1$ to $3\cdot 1$) from his sixteen figure table of $\log_e \frac{1}{2}(1-\alpha_x)$ to argument x by intervals of ·1. We have also to thank Mr Frank Robbins for determining a large number of the values of x.

THE DURATION OF PLAY.

By E. C. FIELLER, B.A.

§ 1. The Problem and its Equations.

Two persons, A and B, play at a game in which their chances of winning are respectively p and q, where

$$p+q=1.$$

A starts playing with a counters, B with b counters, and after each game the loser gives the winner one counter. The set finishes when one of the players loses his last counter.

These are the conditions of the problem that we propose to discuss.

Let $p_{m,n}$ be the chance that after n games B will hold m counters.

If $1 < m < a+b-1$, B must either hold $(m+1)$ counters after the $(n-1)$st game, and lose the nth, the probability of which is $p \times p_{m+1,n-1}$, or else hold $(m-1)$ counters after the $(n-1)$st game, and win the nth, the probability of which is $q \times p_{m-1,n-1}$.

If $m=0$ or 1, B must hold $(m+1)$ counters after the $(n-1)$st game, and lose the nth; if $m=(a+b-1)$ or $(a+b)$, he must hold $(m-1)$ counters after the $(n-1)$st game, and win the nth.

Thus

$$(1) \qquad p_{m,n}=\begin{cases} p \cdot p_{m+1,n-1}+q \cdot p_{m-1,n-1}, & (1 < m < a+b-1) \\ p \cdot p_{m+1,n-1}, & m=0,1 \\ q \cdot p_{m-1,n-1}. & m=a+b-1,\ a+b \end{cases}$$

Since B starts playing with b counters,

$$(1{\cdot}1) \qquad p_{b,0}=1.$$

In n games B can win or lose only n, or $(n-2)$, or $(n-4)$, ... counters, and the number of counters he holds must always lie between 0 and $(a+b)$. Thus

$$(1{\cdot}2) \qquad p_{b+n-2i+1,n}=0, \quad (i=1,2,3,\ldots n)$$

and

$$(1{\cdot}3) \qquad p_{m,n}=0$$

unless $\qquad \mathrm{Max}\,((b-n),\,0) \leqslant m \leqslant \mathrm{Min}\,((b+n),\,(a+b)).$

(1), (1·1), (1·2), and (1·3) are the equations of the problem. Instead of solving them analytically, we may regard them as defining a quantity $p_{m,n}$ whose values we

can set out in a diagram somewhat similar to Pascal's Arithmetic Triangle, which gives the values of the quantity ${}_nC_r$ defined by the equations

$$ {}_nC_r = {}_{n-1}C_r + {}_{n-1}C_{r-1}; \quad {}_0C_0 = 1; \quad {}_nC_r = 0 \text{ unless } 0 \leqslant r \leqslant n. $$

We take a rectangular array of $(a+b+1)$ rows of points, and starting from the bottom, number the rows $0, 1, 2, \ldots (a+b)$, and starting from the left, number the columns $0, 1, 2, 3, \ldots$. Against the point (n, m) common to row m and column n we write the value of $p_{m,n}$. It is not necessary to take more than $(a+b+1)$ rows, by virtue of (1·3).

§ 2. The Solution when a is Infinite.

(a) *The Individual Probabilities.*

For simplicity's sake we consider first the case in which A's fortune is unlimited, and so, consequently, is the number of rows. By (1·1) and (1·3), the number at $(0, b)$ will be 1, and all other entries in column 0 will be zero. By (1·2) and (1·3), the entries in column n will be zero above the $(b+n)$th and below the $(b-n)$th rows, and also in the $(b+n-1)$st, $(b+n-3)$rd, $\ldots$ $(b-n+1)$st rows. By (1), the entry at (n, m) is obtained, for $m > 1$, by adding p times the entry at $(n-1, m+1)$ to q times the entry at $(n-1, m-1)$, and for $m = 0, 1$, by taking p times the entry at $(n-1, m+1)$. Hence (Figure 1*) until n becomes greater than b, the non-vanishing terms in column n are, reading upwards, exactly the successive terms of the binomial expansion

$$(p+q)^n.$$

$$
\begin{array}{r|cccccccc}
 & \cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot \\
b+3 & \cdot & \cdot & \cdot & q^3 & \cdot\cdot & \cdot\cdot & \cdot\cdot & {}_bC_5 p^{b-5} q^5 \\
b+2 & \cdot & \cdot & q^2 & 0 & \cdot\cdot & \cdot\cdot & \cdot\cdot & 0 \\
b+1 & \cdot & q & 0 & 3pq^2 & \cdot\cdot & \cdot\cdot & \cdot\cdot & {}_bC_4 p^{b-4} q^4 \\
b & 1 & 0 & 2pq & 0 & \cdot\cdot & \cdot\cdot & \cdot\cdot & 0 \\
b-1 & \cdot & p & 0 & 3p^2q & \cdot\cdot & \cdot\cdot & \cdot\cdot & {}_bC_3 p^{b-3} q^3 \\
b-2 & \cdot & \cdot & p^2 & 0 & \cdot\cdot & \cdot\cdot & \cdot\cdot & 0 \\
b-3 & \cdot & \cdot & \cdot & p^3 & \cdot\cdot & \cdot\cdot & \cdot\cdot & {}_bC_2 p^{b-2} q^2 \\
 & \cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & 0 \\
2 & \cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & {}_bC_1 p^{b-1} q \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & 0 \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & p^b \\
\hline
 & 0 & 1 & 2 & 3 & \cdot\cdot & \cdot\cdot & \cdot\cdot & b
\end{array}
$$

Figure 1

For $n > b$, these binomial terms become modified on account of the modification for $m = 0, 1$ in the rule for forming the entries in row m. The terms in the columns

* The figure is drawn for $b = 7$; if b has not this value the last column will stand, but the upper portion of it will not fall on the horizontal rows marked by the left-hand scale, as more rows will occur between 2 and $b-3$.

immediately succeeding the bth are easily seen to be those shown in Figure 2, and generalising from these we obtain for the entries in column $(b+2i)$ the values shown in the first column of Figure 3; from this column the next three columns are formed in accordance with equation (1). Since the entries in the last two columns of Figure 3 differ from those in the first two only by having $(i+1)$ in place of i, the correctness of the assumed values follows by induction.

Thus for $(n+m-b)$ even,

$$(2\cdot1)\qquad p_{m,n}=\left({}_nC_{\frac{n+m-b}{2}}-{}_nC_{\frac{n-m-b}{2}}\right)p^{\frac{n-m+b}{2}}q^{\frac{n+m-b}{2}}. \qquad (n\geqslant m+b>b)$$

$$(2\cdot2)\qquad p_{m,n}={}_nC_{\frac{n+m-b}{2}}\,p^{\frac{n-m+b}{2}}q^{\frac{n+m-b}{2}}. \qquad (n\leqslant b,\ \text{or}\ n<m+b>b)$$

$$(2\cdot3)\qquad p_{0,n}=\left({}_nC_{\frac{n-b}{2}}-2\,{}_{n-1}C_{\frac{n-b-2}{2}}\right)p^{\frac{n+b}{2}}q^{\frac{n-b}{2}} \qquad (n>b)$$

Equations (2·1), (2·2), (2·3) give the solution of equations (1), (1·1), (1·2), (1·3).

In Figures 2 and 3 the remaining terms of the unmodified binomial are shown below row 0 in rows numbered $-1, -2, -3, \ldots$. It will be seen that, for $m>0$, the binomial coefficient at (n, m) is modified by the subtraction of that at $(n, -m)$; this result we can obtain directly*.

In any particular sequence of n games, at the end of which B has m counters (we suppose $(n+m-b)$ even), B must lose $\frac{1}{2}(n-m+b)$ times and win $\frac{1}{2}(n+m-b)$ times, so that the probability that this particular sequence will occur is

$$p^{\frac{n-m+b}{2}}\times q^{\frac{n+m-b}{2}}$$

$p_{m,n}$ is this probability multiplied by the number of permissible sequences, and we can find this number by means of an elegant geometrical representation, used by Borel† to determine $p_{0,n}$. Any sequence of n games can be represented by a zigzag path of n steps starting at $(0, b)$ and finishing in column n, and going from (α, β) to $(\alpha+1, \beta+1)$ if B wins the $(\alpha+1)$st game, and to $(\alpha+1, \beta-1)$ if he loses it. For the sequence to last effectively n games, the representative path must never cross or touch row 0, except possibly at $(n, 0)$. Without this restriction, there are ${}_nC_{\frac{n+m-b}{2}}$ zigzag paths joining $(0, b)$ and (n, m), for this is the number of ways in which we can assign the positions of the $\frac{1}{2}(n+m-b)$ upward steps. Now consider any path from $(0, b)$ to (n, m) that comes into contact with row 0. It may have several points $(\nu_j, 0)$ in common with row 0; let $\nu=\text{Min}\,(\nu_j)$. Then if we substitute for that portion of the path that lies between $(\nu, 0)$ and (n, m) its reflection in row 0, we get a path joining $(0, b)$ and $(n, -m)$, and conversely. Thus the paths joining $(0, b)$ and (n, m) and intersecting row 0 are in one-to-one correspondence with the paths joining $(0, b)$ and $(n, -m)$, so that the number of the

* We have indicated the inductive proof, because we believe that it was by this method—which Laplace characterises somewhat contemptuously as "en quelque sort mécanique"—that De Moivre arrived at his results (see § 4).

† *Principes et Formules Classiques du Calcul des Probabilités*, 1925, Ch. v.

former is equal to the number of the latter, or ${}_nC_{\frac{n-m-b}{2}}$. Hence the number of permissible paths is

$$ {}_nC_{\frac{n+m-b}{2}} - {}_nC_{\frac{n-m-b}{2}}, $$

leading to (2·1), while by (1),

$$ p_{0,b+2i} = p \cdot p_{1,b+2i-1} = ({}_{b+2i-1}C_i - {}_{b+2i-1}C_{i-1})\, p^{b+i} q^i, $$

(2·31) $$ \text{i.e. } p_{0,b+2i} = \frac{b\,(b+2i-1)!}{i!\,(b+i)!}\, p^{b+i} q^i; $$

(2·3) and (2·31) are easily seen to be in agreement.

7	•	${}_{b+1}C_4 p^{b-3} q^4$	•	${}_{b+3}C_5 p^{b-2} q^5$	•
6	${}_bC_3 p^{b-3} q^3$	•	${}_{b+2}C_4 p^{b-2} q^4$	•	${}_{b+4}C_5 p^{b-1} q^5$
5	•	${}_{b+1}C_3 p^{b-2} q^3$	•	${}_{b+3}C_4 p^{b-1} q^4$	•
4	${}_bC_2 p^{b-2} q^2$	•	${}_{b+2}C_3 p^{b-1} q^3$	•	$({}_{b+4}C_4 - 1) p^b q^4$
3	•	${}_{b+1}C_2 p^{b-1} q^2$	•	$({}_{b+3}C_3 - 1) p^b q^3$	•
2	${}_bC_1 p^{b-1} q$	•	$({}_{b+2}C_2 - 1) p^b q^2$	•	$({}_{b+4}C_3 - {}_{b+4}C_1) p^{b+1} q^3$
1	•	$({}_{b+1}C_1 - 1) p^b q$	•	$({}_{b+3}C_2 - {}_{b+3}C_1) p^{b+1} q^2$	•
0	p^b	•	$({}_{b+2}C_1 - 2) p^{b+1} q$	•	$({}_{b+4}C_2 - 2\,{}_{b+3}C_1) p^{b+2} q^2$
−1		p^{b+1}	•	${}_{b+3}C_1 p^{b+2} q$	•
−2			p^{b+2}	•	${}_{b+4}C_1 p^{b+3} q$
−3				p^{b+3}	•
−4					p^{b+4}

Figure 2.

(b) B's Chance of Ruin.

Let ${}_nP_b$ be the chance that B will lose his last counter on or before the nth game. The chance that he will lose it on the $(b+2i+1)$st game being zero, we have, by (2·31),

(3) $$ {}_{b+2i+1}P_b = {}_{b+2i}P_b = p^b \left[1 + b \,.\, pq + \frac{b\,(b+3)}{2}\, p^2 q^2 + \dots + \frac{b\,(b+2i-1)!}{i!\,(b+i)!}\, p^i q^i \right]. $$

It is convenient to transform this result by noting that B, if he has not previously lost all his counters, must have, at the end of $(b+2i)$ games, either 0, 2, 4, ... or $(2b+2i)$ counters, and at the end of $(b+2i+1)$ games, either 1, 3, 5, ... or $(2b+2i+1)$ counters. Thus

(3·1) $$ {}_{b+2i}P_b + p_{2,b+2i} + p_{4,b+2i} + \dots + p_{2b+2i,\,b+2i} = 1, $$

whence

(3·11) $$ {}_{b+2i}P_b + \sum_{r=1}^{b+i} {}_{b+2i}C_{i+r}\, p^{b+i-r} q^{i+r} - \sum_{r=1}^{i} {}_{b+2i}C_{i-r}\, p^{b+i-r} q^{i+r} = 1, $$

or

(3·12) ${}_{b+2i}P_b =$ 1st $(i+1)$ terms of $(p+q)^{b+2i}$ + 1st i terms of $\left(\frac{p}{q}\right)^b (q+p)^{b+2i}$.

Similarly

(3·2) $${}_{b+2i+1}P_b + p_{1,b+2i+1} + p_{3,b+2i+1} + \ldots + p_{2b+2i+1,b+2i+1} = 1,$$

whence

(3·21) $${}_{b+2i+1}P_b + \sum_{r=1}^{b+i+1} {}_{b+2i+1}C_{i+r}\, p^{b+i-r+1} q^{i+r} - \sum_{r=1}^{i+1} {}_{b+2i+1}C_{i-r+1}\, p^{b+i-r+1} q^{i+r} = 1,$$

or

(3·22)

$${}_{b+2i+1}P_b = \text{1st } (i+1) \text{ terms of } (p+q)^{b+2i+1} + \text{1st } (i+1) \text{ terms of } \left(\frac{p}{q}\right)^b (q+p)^{b+2i+1}.$$

6	$p^{b+i-3}q^{i+3}({}_{b+2i}C_{i+3} - {}_{b+2i}C_{i-3})$	•	$p^{b+i-2}q^{i+4}({}_{b+2i+2}C_{i+4} - {}_{b+2i+2}C_{i-2})$	•
5	•	$p^{b+i-2}q^{i+3}({}_{b+2i+1}C_{i+3} - {}_{b+2i+1}C_{i-2})$	•	$p^{b+i-1}q^{i+4}({}_{b+2i+3}C_{i+4} - {}_{b+2i+3}C_{i-1})$
4	$p^{b+i-2}q^{i+2}({}_{b+2i}C_{i+2} - {}_{b+2i}C_{i-2})$	•	$p^{b+i-1}q^{i+3}({}_{b+2i+2}C_{i+3} - {}_{b+2i+2}C_{i-1})$	•
3	•	$p^{b+i-1}q^{i+2}({}_{b+2i+1}C_{i+2} - {}_{b+2i+1}C_{i-1})$	•	$p^{b+i}q^{i+3}({}_{b+2i+3}C_{i+3} - {}_{b+2i+3}C_{i})$
2	$p^{b+i-1}q^{i+1}({}_{b+2i}C_{i+1} - {}_{b+2i}C_{i-1})$	•	$p^{b+i}q^{i+2}({}_{b+2i+2}C_{i+2} - {}_{b+2i+2}C_{i})$	•
1	•	$p^{b+i}q^{i+1}({}_{b+2i+1}C_{i+1} - {}_{b+2i+1}C_{i})$	•	$p^{b+i+1}q^{i+2}({}_{b+2i+3}C_{i+2} - {}_{b+2i+3}C_{i+1})$
0	$p^{b+i}q^{i}({}_{b+2i}C_{i} - 2\,{}_{b+2i-1}C_{i-1})$	•	$p^{b+i+1}q^{i+1}({}_{b+2i+2}C_{i+1} - 2\,{}_{b+2i+1}C_{i})$	•
−1	•	$p^{b+i+1}q^{i}\,{}_{b+2i+1}C_{i}$	•	$p^{b+i+2}q^{i+1}\,{}_{b+2i+3}C_{i+1}$
−2	$p^{b+i+1}q^{i-1}\,{}_{b+2i}C_{i-1}$	•	$p^{b+i+2}q^{i}\,{}_{b+2i+2}C_{i}$	•
−3	•	$p^{b+i+2}q^{i-1}\,{}_{b+2i+1}C_{i-1}$	•	$p^{b+i+3}q^{i}\,{}_{b+2i+3}C_{i}$
−4	$p^{b+i+2}q^{i-2}\,{}_{b+2i}C_{i-2}$	•	$p^{b+i+3}q^{i-1}\,{}_{b+2i+2}C_{i-1}$	•
−5	•	$p^{b+i+3}q^{i-2}\,{}_{b+2i+1}C_{i-2}$	•	$p^{b+i+4}q^{i-1}\,{}_{b+2i+3}C_{i-1}$
−6	$p^{b+i+3}q^{i-3}\,{}_{b+2i}C_{i-3}$	•	$p^{b+i+4}q^{i-2}\,{}_{b+2i+2}C_{i-2}$	•
	$(p+q)^{b+2i}$	$(p+q)^{b+2i+1}$	$(p+q)^{b+2i+2}$	$(p+q)^{b+2i+3}$

Figure 3.

Now*

(3·3) $$\text{1st } s \text{ terms of } \{x + (1-x)\}^n = I_x(n-s+1, s)$$
$$= \int_0^x x^{n-s}(1-x)^{s-1}\,dx \Big/ \int_0^1 x^{n-s}(1-x)^{s-1}\,dx,$$

so that (3·22) gives

$${}_{b+2i+1}P_b = I_p(b+i+1, i+1) + \left(\frac{p}{q}\right)^b I_q(b+i+1, i+1),$$

* This theorem is due to Professor Pearson (*Biometrika*, Vol. XVI. p. 202). It can be proved thus: If m be integral, and $x = py$,

$$\int_0^p x^l (1-x)^m\,dx = \int_0^1 p^{l+1} y^l \{(1-p) + p(1-y)\}^m\,dy$$
$$= \sum_{r=0}^{m} p^{l+m-r+1}(1-p)^r \frac{m!}{r!\,(m-r)!} \frac{\Gamma(l+1)\,\Gamma(m-r+1)}{\Gamma(l+m-r+2)}$$
$$= \frac{\Gamma(l+1)\,\Gamma(m+1)}{\Gamma(l+m+2)} \sum_{r=0}^{m} \frac{(l+m+1)(l+m)\ldots(l+m-r+2)}{r!} p^{l+m-r+1}(1-p)^r$$
$$= \int_0^1 x^l (1-x)^m\,dx \times \text{sum of 1st } (m+1) \text{ terms of } \{p + (1-p)\}^{l+m+1}.$$

whence, using the first of equations (3),

$$(4) \qquad {}_nP_b = I_p\left(\frac{n+b+1}{2}, \frac{n-b+1}{2}\right) + \left(\frac{p}{q}\right)^b I_q\left(\frac{n+b+1}{2}, \frac{n-b+1}{2}\right),$$

or

$$I_p\left(\frac{n+b+2}{2}, \frac{n-b+2}{2}\right) + \left(\frac{p}{q}\right)^b I_q\left(\frac{n+b+2}{2}, \frac{n-b+2}{2}\right),$$

according as $n-b$ is odd or even.

Equation (4) expresses ${}_nP_b$ in terms of proportional areas under a Pearson's Type I Curve. The mode and mean of this curve are at, respectively*,

$$x = \frac{n+b-1}{2(n-1)} \text{ and } x = \frac{n+b+1}{2(n+1)} \text{ if } (n-b) \text{ be odd,}$$

$$x = \frac{n+b}{2n} \quad \text{ and } x = \frac{n+b+2}{2(n+2)} \text{ if } (n-b) \text{ be even.}$$

As n increases, the curve becomes more and more nearly symmetrical, and the area under it is concentrated more and more closely round the mode, which tends to the limiting position $x = \frac{1}{2}$.

Thus

$$(5) \quad \underset{n\to\infty}{\mathrm{Lt}}\, I_x\left(\frac{n+b+1}{2}, \frac{n-b+1}{2}\right) = \underset{n\to\infty}{\mathrm{Lt}}\, I_x\left(\frac{n+b+2}{2}, \frac{n-b+2}{2}\right) = 0, \tfrac{1}{2}, \text{ or } 1,$$

according as $x < \frac{1}{2}$, $x = \frac{1}{2}$, or $x > \frac{1}{2}$.

Let P_b be the probability that B will ultimately be ruined. Then by (4) and (5),

$$(6) \qquad P_b = \underset{n\to\infty}{\mathrm{Lt}}\, {}_nP_b = \left(\frac{p}{q}\right)^b \text{ if } q > \tfrac{1}{2},$$

$$= 1 \quad \text{ if } q \leqslant \tfrac{1}{2}.$$

Anybody, therefore, who plays continually at a game in which he has not a definite advantage is morally certain† to be ruined.

(c) *The Value of B's Expectation.*

Let B's expectation, when it is agreed to limit the set to n games, be ${}_nE_b$. Then

$$(7) \qquad {}_nE_b = \sum_{m=1}^{b+n} m p_{m,n}.$$

If B has m counters, his expectation on the next game is $m+q-p$, and accordingly

$$(7\cdot1) \qquad {}_{n+1}E_b = \sum_{m=1}^{b+n} (m+q-p)\, p_{m,n} = {}_nE_b + (q-p)(1-{}_nP_b).$$

If $p = q = \frac{1}{2}$, ${}_nE_b$ is constant from game to game, and equal to b.

* See p. 1 of "The Numerical Evaluation of the Incomplete B-Function," by H. E. Soper (*Tracts for Computers*, No. VII).

† "En représentant, comme on le fait ordinairement, par l'unité la certitude absolue, celle par exemple qui résulte d'une démonstration rigoureuse, on pourra regarder comme une certitude morale toute fraction variable qui, sans devenir jamais égale à l'unité, peut en approcher d'assez près pour surpasser toute fraction déterminée." AMPÈRE.

Accordingly, if a man embarks on a career of inveterate gambling at a fair game, his expectation at the outset is equal to the amount that he has decided to risk, and at any subsequent moment, to the amount that he has still in hand. This result does not contradict the fact that he is morally certain to lose it. The chance that he will have anything left after n games becomes infinitely small as n increases, but the amount that he can expect to have, if he has anything, becomes correspondingly infinitely great. The objection to inveterate gambling lies in the practical consideration, that it is not worth while to face a very strong risk of being penniless, for the sake of a very slight chance of becoming a millionaire.

These remarks apply only to the case of $q=\frac{1}{2}$. For B's expectation on the $(b+2i)$th game, when p and q are not equal, we have

$$\begin{aligned}
{}_{b+2i}E_b &= 2p^{b+i-1}q^{i+1}\left\{\sum_{r=1}^{b+i} r\,{}_{b+2i}C_{i+r}\left(\frac{q}{p}\right)^{r-1} - \sum_{r=1}^{i} r\,{}_{b+2i}C_{i-r}\left(\frac{q}{p}\right)^{r-1}\right\} \\
&= 2p^{b+i-1}q^{i+1}\frac{d}{dt}\left(\sum_{r=1}^{b+i} {}_{b+2i}C_{i+r}\,t^r - \sum_{r=1}^{i} {}_{b+2i}C_{i-r}\,t^r\right) \qquad \left(t=\frac{q}{p}\right) \\
&= 2p^{b+i-1}q^{i+1}\frac{d}{dt}\left\{\frac{1}{p^{b+i}q^i}(1-{}_{b+2i}P_b)\right\} \text{ by (3·11)} \\
&= \frac{2t^{i+1}}{(1+t)^{b+2i}B(b+i+1,\, i+1)}\frac{d}{dt}\left\{\frac{(1+t)^{b+2i}}{t^i}B(b+i+1,\, i+1)\right. \\
&\qquad \left. - \frac{(1+t)^{b+2i}}{t^i}\int_0^{\frac{1}{1+t}} x^{b+i}(1-x)^i\,dx - \frac{(1+t)^{b+2i}}{t^{b+i}}\int_0^{\frac{t}{1+t}} x^{b+i}(1-x)^i\,dx\right\} \text{ by (4)} \\
&= 2\{(b+2i)q-i\}\{1-I_p(b+i+1,\, i+1)\} \\
&\qquad - 2\left(\frac{p}{q}\right)^b\{(b+2i)q-(b+i)\}I_q(b+i+1,\, i+1),
\end{aligned}$$

or, by (4),

$$(7{\cdot}2)\quad {}_{b+2i}E_b = 2\{bq+i(q-p)\}\{1-{}_{b+2i}P_b\} + 2\left(\frac{p}{q}\right)^b b\,.\,I_q(b+i+1,\, i+1),$$

whence, by (7·1),

$$(7{\cdot}3)\quad {}_{b+2i+1}E_b = 2\{bq+(i+\tfrac{1}{2})(q-p)\}\{1-{}_{b+2i}P_b\} + 2\left(\frac{p}{q}\right)^b b\,.\,I_q(b+i+1,\, i+1).$$

For $q>\frac{1}{2}$, $$\operatorname*{Lt}_{i\to\infty} {}_{b+2i}P_b = \left(\frac{p}{q}\right)^b,$$

and ${}_{b+2i}E_b$ and ${}_{b+2i+1}E_b$ tend to infinity with

$$i(q-p)(1-{}_{b+2i}P_b).$$

For $q<\frac{1}{2}$, $$\operatorname*{Lt}_{i\to\infty}(1-{}_{b+2i}P_b) = \operatorname*{Lt}_{i\to\infty} I_q(b+i+1,\, i+1) = 0,$$

and accordingly

$$\operatorname*{Lt}_{i\to\infty} {}_{b+2i}E_b = \operatorname*{Lt}_{i\to\infty} {}_{b+2i+1}E_b = \operatorname*{Lt}_{i\to\infty} i(q-p)(1-{}_{b+2i}P_b) = 0,$$

since $i(q-p)(1-{}_{b+2i}P_b)$ is negative for $q<\frac{1}{2}$, while the expectations are essentially positive.

Denoting by E_b B's expectation at the outset, when no limit is imposed on the number of games, we have, therefore,

$$(8)\qquad E_b = 0,\ b, \text{ or } \infty \text{ according as } q <, =, \text{ or } > \tfrac{1}{2}.$$

(d) *The Probable Length of the Set.*

If q be greater than p, there is a finite probability that B will never be ruined and the probable length of the set is infinite.

If $q \leqslant p$, the probable number of games is

$$\sum_{i=0}^{\infty}(b+2i)p_{0,b+2i}=bp^b\sum_{i=0}^{\infty}\frac{(b+2i)!}{i!(b+i)!}p^iq^i.$$

The ratio of the ith and $(i+1)$st terms of this series is

$$\frac{1}{pq}\frac{i(b+i)}{(b+2i-1)(b+2i)}=\frac{1}{4pq}\left\{1+\frac{1}{2i}+O\left(\frac{1}{i^2}\right)\right\};$$

for $p=q$, $4pq=1$, and the series therefore diverges, so that the probable length of the set is again infinite.

For $q<p$, the series converges to the limit

$$\begin{aligned}\sum_{i=0}^{\infty}(b+2i)p_{0,b+2i}&=bP_b+2pq\frac{d}{d(pq)}\frac{P_b}{p^b}\\&=b+2pq\frac{d(p)}{d(pq)}\cdot\frac{d}{dp}\cdot\frac{1}{p^b}\\&=b\left[1+\frac{2q}{p^b(p-q)}\right];\end{aligned}$$

this, when $q<p$, is the probable number of games that will be played before B is ruined.

§ 8. The Solution when a and b are Finite.

(a) *The Individual Probabilities.*

We proceed to the general case, in which A, as well as B, starts playing with a limited number of counters. Further modifications must now be made in our table of probabilities. In the case first considered, the set stopped if B lost b counters; now it stops also, if he wins a counters. The path representing any permissible sequence of games must now lie below row $(a+b)$, as well as above row 0. The binomial coefficients in our table, instead of being modified, as it were by reflection, only in row 0 after column b, will now be modified also, in the same way, in row $(a+b)$ after column a. Then, after column $(a+2b)$, (we suppose $b \leqslant a$), the modified coefficients will again be modified in row $(a+b)$, after column $(2a+b)$ and again after column $(2a+3b)$ in row 0, after column $(3a+2b)$ and again after column $(3a+4b)$ in row $(a+b)$, and so on.

Let V be the point $(0, b)$, and let $A_1, A_2, A_3, \ldots, B_1, B_2, B_3, \ldots$ be its two sets of reflections in row 0 and row $(a+b)$, so that $A_1, A_2, A_3, \ldots$ are the points $(0, 2a+b)$, $(0, -(2a+b))$, $(0, 4a+3b), \ldots$ and $B_1, B_2, B_3, \ldots$ the points $(0, -b)$, $(0, 2a+3b)$, $(0, -(2a+3b)), \ldots$.

We have seen that, for $n \leqslant b$, the entries in column b are the terms of the binomial expansion $(p+q)^n$. We call this array, starting from V, of the terms of the successive powers of $(p+q)$, the binomial table $\{(p+q)^n, V\}$. For rows above row 0, the modification of table $\{(p+q)^n, V\}$ in row 0 consists in the introduction

of the table $\left\{-\left(\frac{p}{q}\right)^b (q+p)^n, B_1\right\}$; accordingly, for rows between row 0 and row $(a+b)$, the modification of these two tables in row $(a+b)$ will consist in the introduction of tables $\left\{-\left(\frac{q}{p}\right)^a (p+q)^n, A_1\right\}$, $\left\{\left(\frac{q}{p}\right)^{a+b} (p+q)^n, B_2\right\}$, the modification of these, in row 0, in the introduction of tables

$$\left\{\left(\frac{p}{q}\right)^{a+b} (q+p)^n, A_2\right\}, \quad \left\{-\left(\frac{p}{q}\right)^{a+2b} (q+p)^n, B_3\right\},$$

and so on (see Figure 4).

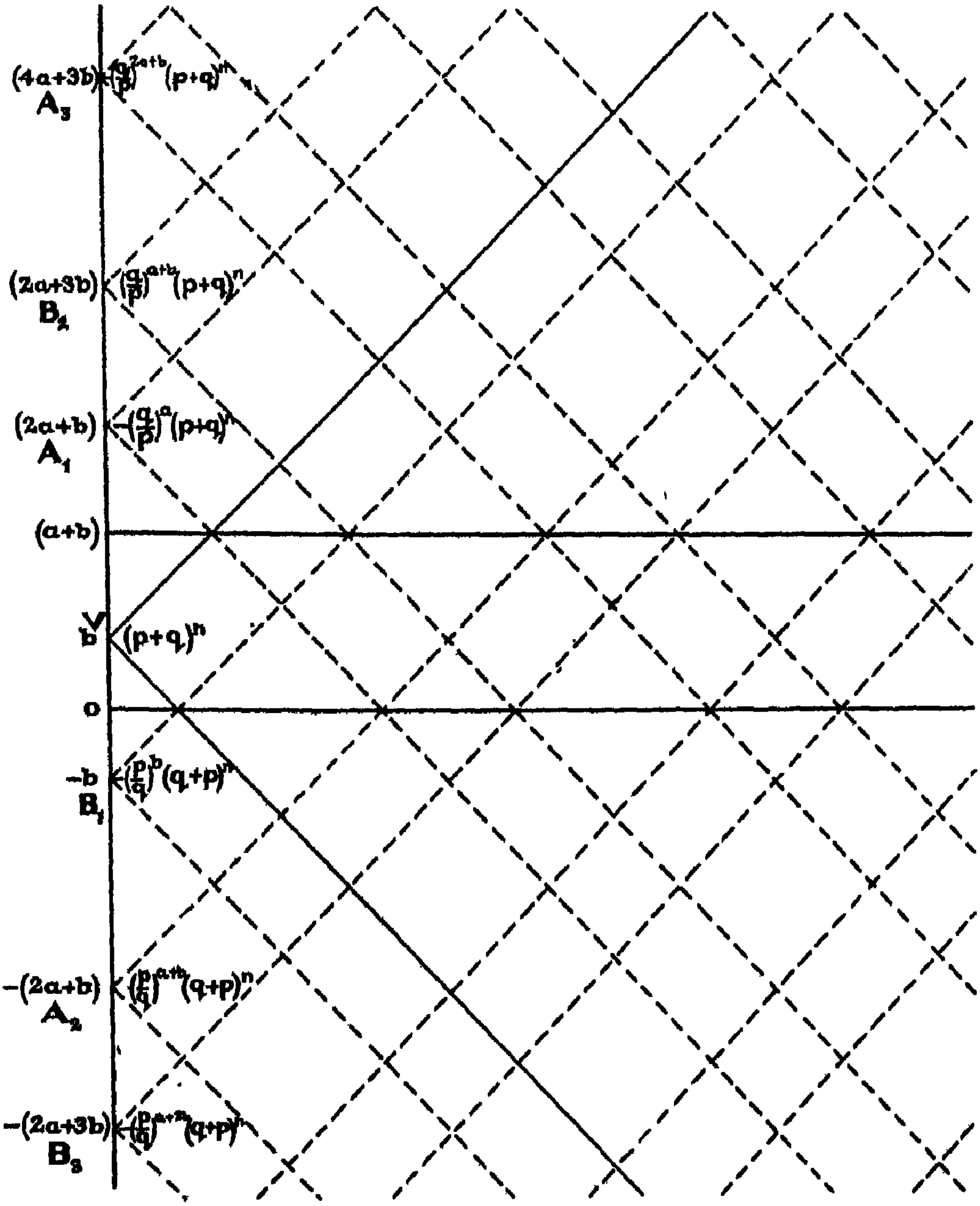


Figure 4.

For the entry at $(b+2i, 0)$ we have, associated with the tables whose vertices are at V and B_1, a term

$$\frac{b(b+2i-1)!}{i!\,(b+i)!}\,p^{b+i}q^{i},$$

with the tables vertices A_1 and A_2 a term $\dfrac{-(b+2a)(b+2i-1)!}{(i-a)!\,(a+b+i)!}\,p^{b+i}q^{i}$,

" " " B_2 and B_3 " $\dfrac{+(3b+2a)(b+2i-1)!}{(i-a-b)!\,(a+2b+i)!}\,p^{b+i}q^{i}$,

and so on.

The probability that B will lose his last counter on the $(b+2i)$th game is therefore

$$(9{\cdot}1)\quad p_{0,b+2i}=(b+2i-1)!\left\{\sum_{r=0}^{r(a+b)\leqslant i}\frac{b+2r(a+b)}{(i-r(a+b))!\,(b+r(a+b)+i)!}\right.$$
$$\left.-\sum_{r=1}^{r(a+b)\leqslant i+b}\frac{2r(a+b)-b}{(i-r(a+b)+b)!\,(r(a+b)+i)!}\right\}p^{b+i}q^{i},$$

a result which we may write

$$(9{\cdot}2)\quad p_{0,n}=\left\{\sum_{r=0}^{r(a+b)\leqslant\frac{n-b}{2}}\frac{b+2r(a+b)}{n}\,{}_nC_{\frac{n-b-2r(a+b)}{2}}\right.$$
$$\left.-\sum_{r=1}^{r(a+b)\leqslant\frac{n+b}{2}}\frac{a+(2r-1)(a+b)}{n}\,{}_nC_{\frac{n+b-2r(a+b)}{2}}\right\}p^{\frac{n+b}{2}}q^{\frac{n-b}{2}},$$

if $n-b$ be even and non-negative, and zero otherwise.

The probability that B will have m counters left after n games have been played is zero if $(n+m-b)$ be odd; if $(n+m-b)$ be even, it is, for $0<m<a+b$,

$$(9{\cdot}3)\quad p_{m,n}$$
$$=\left[\sum_{r=0}^{2r(a+b)\leqslant n+m-b}{}_nC_{\frac{n+m-b-2r(a+b)}{2}}-\sum_{r=0}^{2r(a+b)\leqslant n-m-b}{}_nC_{\frac{n-m-b-2r(a+b)}{2}}\right.$$
$$\left.-\sum_{r=1}^{2r(a+b)\leqslant n+m+b}{}_nC_{\frac{n+m+b-2r(a+b)}{2}}+\sum_{r=1}^{2r(a+b)\leqslant n-m+b}{}_nC_{\frac{n-m+b-2r(a+b)}{2}}\right]p^{\frac{n-m+b}{2}}q^{\frac{n+m-b}{2}}$$

(b) *Ellis's Theorem.*

Since ${}_nC_{\frac{n+m-b}{2}}={}_nC_{\frac{n-m+b}{2}}$, the second member of (9·3) is unaltered by interchanging p with q, b with m, and a with $(a+b-m)$; thus in the course of the set, during which the sum of A's and B's possessions must remain constant, the probability, when B holds b counters, that after n more games he will hold m counters, is the same as the probability, when A holds m counters, that after n more games he will hold b counters. This symmetry is readily explained by the geometrical representation; for if we take any path permissible in one case and reflect it first in the line $x=\dfrac{n}{2}$ and then in the line $y=\dfrac{a+b}{2}$, we arrive at a path permissible in the other case.

By (2·1), a like symmetry exists when a is infinite—a player B', whose chance of winning is equal to B's chance of losing, stands the same chance of having b counters after n games if he starts with m, as B does, of having m counters after n games if he starts with b. It is clear that if we use column $(n-r)$ in the table of B's chances as column r in a table of B''s chances, any path joining $(0, b)$ and (n, m) represents corresponding sequences of games, and its chance of being described is the same, whether we are considering B's or B''s possessions.

(c) *Other Expressions for the Probabilities.*

We return to equation (9·3); it gives, for $(n+m-b)$ even, and $0 < m < a+b$:

$$p_{m,n}$$

$$=\left[\begin{array}{l}\sum\limits_{r=0}^{2r(a+b)\leqq n+m-b} {}_nC_{\frac{n+m-b-2r(a+b)}{2}} + \sum\limits_{r=1}^{2r(a+b)\leqq n-m+b} {}_nC_{\frac{n+m-b+2r(a+b)}{2}} \\ -\sum\limits_{r=0}^{2r(a+b)\leqq n-m-b} {}_nC_{\frac{n-m-b-2r(a+b)}{2}} - \sum\limits_{r=1}^{2r(a+b)\leqq n+m+b} {}_nC_{\frac{n-m-b+2r(a+b)}{2}}\end{array}\right] p^{\frac{n-m+b}{2}} q^{\frac{n+m-b}{2}}$$

or

$$(9·4)\qquad p_{m,n} = \left[\Sigma\, {}_nC_{\frac{n+m-b\pm 2r(a+b)}{2}} - \Sigma\, {}_nC_{\frac{n-m-b\pm 2r(a+b)}{2}}\right] p^{\frac{n-m+b}{2}} q^{\frac{n+m-b}{2}};$$

hence

$$(9·5)\qquad p_{0,n} = p\cdot p_{1,n-1} = \left[\Sigma\, {}_{n-1}C_{\frac{n-b\pm 2r(a+b)}{2}} - \Sigma\, {}_{n-1}C_{\frac{n-2-b\pm 2r(a+b)}{2}}\right] p^{\frac{n+b}{2}} q^{\frac{n-b}{2}},$$

$$(9·6)\qquad p_{a+b,n} = q\cdot p_{a+b-1,n-1} = \left[\Sigma\, {}_{n-1}C_{\frac{n-a\pm 2r(a+b)}{2}} - \Sigma\, {}_{n-1}C_{\frac{n-2-a\pm 2r(a+b)}{2}}\right] p^{\frac{n-a}{2}} q^{\frac{n+a}{2}}$$

In (9·4), (9·5), and (9·6) the summations extend over all possible integral non-negative values of r.

Now the term independent of x in the expansion of $x^{-\frac{n+m-b}{2}}(1+x)^n$ is ${}_nC_{\frac{n+m-b}{2}}$ or zero, according as $(n+m-b)$ is even or odd. Accordingly, if we give x the $(a+b)$ values

$$\cos\frac{2r\pi}{a+b} + \sqrt{-1}\sin\frac{2r\pi}{a+b} \qquad (r = 0, 1, 2 \ldots (a+b-1)),$$

which are the $(a+b)$ several $(a+b)$th roots of unity, and add the corresponding values of $x^{-\frac{n+m-b}{2}}(1+x)^n$, we get $(a+b)\times\Sigma\, {}_nC_{\frac{n+m-b\pm 2r(a+b)}{2}}$ or zero, according as $(n+m-b)$ is even or odd. The sum is

$$\sum_{r=0}^{a+b-1}\left(\cos\frac{2r\pi}{a+b} + \sqrt{-1}\sin\frac{2r\pi}{a+b}\right)^{-\frac{n+m-b}{2}}\left(1+\cos\frac{2r\pi}{a+b} + \sqrt{-1}\sin\frac{2r\pi}{a+b}\right)^n$$

$$= \sum_{r=0}^{a+b-1}\left(\cos\frac{b-m}{a+b}r\pi + \sqrt{-1}\sin\frac{b-m}{a+b}r\pi\right)\left(2\cos\frac{r\pi}{a+b}\right)^n.$$

The imaginary part must vanish, and

$$\frac{1}{a+b}\sum_{r=0}^{a+b-1}\cos\frac{b-m}{a+b}r\pi\left(2\cos\frac{r\pi}{a+b}\right)^n=\Sigma\,{}_nC_{\frac{n+m-b\pm2r(a+b)}{2}}\quad\text{or}\quad 0;$$

similarly

$$\frac{1}{a+b}\sum_{r=0}^{a+b-1}\cos\frac{b+m}{a+b}r\pi\left(2\cos\frac{r\pi}{a+b}\right)^n=\Sigma\,{}_nC_{\frac{n-m-b\pm2r(a+b)}{2}}\quad\text{or}\quad 0,$$

according as $(n+m-b)$ is even or odd. In the latter case $p_{m,n}$ vanishes, and the last two equations, with (9·4), (9·5), and (9·6), accordingly give

$$(9{\cdot}7)\qquad p_{m,n}=\frac{2^{n+1}p^{\frac{n-m+b}{2}}q^{\frac{n+m-b}{2}}}{a+b}\sum_{r=0}^{a+b-1}\sin\frac{br\pi}{a+b}\sin\frac{mr\pi}{a+b}\left(\cos\frac{r\pi}{a+b}\right)^n,$$

$$(9{\cdot}8)\qquad p_{0,n}=\frac{2^{n}p^{\frac{n+b}{2}}q^{\frac{n-b}{2}}}{a+b}\sum_{r=0}^{a+b-1}\sin\frac{br\pi}{a+b}\sin\frac{r\pi}{a+b}\left(\cos\frac{r\pi}{a+b}\right)^{n-1}$$

and

$$(9{\cdot}9)\qquad p_{a+b,n}=\frac{2^{n}p^{\frac{n-a}{2}}q^{\frac{n+a}{2}}}{a+b}\sum_{r=0}^{a+b-1}\sin\frac{br\pi}{a+b}\sin\frac{a+b-1}{a+b}r\pi\left(\cos\frac{r\pi}{a+b}\right)^{n-1}$$

(9·7) holding for all values of m in $0<m<a+b$.

In (9·7), (9·8), and (9·9), the sums may be taken from 1 to $(a+b-1)$, since the terms corresponding to $r=0$ vanish.

If we give to a any value larger that n, then equations (9·4) and (9·5) reduce to (2·1) and (2·3) respectively; (9·4) and (9·5) thus contain the solution to the problem, both when a is finite and when a is infinite. The equivalent equations, (9·7) and (9·8), accordingly give, in an infinite number of forms, the solution to the case dealt with in §2.

(*d*) *B's chance of Ruin.*

As before, let ${}_nP_b$ be B's chance of losing his last counter on or before the nth game. Then

$$(10)\qquad {}_{b+2i+1}P_b={}_{b+2i}P_b=\sum_{j=0}^{i}p_{0,b+2j},$$

where $p_{0,p+2j}$ is given by (9·1).

By (3) and (3·22) we have

$$\sum_{j=0}^{i}\frac{b(b+2j-1)!}{(b+j)!\,j!}p^{b+j}q^j=\text{1st }(i+1)\text{ terms of }(p+q)^{b+2i+1}$$

$$+\text{1st }(i+1)\text{ terms of }\left(\frac{p}{q}\right)^b(q+p)^{b+2i+1}.$$

Thus (10) gives

$$(10\cdot1)\quad {}_{b+2i}P_b = {}_{b+2i+1}P_b$$

$$= \text{1st } (i+1) \text{ terms of } (p+q)^{b+2i+1} + \text{1st } (i+1) \text{ terms of } \left(\frac{p}{q}\right)^b (q+p)^{b+2i+1}$$

$$- \text{1st } (i-a+1) \ldots \left(\frac{q}{p}\right)^a (p+q)^{b+2i+1} - \text{1st } (i-a+1) \ldots \left(\frac{p}{q}\right)^{a+b} (q+p)^{b+2i+1}$$

$$+ \text{1st } (i-a-b+1) \ldots \left(\frac{q}{p}\right)^{a+b} (p+q)^{b+2i+1}$$

$$+ \text{1st } (i-a-b+1) \ldots \left(\frac{p}{q}\right)^{a+2b} (q+p)^{b+2i+1}$$

$$- \text{1st } (i-2a-b+1) \ldots \left(\frac{q}{p}\right)^{2a+b} (p+q)^{b+2i+1}$$

$$- \text{1st } (i-2a-b+1) \ldots \left(\frac{p}{q}\right)^{2a+2b} (q+p)^{b+2i+1}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

or, by (3·3),

$$(10\cdot2)\quad {}_{b+2i}P_b = {}_{b+2i+1}P_b$$

$$= \sum_{r=0}^{r(a+b)\leqslant i} \left(\frac{q}{p}\right)^{r(a+b)} I_p(r(a+b)+b+i+1,\ i-r(a+b)+1)$$

$$- \left(\frac{q}{p}\right)^a \sum_{r=1}^{r(a+b)\leqslant i+b} \left(\frac{q}{p}\right)^{(r-1)(a+b)} I_p(r(a+b)+i+1,\ i+b-r(a+b)+1)$$

$$+ \left(\frac{p}{q}\right)^b \sum_{r=0}^{r(a+b)\leqslant i} \left(\frac{p}{q}\right)^{r(a+b)} I_q(r(a+b)+b+i+1,\ i-r(a+b)+1)$$

$$- \sum_{r=1}^{r(a+b)\leqslant i+b} \left(\frac{p}{q}\right)^{r(a+b)} I_q(r(a+b)+i+1,\ i+b-r(a+b)+1).$$

From (10·2) we can derive the well-known expressions for P_b, the chance that B will ultimately be ruined.

For if $q < \frac{1}{2}$,

$$\left(\frac{p}{q}\right)^b \sum_{r=0}^{r(a+b)\leqslant i} \left(\frac{p}{q}\right)^{r(a+b)} I_q(r(a+b)+b+i+1,\ i-r(a+b)+1)$$

$$= \int_0^q \sum_{r=0} \left(\frac{p}{q}\right)^{r(a+b)+b} \frac{(b+2i+1)\,(b+2i)!}{(r(a+b)+b+i)!\,(i-r(a+b))!} x^{r(a+b)+b+i} (1-x)^{i-r(a+b)}\, dx$$

$$< \int_0^q (b+2i+1) \left(\frac{q}{p}\right)^i \left(1-x+\frac{px}{q}\right)^{b+2i} dx = \frac{q}{p-q}\left(\frac{q}{p}\right)^i \left[\left(1+\frac{p-q}{q}\,x\right)^{b+2i+1}\right]_0^q$$

$$= \frac{q}{p-q}\left\{(2p)^{b+1}(4pq)^i - \left(\frac{q}{p}\right)^i\right\} \to 0 \text{ as } i \to \infty.$$

Similarly, the last sum in (10·2) tends to zero if $q < \frac{1}{2}$, and the first two sums tend to zero if $q > \frac{1}{2}$.

If $q > \frac{1}{2}$,

$$\operatorname*{Lt}_{i\to\infty}\left\{\left(\frac{p}{q}\right)^{r(a+b)} I_q(r(a+b)+b+i+1,\, i-r(a+b)+1)\right\} = \left(\frac{p}{q}\right)^{r(a+b)},$$

$$\left(\frac{p}{q}\right)^{r(a+b)} I_q(r(a+b)+b+i+1,\, i-r(a+b)+1) < \left(\frac{p}{q}\right)^{r(a+b)}$$

and $\sum\limits_{r=1}^{\infty}\left(\frac{p}{q}\right)^{r(a+b)}$ converges to $\dfrac{1}{1-\left(\frac{p}{q}\right)^{a+b}}$.

Hence, by Tannery's Theorem*,

$$\sum_{r=0}^{r(a+b)<i}\left(\frac{p}{q}\right)^{r(a+b)} I_q(r(a+b)+b+i+1,\, i-r(a+b)+1) \to \frac{1}{1-\left(\frac{p}{q}\right)^{a+b}} \text{ as } i\to\infty.$$

Similarly, for $q > \frac{1}{2}$ the last sum in (10·2) tends to $\dfrac{\left(\frac{p}{q}\right)^{a+b}}{1-\left(\frac{p}{q}\right)^{a+b}}$, and for $q < \frac{1}{2}$ the first two sums tend to $\dfrac{1}{1-\left(\frac{q}{p}\right)^{a+b}}$ and $\dfrac{\left(\frac{q}{p}\right)^{a}}{1-\left(\frac{q}{p}\right)^{a+b}}$ respectively.

Thus for $q < \frac{1}{2}$,
$$P_b = \frac{1-\left(\frac{q}{p}\right)^{a}}{1-\left(\frac{q}{p}\right)^{a+b}},$$

and for $q > \frac{1}{2}$,
$$P_b = \frac{\left(\frac{p}{q}\right)^{b}-\left(\frac{p}{q}\right)^{a+b}}{1-\left(\frac{p}{q}\right)^{a+b}} = \frac{\left(\frac{q}{p}\right)^{a}-1}{\left(\frac{q}{p}\right)^{a+b}-1},$$

so that

$$P_b = \frac{1-\left(\frac{q}{p}\right)^{a}}{1-\left(\frac{q}{p}\right)^{a+b}} \qquad (p \neq q). \tag{11·1}$$

P_b must clearly be a decreasing function of $\frac{q}{p}$, so that its value for $p = q$ must lie between the limits to which P_b tends as $\frac{q}{p}$ tends to unity from above and from below. But these limits are both $\frac{a}{a+b}$. Accordingly

$$P_b = \frac{a}{a+b} \qquad (p = q). \tag{11·2}$$

The values given in (6) are the limits of those given in (11·1) and (11·2), when $a\to\infty$.

* See Bromwich, *Infinite Series*, p. 136.

§ 4. SOME APPROXIMATE RESULTS.

So far we have been concerned with exact results. B's chance of being ruined in a given number of games, when A's fortune is unlimited, is given very simply by equation (4), and until n and b become considerable its numerical value may be found from the Tables of the Incomplete Beta-Function shortly to be published. Outside the range of these tables, the integrals can be evaluated approximately, by Weddling, by Dr Müller's continued fraction, or by the methods dealt with by Dr Wishart*.

If $p = q = \frac{1}{2}$, B's chance of being ruined in n games is, if $n + b$ be even,

$$(12) \qquad {}_nP_b = 2I_{\frac{1}{2}}\left(\frac{n+b+2}{2}, \frac{n-b+2}{2}\right).$$

We proceed to examine some methods of approximating to this chance, when n is large compared with b.

Method A.

The mode and the mean of the Type I Curve

$$(12{\cdot}1) \qquad y = \frac{\Gamma(n+2)}{\Gamma\left(\frac{n+b+2}{2}\right)\Gamma\left(\frac{n-b+2}{2}\right)} x^{\frac{n+b}{2}}(1-x)^{\frac{n-b}{2}}$$

being at $\frac{1}{2} + \frac{b}{2n}$, $\frac{1}{2} + \frac{b}{2(n+2)}$ respectively, the median is at $\frac{1}{2} + \frac{b(3n+2)}{6n(n+2)}$ approximately; accordingly we write (12) in the form

$$\tfrac{1}{2}(1 - {}_nP_b) = \frac{\Gamma(n+2)}{\Gamma\left(\frac{n+b+2}{2}\right)\Gamma\left(\frac{n-b+2}{2}\right)}\left\{\int_{\frac{1}{2}}^{\frac{1}{2}+\frac{b}{2n}} x^{\frac{n+b}{2}}(1-x)^{\frac{n-b}{2}}\,dx\right.$$
$$\left. - \int_{\frac{1}{2}+\frac{b(3n+2)}{6n(n+2)}}^{\frac{1}{2}+\frac{b}{2n}} x^{\frac{n+b}{2}}(1-x)^{\frac{n-b}{2}}\,dx\right\}.$$

The second member of Dr Wishart's equation (27) (*Biometrika*, Vol. XIX. p. 29) gives an approximation to the modal integral† of a Type I Curve in a series of Incomplete Normal Moment Functions. Using his result we have

$$(13) \qquad \tfrac{1}{2}(1 - {}_nP_b) = k_0\{M(u_1) - M(u_2)\},$$

where $\quad u_1 = \dfrac{b\sqrt{n}}{\sqrt{n^2 - b^2}}, \quad u_2 = \dfrac{4}{3(n+2)}\dfrac{b\sqrt{n}}{\sqrt{n^2-b^2}},$

$$k_0 = \left(1 + \frac{1}{n}\right)\left\{1 - \frac{1}{12n}\left(\frac{3n^2+b^2}{n^2-b^2}\right) + \frac{1}{288n^2}\left(\frac{3n^2+b^2}{n^2-b^2}\right)^2\right\},$$

* "The Approximate Quadrature of Certain Skew Curves," *Biometrika*, Vol. XIX.

† Not, as Dr Wishart states, to the Incomplete Beta-Function.

and $$M(u) = m_0(u) + \frac{4b}{3\sqrt{n(n^2-b^2)}}\, m_3(u) - \frac{3}{4}\frac{n^2+3b^2}{n(n^2-b^2)}\, m_4(u)$$
$$+ \frac{32}{5}\frac{b(n^2+b^2)}{\{n(n^2-b^2)\}^{\frac{3}{2}}}\, m_5(u) + \frac{5}{6}\left\{\frac{4b^2}{n(n^2-b^2)} - \frac{3(n^4+10n^2b^2+5b^4)}{n^2(n^2-b^2)^2}\right\} m_6(u)$$
$$- 8\frac{b(n^2+3b^2)}{\{n(n^2-b^2)\}^{\frac{3}{2}}}\, m_7(u) + \frac{7}{32}\frac{15n^4+346n^2b^2+109b^4}{n^2(n^2-b^2)^2}\, m_8(u)$$
$$+ \frac{512}{27}\frac{b^3}{\{n(n^2-b^2)\}^{\frac{3}{2}}}\, m_9(u) - \frac{105}{4}\frac{b^2(n^2+3b^2)}{n^2(n^2-b^2)^2}\, m_{10}(u) + \frac{385}{18}\frac{b^4}{n^2(n^2-b^2)^2}\, m_{12}(u);$$

here $m_0(u) = \int_0^u \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx$ and $m_3(u), m_4(u), \ldots$ are the 3rd, 4th, ... Incomplete Normal Moment Functions tabled in *Tables for Statisticians and Biometricians.*

This expansion is valid only when u_1 and u_2 are not much greater than 1, that is, roughly, when $n > b^2$. In this case, since the coefficients in $M(u)$ are given only as far as terms in $\frac{1}{n^2}$, we lose nothing in accuracy by taking

$$(13{\cdot}1) \qquad k_0 = 1 + \frac{3}{4}\frac{1}{n} - \left(\frac{7}{32} + \frac{b^2}{3n}\right)\frac{1}{n^2},$$

and

$$(13{\cdot}2) \quad M(u) = m_0(u) + \frac{4}{3}\frac{b}{\sqrt{n}}\frac{1}{n}\left(1 + \frac{1}{2}\frac{b^2}{n^2}\right) m_3(u) - \frac{3}{4}\frac{1}{n}\left(1 + \frac{4b^2}{n^2}\right) m_4(u)$$
$$+ \frac{32}{5}\frac{b}{\sqrt{n}}\frac{1}{n^2} m_5(u) + \frac{5}{6}\frac{1}{n^2}\left(\frac{4b^2}{n} - 3\right) m_6(u)$$
$$- 8\,\frac{b}{\sqrt{n}}\frac{1}{n^2} m_7(u) + \frac{105}{32}\frac{1}{n^2} m_8(u).$$

Method B.

If we replace the Type I Curve (12·1) by a normal curve of the same mean and standard deviation, $\frac{1}{2} + \frac{b}{2(n+2)}$ and $\frac{1}{2(n+2)}\sqrt{\frac{(n+2)^2-b^2}{n+3}}$ respectively, then instead of (12) we have the approximate result

$$(14) \qquad \tfrac{1}{2}(1 - {}_nP_b) = \int_0^{b\sqrt{\frac{n+3}{(n+2)^2-b^2}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx.$$

Method C.

Bertrand* has pointed out that if we approximate to the factorials in (2·31) by means of Stirling's Theorem, and make $p = q = \frac{1}{2}$, then that equation becomes

$$(15) \qquad p_{0,b+2i} = \frac{2b}{\sqrt{2\pi}\,(b+2i)^{\frac{3}{2}}}\, e^{-\frac{1}{2}\frac{b^2}{b+2i}};$$

thus, approximately,

$$1 - {}_{b+2i}P_b = \sum_{i+1}^{\infty} p_{0,b+2i} = \frac{2b}{\sqrt{2\pi}} \int_{i+1}^{\infty} \frac{e^{-\frac{1}{2}\frac{b^2}{b+2i}}}{(b+2i)^{\frac{3}{2}}}\,di.$$

* *Calcul des Probabilités*, Ch. VI.

Writing $b+2i=n$, $nx^2=b^2$, we find

$$(16) \qquad \tfrac{1}{2}(1-{}_nP_b)=\int_0^{\frac{b}{\sqrt{n+2}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\,dx.$$

Method D.

We have seen that (9·8) gives the value of $p_{0,n}$ when a is infinite, if in that equation we give a any value $>n$. Accordingly

$${}_nP_b=P_b-\sum_{n+1}^{\infty}p_{0,n}=P_b-\frac{(4pq)^{\frac{n+1}{2}}}{a+b}\left(\frac{p}{q}\right)^{\frac{b}{2}}\sum_{r=1}^{a+b-1}\frac{\sin\dfrac{r\pi}{a+b}\sin\dfrac{rb\pi}{a+b}\left(\cos\dfrac{r\pi}{a+b}\right)^n}{1-2\sqrt{pq}\cos\dfrac{r\pi}{a+b}}.$$

When a is large compared with b, we may replace the sum by an integral, and write

$${}_nP_b=P_b-\frac{(4pq)^{\frac{n+1}{2}}}{\pi}\left(\frac{p}{q}\right)^{\frac{b}{2}}\int_0^{\pi}\frac{\sin r\phi\sin rb\phi\cos^n\phi}{1-\sqrt{4pq}\cos\phi}\,d\phi.$$

Putting $\phi+\phi'=\pi$, we have, integrating separately over the ranges 0 to $\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ to π,

$$\int_{\frac{\pi}{2}}^{\pi}\frac{\sin\phi\sin b\phi\cos^n\phi}{1-\sqrt{4pq}\cos\phi}\,d\phi=\int_0^{\frac{\pi}{2}}\frac{\sin\phi'(-1)^{b-1}\sin b\phi'(-1)^n\cos^n\phi'}{1+\sqrt{4pq}\cos\phi'}\,d\phi'$$

$$=(-1)^{n+b-1}\int_0^{\frac{\pi}{2}}\frac{\sin\phi\sin b\phi\cos^n\phi}{1+\sqrt{4pq}\cos\phi}\,d\phi,$$

whence

$$(17) \qquad {}_nP_b=P_b-(4pq)^{\frac{n'+2}{2}}\left(\frac{p}{q}\right)^{\frac{b}{2}}\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{\sin\phi\sin b\phi\cos^{n'+1}\phi}{1-4pq\cos^2\phi}\,d\phi,$$

where $n'=n-1$ or n, according as $n+b$ is odd or even.

For $p=q=\frac{1}{2}$, (17) becomes

$$(17{\cdot}1)^* \qquad {}_nP_b=1-\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{\cos^{n'+1}\phi}{\sin\phi}\sin b\phi\,d\phi.$$

Let

$$\Phi=\int_0^{\frac{\pi}{2}}\frac{\cos^s\phi}{\sin\phi}\sin b\phi\,d\phi.$$

The integrand in Φ has its maximum value, b, at $\phi=0$, and if s is large decreases very rapidly in numerical value as ϕ increases. The value of the integral is therefore due almost entirely to the contribution of a small range of ϕ near $\phi=0$, and for this range we have, neglecting powers of ϕ above the fifth,

$$\log\frac{\cos^s\phi}{\sin\phi}=s\log\left(1-\frac{\phi^2}{2}+\frac{\phi^4}{24}\right)-\log\phi-\log\left(1-\frac{\phi^2}{6}+\frac{\phi^4}{120}\right)$$

$$=-s\left(\frac{\phi^2}{2}-\frac{\phi^4}{24}\right)-s\frac{\phi^4}{8}+\left(\frac{\phi^2}{6}-\frac{\phi^4}{120}\right)+\frac{\phi^4}{72}-\log\phi$$

$$=-\tfrac{1}{2}\phi^2(s-\tfrac{1}{3})-\tfrac{1}{12}\phi^4(s-\tfrac{1}{15})-\log\phi$$

$$=-\tfrac{1}{2}\phi^2(s-\tfrac{1}{3})+\log\{1-\tfrac{1}{12}\phi^4(s-\tfrac{1}{15})\}-\log\phi,$$

* The method by which we proceed to equation (19) is due to Laplace (*Théorie Analytique des Probabilités*).

so that, approximately,

$$\frac{\cos^s\phi}{\sin\phi} = \frac{e^{-\frac{1}{2}\phi^2(s-\frac{1}{2})}}{\phi}\{1 - \tfrac{1}{12}\phi^4(s - \tfrac{1}{15})\}.$$

The second member of this equation decreases very rapidly as ϕ increases; accordingly, treating

$$\int_{\frac{\pi}{2}}^{\infty} \frac{e^{-\frac{1}{2}\phi^2(s-\frac{1}{2})}}{\phi}\{1 - \tfrac{1}{12}(s - \tfrac{1}{15})\,\phi^4\}\sin b\phi\, d\phi$$

as negligible, we have, approximately,

$$(18) \qquad \Phi = \int_0^{\infty} \frac{e^{-\frac{1}{2}\phi^2(s-\frac{1}{2})}}{\phi}\{1 - \tfrac{1}{12}(s - \tfrac{1}{15})\,\phi^4\}\sin b\phi\, d\phi,$$

whence*

$$(18{\cdot}1) \qquad \frac{d\Phi}{db} = \int_0^{\infty} e^{-\frac{1}{2}\phi^2(s-\frac{1}{2})}\{1 - \tfrac{1}{12}(s - \tfrac{1}{15})\,\phi^4\}\cos b\phi\, d\phi.$$

Now
$$\int_0^{\infty} e^{-\lambda^2\phi^2}\cos b\phi\, d\phi = \frac{\sqrt{\pi}}{2\lambda}e^{-\frac{b^2}{4\lambda^2}},$$

whence
$$\int^{\infty} \phi^4 e^{-\lambda^2\phi^2}\cos b\phi\, d\phi = \frac{\sqrt{\pi}}{2\lambda}\frac{d^4}{db^4}\left(e^{-\frac{b^2}{4\lambda^2}}\right)$$
$$= \frac{3\sqrt{\pi}}{8\lambda^5}\left(1 - \frac{b^2}{\lambda^2} + \frac{1}{12}\frac{b^4}{\lambda^4}\right)e^{-\frac{b^2}{4\lambda^2}}$$

Thus we have from (18·1), writing $\frac{1}{2}(s - \frac{1}{2}) = \lambda^2 = \frac{1}{2}\sigma^2$,

$$\frac{d\Phi}{db} = \frac{\sqrt{\pi}}{2\lambda}\left\{1 - \frac{1}{8\lambda^4}\left(\lambda^2 + \frac{2}{15}\right)\left(1 - \frac{b^2}{\lambda^2} + \frac{1}{12}\frac{b^4}{\lambda^4}\right)\right\}e^{-\frac{b^2}{4\lambda^2}}$$
$$= \frac{1}{\sigma}\sqrt{\frac{\pi}{2}}\left\{\left(1 - \frac{1}{4\sigma^2} - \frac{1}{15\sigma^4}\right) + \frac{b^2}{\sigma^2}\left(\frac{1}{2\sigma^2} + \frac{2}{15\sigma^4}\right) - \frac{1}{6}\frac{b^4}{\sigma^4}\left(\frac{1}{2\sigma^2} + \frac{2}{15\sigma^4}\right)\right\}e^{-\frac{1}{2}\frac{b^2}{\sigma^2}}$$

Now
$$\int_0^b \frac{b^2}{\sigma^2}e^{-\frac{1}{2}\frac{b^2}{\sigma^2}}db = -be^{-\frac{1}{2}\frac{b^2}{\sigma^2}} + \int_0^b e^{-\frac{1}{2}\frac{b^2}{\sigma^2}}db,$$
$$\int_0^b \frac{b^4}{\sigma^4}e^{-\frac{1}{2}\frac{b^2}{\sigma^2}}db = -\frac{b^3}{\sigma^2}e^{-\frac{1}{2}\frac{b^2}{\sigma^2}} - 3b\, e^{-\frac{1}{2}\frac{b^2}{\sigma^2}} + 3\int_0^b e^{-\frac{1}{2}\frac{b^2}{\sigma^2}}db;$$

thus the equation

$$\Phi = \frac{1}{\sigma}\sqrt{\frac{\pi}{2}}\int_0^b\left\{\left(1 - \frac{1}{4\sigma^2} - \frac{1}{15\sigma^4}\right) + \frac{b^2}{\sigma^2}\left(\frac{1}{2\sigma^2} + \frac{2}{15\sigma^4}\right) - \frac{1}{6}\frac{b^4}{\sigma^4}\left(\frac{1}{2\sigma^2} + \frac{2}{15\sigma^4}\right)\right\}e^{-\frac{1}{2}\frac{b^2}{\sigma^2}}db,$$

gives, after some reduction,

$$(18{\cdot}2) \quad \Phi = \pi\int_0^b \frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{1}{2}\frac{b^2}{\sigma^2}}db - \frac{\pi}{4}\frac{b}{\sigma^2}\left(1 + \frac{4}{15\sigma^2}\right)\left(1 - \frac{b^2}{3\sigma^2}\right)\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{1}{2}\frac{b^2}{\sigma^2}}.$$

* The differentiation under the integral sign is legitimate if the resulting integral is uniformly convergent; but this is so, for the integral obtained by omitting the trigonometrical factor in the integrand is absolutely convergent. A similar remark applies to the differentiations that follow.

In (18·2) take $s = n' + 1$, or $\sigma^2 = n' + \frac{2}{3}$; substitute in (17·1), and we have the approximate result

$$(19) \qquad \tfrac{1}{2}(1 - {}_nP_b) = \int_0^{\frac{b}{\sqrt{n'+\frac{2}{3}}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx$$
$$- \frac{1}{4}\frac{b}{n'+\frac{2}{3}}\left(1 + \frac{4}{15}\frac{1}{n'+\frac{2}{3}}\right)\left(1 - \frac{1}{3}\frac{b^2}{n'+\frac{2}{3}}\right)\frac{1}{\sqrt{2\pi(n'+\frac{2}{3})}} e^{-\frac{1}{2}\frac{b^2}{n'+\frac{2}{3}}}.$$

(19) is a more accurate form of the equation given by Laplace (*loc. cit.* p. 259), which is, in our notation,

$$\tfrac{1}{2}(1 - {}_nP_b) = \int_0^{\frac{b}{\sqrt{n+\frac{2}{3}}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx - \frac{1}{4}\frac{b}{n+\frac{2}{3}}\left(1 - \frac{1}{3}\frac{b^2}{n+\frac{2}{3}}\right)\frac{1}{\sqrt{2\pi(n+\frac{2}{3})}} e^{-\frac{1}{2}\frac{b^2}{n+\frac{2}{3}}}.$$

If we sum from b to ∞ the expression for $p_{0,n}$ given by (9·8), and then make a infinite, we get

$$P_b = \frac{(4pq)^{\frac{b}{2}}}{\pi}\left(\frac{p}{q}\right)^{\frac{b}{2}} \int_0^{\pi} \frac{\sin\phi \sin b\phi (\cos\phi)^{b-1}}{1 - \sqrt{4pq}\cos\phi}\,d\phi,$$

whence, as for equation (17),

$$(20) \qquad P_b = (2p)^b \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin\phi \sin b\phi (\cos\phi)^{b-1}}{1 - 4pq\cos^2\phi}\,d\phi.$$

If we substitute from (20) in (17), make $p = q$, and then approximate to the two integrals by means of (18·2), we arrive at

$$(21) \qquad \tfrac{1}{2}\,{}_nP_b = \int_0^{\frac{b}{\sqrt{b-\frac{1}{3}}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx - \int_0^{\frac{b}{\sqrt{n'+\frac{2}{3}}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx$$
$$- \frac{1}{4}\frac{b}{b-\frac{1}{3}}\left(1 + \frac{4}{15}\frac{1}{b-\frac{1}{3}}\right)\left(1 - \frac{1}{3}\frac{b^2}{b-\frac{1}{3}}\right)\frac{1}{\sqrt{2\pi(b-\frac{1}{3})}} e^{-\frac{1}{2}\frac{b^2}{b-\frac{1}{3}}}$$
$$+ \frac{1}{4}\frac{b}{n'+\frac{2}{3}}\left(1 + \frac{4}{15}\frac{1}{n'+\frac{2}{3}}\right)\left(1 - \frac{1}{3}\frac{b^2}{n'+\frac{2}{3}}\right)\frac{1}{\sqrt{2\pi(n'+\frac{2}{3})}} e^{-\frac{1}{2}\frac{b^2}{n'+\frac{2}{3}}},$$

where, as in (19), $n' = n - 1$ or n, according as $n + b$ is odd or even.

Except for small values of b, (21) will not give results differing significantly from those provided by (19); for $b = 6$, $n = 78$, the true value of ${}_nP_b$ is ·499897; (19) gives ·499895, and (21), ·499710, so that (21) appears to be less accurate, as well as less simple, than (19).

Incidentally, by comparing (20) with (6), we have the analytical theorem

$$(22) \qquad \int_0^{\frac{\pi}{2}} \frac{\sin\phi \sin b\phi (\cos\phi)^{b-1}}{1 - \lambda\cos^2\phi}\,d\phi = (2\mu)^{-b}\frac{\pi}{2},$$

where $0 < \lambda \leqslant 1$, and μ is the greater root of

$$4\mu^2 - 4\mu + \lambda = 0.$$

We turn now to the converse problem: when B's fortune is known, A's unlimited, and $p=q$, what number of games must we assign to make B's chance of ruin assume any given value?

In general, of course, the question cannot, strictly speaking, be answered: all that we can hope to do is to find an integer n, even or odd with b, such that ${}_nP_b < P < {}_{n+2}P_b$, where P is the given value.

Approximately, n is given by the equation

(23) $$I_{\frac{1}{2}}\left(\frac{n+b+2}{2}, \frac{n-b+2}{2}\right)=\tfrac{1}{2}P;$$

outside the range of the Incomplete Beta-Function Tables, approximations to n can be found by replacing (23) by equations (14) and (16), which can be solved by means of tables giving the abscissa of the normal curve in terms of its area*.

For $P=\frac{1}{2}$, the case discussed by Laplace (*loc. cit.* pp. 257—260), the problem is the same as that of finding a Type I Curve, with a given difference between its indices, and having one quartile at $x=\cdot 5$. If, as in equation (14), we replace the Type I Curve by a normal curve having the same mean and standard deviation, we reach the approximate result

$$n+2=\frac{b^2}{2t^2}\left(1+\sqrt{1+\frac{4t^2}{b^2}+\frac{4t^4}{b^2}}\right) \qquad (t=\cdot 6744897502)$$

$$=1\cdot 09905467\,b^2\left(1+\sqrt{1+2\cdot 64761429\,b^{-2}}\right),$$

or, slightly less exactly,

(24) $$n=2\cdot 1981093\,b^2-\cdot 545064.$$

If we put ${}_nP_b=\frac{1}{2}$ in equation (13), take k_0 to be unity, neglect $M(u_2)$ because u_2 is small, and retain only the first term of $M(u_1)$, we get

$$\int_0^{\frac{b\sqrt{n}}{\sqrt{n^2-b^2}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\,dx=\tfrac{1}{4},$$

whence $$n=\frac{b^2}{2t^2}\left(1+\sqrt{1+\frac{4t^4}{b^2}}\right),$$

or, approximately,

(25) $$n=2\cdot 1981093\,b^2+\cdot 454936,$$

a result differing by 1 from that of (24).

If we put ${}_nP_b=\frac{1}{2}$ in (16), we have, at once,

(26) $$n=2\cdot 1981093\,b^2-2.$$

Finally, we may apply the equation of Method D to the converse problem; there we take $n'=n$, because the value of n required is even or odd with b.

* *Tables for Statisticians and Biometricians*, Part I, Table III.

If, as a first approximation, we retain only the integral term in the second member of (19), we obtain, for the value of n which makes ${}_nP_b$ equal to a half,

$$\int_0^{\frac{b}{\sqrt{n+\frac{2}{3}}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \tfrac{1}{4},$$

whence

(27) $$n = 2{\cdot}1981093\, b^2 - \tfrac{2}{3} \text{ approximately,}$$

an equation not very different from (24), (25), and (26).

If we substitute this value of n in the remaining term in (19) we get

$$-\frac{1}{4}\frac{t^2}{b}\left(1+\frac{4}{15}\frac{t^2}{b^2}\right)\left(1-\frac{1}{3}t^2\right)\frac{t}{b}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2} \quad (t = {\cdot}6744897502)$$

$$= \frac{t^2-3}{12}\frac{t^3}{b^2}\left(1+\frac{4}{15}\frac{t^2}{b^2}\right) \times {\cdot}3177765727$$

$$= -{\cdot}0206807\, b^{-2} - {\cdot}0025089\, b^{-4}.$$

We have, therefore, for a second approximation to the value of n that makes ${}_nP_b$ equal to a half,

(28) $$\int_0^{\frac{b}{\sqrt{n+\frac{2}{3}}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = {\cdot}25 + {\cdot}0206807\, b^{-2} + {\cdot}0025089\, b^{-4}.$$

We may solve this equation either by means of Table III in Part I of *Tables for Statisticians*, or by the following approximate process.

Let $$\alpha = 2\int_0^x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx,$$

so that $$\frac{dx}{d\alpha} = \frac{1}{2}\Big/\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\right), \quad \frac{d^2x}{d\alpha^2} = \frac{1}{4}x\Big/\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\right)^2.$$

Take $\alpha = {\cdot}5$, so that* $$x = {\cdot}6744897502 = t,$$

and $$\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2} = {\cdot}3177765727 = \lambda \text{ say.}$$

Then for small ϵ, $$\alpha + \epsilon = 2\int_0^{t+\eta} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx.$$

where $$t + \eta = t + \frac{1}{2\lambda}\epsilon + \frac{t}{8\lambda^2}\epsilon^2 \text{ approximately.}$$

Take $$\epsilon = \frac{3-t^2}{6}\frac{\lambda t^3}{b^2}\left(1+\frac{4}{15}\frac{t^2}{b^2}\right).$$

Then as far as terms in b^{-4},

$$t + \eta = t\left\{1 + \frac{3-t^2}{12}\frac{t^2}{b^2}\left(1+\frac{4}{15}\frac{t^2}{b^2}\right) + \frac{(3-t^2)^2}{288}\frac{t^6}{b^4}\right\},$$

* These values are taken from Kondo and Elderton's Tables of the Normal Curve.

so that (21) gives

$$n+\frac{2}{3}=\frac{b^2}{t^2}\left\{1+\frac{3-t^2}{12}\frac{t^2}{b^2}\left(1+\frac{4}{15}\frac{t^2}{b^2}\right)+\frac{(3-t^2)^2}{288}\frac{t^4}{b^4}\right\}^{-2}$$

$$=\frac{b^2}{t^2}-\frac{3-t^2}{6}-\frac{3-t^2}{6}\frac{t^2}{b^2}\left(\frac{4}{15}+\frac{3-t^2}{24}t^2-\frac{3-t^2}{8}\right)$$

$$=\frac{b^2}{t^2}-\frac{3-t^2}{6}-\frac{3-t^2}{6}\left\{\frac{4}{15}-\frac{(3-t^2)^2}{24}\right\}\frac{t^2}{b^2},$$

whence

$$(28{\cdot}1)\qquad n=2{\cdot}1981093\,b^2-1{\cdot}090844+{\cdot}0006219\,b^{-2}.$$

In the table of numerical illustrations, the second and third columns show, for different values of b, the values of n provided by equations (24) and (28) respectively; the approximate constancy of the difference between these values indicates that for our purpose (28·1) is just as good as (28). To estimate the degree of accuracy to which (28·1) gives the value of n satisfying

$$(29)\qquad \int_0^{\frac{b^2}{\sqrt{n+\frac{2}{3}}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx-{\cdot}25$$

$$-\frac{1}{4}\frac{b}{n+\frac{2}{3}}\left(1+\frac{4}{15}\frac{1}{n+\frac{2}{3}}\right)\left(1-\frac{1}{3}\frac{b^2}{n+\frac{2}{3}}\right)\frac{1}{\sqrt{2\pi}\,(n+\frac{2}{3})}e^{-\frac{1}{2}\frac{b^2}{n+\frac{2}{3}}}=0$$

we write $\dfrac{b}{\sqrt{n+\frac{2}{3}}}=t+\eta$, where η will be small.

By Taylor's Theorem,

$$\int_0^{t+\eta}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx-{\cdot}25=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}\left(\eta-\frac{t}{2}\eta^2+\frac{t^2-1}{6}\eta^3+\frac{3t-t^3}{24}\eta^4+\dots\right),$$

$$\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(t+\eta)^2}=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}\left(1-t\eta+\frac{t^2-1}{2}\eta^2+\frac{3t-t^3}{6}\eta^3+\dots\right),$$

and

$$\frac{1}{4}\frac{(t+\eta)^3}{b^2}\left\{1+\frac{4}{15}\frac{(t+\eta)^2}{b^2}\right\}\left\{1-\frac{1}{3}(t+\eta)^2\right\}=\frac{1}{4}\frac{t^3}{b^2}\left(\frac{3-t^2}{3}+\frac{4}{15}\frac{3-t^2}{3}\frac{t^2}{b^2}\right)$$

$$+\frac{1}{4}\frac{t^2}{b^2}\left(\frac{9-5t^2}{3}+\frac{4}{15}\frac{15-7t^2}{3}\frac{t^2}{b^2}\right)\eta+\frac{1}{4}\frac{t}{b^2}\left(\frac{9-10t^2}{3}+4\frac{10-7t^2}{15}\frac{t^2}{b^2}\right)\eta^2$$

$$+\dots\dots;$$

as far as terms in η^2, therefore,

$$\frac{1}{4}\frac{t^3}{b^2}\frac{3-t^2}{3}\left(1+\frac{4}{15}\frac{t^2}{b^2}\right)-\eta\left\{1-\frac{1}{4}\frac{t^2}{b^2}\left(\frac{9-8t^2+t^4}{3}+\frac{4}{15}\frac{15-10t^2+t^4}{3}\frac{t^2}{b^2}\right)\right\}$$

$$+\eta^2\frac{t}{2}\left\{1+\frac{1}{4b^2}\left(\frac{18-41t^2+14t^4-t^6}{3}+\frac{4}{15}\frac{60-75t^2+18t^4-t^6}{3}\frac{t^2}{b^2}\right)\right\}=0.$$

Neglecting terms in η^2 and $\dfrac{1}{b^4}$, we find from this equation

$$n=2{\cdot}1981093\,b^2-1{\cdot}090844;$$

solving the quadratic, we get, as far as terms in b^{-4},

$$\eta = \cdot 06507946^{-3} + \cdot 2439733b^{-4},$$

whence

$$n = 2\cdot 1981093b^2 - 1\cdot 090844 - 1\cdot 528788b^{-2}. \tag{30}$$

The terms that we have neglected in finding the last value of η are of the order η^2, or b^{-6}; the value of η may therefore be regarded as accurate as far as terms in b^{-4}, so that (30) gives the solution of (29) as far as terms in b^{-2}.

In the accompanying table, Columns 2 and 3 show the approximations to n, for different values of b, given by equations (24) and (28) respectively. Equation (27) would give values less by $\cdot 122$, equation (25) values greater by 1, than those in the second column. Except for small values of b, the values from equations (28·1) and (30) coincide with those in the third column, and those from equation (26) are less than the latter by $\cdot 901$. In the fourth column are given the values of n that we set out to find, in the remaining columns the values of ${}_nP_b$ provided by equations (16), (13), and (19) respectively.

b	Approximations to n		n	Approximations to ${}_nP_b$		
	(i) by (24)	(ii) by (28)		(i) by (16)	(ii) by (13)	(iii) by (19)
1	1·653	1·114	1	—	—	—
2	8·247	7·702	6	—	—	—
3	19·238	18·692	17	—	*·4806824*	·4806481
4	34·625	34·079	34	—	*·4995508*	·4995520
5	54·408	53·863	53	—	*·4966174*	·4966140
6	78·587	78·041	78	—	*·4998968*	·4998953
7	107·162	106·617	105	·49859	·4967030	·4967534
8	140·134	139·588	138	·49896	·4975330	·4975636
9	177·502	176·956	175	·49873	·4976108	·4976290
10	219·266	218·720	218	·50018	·4992850	·4992970
20	878·699	878·153	878	·50018	·4999620	·4999627
30	1977·753	1977·207	1976	·49997	·4998690	·4998691
40	3516·429	3515·883	3514	·49994	·4998850	·4998851
50	5494·727	5494·182	5494	·50003	·4999929	·4999928
60	7912·647	7912·101	7912	·50002	·4999971	·4999972
70	10770·189	10769·643	10768	·49995	·4999672	·4999673
80	14067·353	14066·807	14066	·49997	·4999876	·4999877
90	17804·138	17803·593	17802	·499980	·4999809	·4999808
100	21980·545	21980·000	21980	·500008	·4999999	·49999996

Until $n+b$ becomes greater than 100, we can find the true value of ${}_nP_b$ from tables of the Incomplete Beta-Function; these true values are shown in italics in Column 6 for the values 3, 4, 5, and 6 of b. It will be seen that even for such small values of b (19) provides a very close approximation, and we may expect it to improve as b increases. The approximations to ${}_nP_b$ provided by (16) and (13) diverge somewhat widely from the true values when b is very small, but the close agreement between the lower portions of Columns 6 and 7 indicates that the error in (13) very quickly disappears.

Some interest attaches to the value of n corresponding to $b=100$. It is for this value of b that Laplace says: "Il y a donc alors du désavantage à parier un contre un que A gagnera la partie dans 23780 coups; mais il y a de l'avantage à parier qu'il la gagnera dans 23781 coups." The difference between Laplace's result and our own is due in part to the fact that he was not able to refer to the tables that are nowadays available; but in any case Laplace would appear to have lost sight of the approximate nature of his result, since the odds are exactly the same that the set will be ended in 23780 games, as in 23781. Actually, we find from equation (10),

$$_{23780}P_{100}=\cdot 516687.$$

§ 5. Historical Note.

The problem of the Duration of Play is one of the oldest in the Calculus of Probabilities*, and several of the results given above are by no means new. Perhaps the best known of them are the expressions for B's chance of ultimately being ruined, namely

$$(11\cdot 1)\qquad P_b=\frac{1-\left(\frac{q}{p}\right)^a}{1-\left(\frac{q}{p}\right)^{a+b}},\qquad (p\neq q)$$

$$(11\cdot 2)\qquad P_b=\frac{a}{a+b},\qquad (p=q)$$

$$(6)\qquad \begin{cases} P_b=\left(\frac{p}{q}\right)^b, & (a=\infty,\ q>p)\\ P_b=\ 1\ , & (a=\infty,\ q\leqslant p)\end{cases}$$

and the expression for B's chance of being ruined in a given number of games, when a is infinite, in the form

$$(8)\quad {}_{b+2i+1}P_b={}_{b+2i}P_b=p^b\left[1+b.pq+\frac{b(b+3)}{2}p^2q^2+\ ..+\frac{b(b+2i-1)!}{i!(b+i)!}p^iq^i\right].$$

* I have to thank Professor Pearson for pointing out that it was in Huygens' small tract, *Van Rekeningh in Spelen van Geluck*, in the course of which he considers the somewhat similar Problem of Points, that the Calculus of Probabilities originated. Huygens' problem is this: Several players engage in a set, he that first gains a certain number of games being the winner; given the number of games still required by the various players, determine their chances of winning. Huygens communicated his tract to his teacher, Franciscus van Schooten, who published a Latin translation of it in 1657 as an appendix to his *Exercitationum Mathematicarum Libri Quinque*; the vernacular version appeared in 1660, and English translations were published in 1692 or so by Dr Arbuthnot, and in 1714 by W. Browne. It is true that Pascal and Fermat were discussing questions of chance in their correspondence three or four years before Huygens' tract appeared, and Huygens himself says in his preface, "Sciendum vero, quod jam pridem inter praestantissimos tota Gallia Geometras calculus hic agitatus fuerit, ne quis indebitam mihi primae inventionis gloriam hac in re tribuat." But the Pascal-Fermat letters remained unpublished for another twenty years, and it was Huygens' tract that inspired the work of Montmort and de Moivre. In his preface to *The Doctrine of Chances*, de Moivre explicitly states that when he wrote his "Specimen" he "had not at that time read anything concerning this Subject, but Mr Huygens' Book *de Ratiociniis in Ludo Aleae*, and a little English piece which was properly a Translation of it."

This last result was first given by de Moivre, in his discussion of Problem LXV in *The Doctrine of Chances* (3rd edition, 1756). The equivalent result ((3·12) and (3·22)), and the expression (9·1) for the probability when a is not infinite that B will lose his last counter on the $(b+2i)$th game, were also given by de Moivre (*loc. cit.* Problems LXV and LXIV), although they were developed for the present paper before his work on the subject was consulted. De Moivre's solutions to Problems LXIV and LXV are given without any demonstration, but his solutions to Problems LVIII to LXIII, which also deal with the Duration of Play, leave little doubt that he reached his results by the inductive method indicated above (§ 2). Lagrange (see below) supplied proofs for equations (3), (3·12), and (3·22), but we have not been able to find any previous demonstration of (9·1).

Lagrange discusses the problem in the latter part of his "Recherches sur les suites récurrentes..., ou sur l'intégration des équations linéaires aux différences finies et partielles; et sur l'usage de ces équations dans la théorie des hazards" (*Nouveaux Mémoires de l'Académie Royale*, Berlin, 1775*). In his solution to Problème V (pp. 253—256, and pp. 258—261), Lagrange obtains de Moivre's two values for ${}_{b+2i}P_b$ in the case of a infinite; in Problème VI he finds for the general case the probability that either A or B will be broken in a given number of games, and indicates, without actually arriving at, a solution "qui répond à la méthode du Prob. LXIII...de Moivre" (pp. 261—265). Lagrange's solutions undoubtedly have, as he claims, the advantage of being "plus analytiques" than de Moivre's, but whether they are also "plus directes" seems a more open question.

Following Lagrange, Laplace demonstrated our equation (3) by means of his generating functions (*Théorie Analytique des Probabilités*, 1847 edition, p. 256) then Ampère, saying "j'ai banni de ces démonstrations les méthodes d'induction, dont on fait, à ce qu'il semble, trop d'usage dans la théorie des probabilités," established equation (2·31), and thence equation (3), by purely algebraic considerations of the possible combinations of gains and losses (*Considérations sur la Théorie Mathématique du Jeu*, Lyon, 1802). The probabilities (equations (6), (11·1), and (11·2)) that B will ultimately be ruined were first given by Ampère in this same memoir, but it is worth noticing that they follow immediately from one of Laplace's results (*loc. cit.* p. 254, equation H). He finds

$$
{}_{b+2i}P_b = \frac{p^b(p^a-q^a)}{p^{a+b}-q^{a+b}} - \frac{2^{b+2i+2}\,p^b(pq)^{i+1}}{a+b}
$$

$$
\times \sum_{r=0}^{\left[\frac{a+b-2}{2}\right]} \left\{ \frac{\sin\dfrac{2(r+1)\pi}{a+b}\,\sin\dfrac{(r+1)b\pi}{a+b}\left(\cos\dfrac{(r+1)\pi}{a+b}\right)^{b+2i}}{p^2-2pq\cos\dfrac{2(r+1)\pi}{a+b}+q^2} \right\} \dagger.
$$

* Lagrange's paper was read in 1776 and published in 1777.

† Laplace gives the upper limit of r in the sum, when $(a+b)$ is odd, as $\frac{1}{2}(a+b-1)$, but this is really the upper limit of $(r+1)$.

For all non-vanishing terms in the sum,

$$4pq\cos^2\frac{(r+1)\pi}{a+b}<1\,;$$

accordingly, as $i\to\infty$ each term of the sum tends to zero, giving (11·1) and (11·2), and thence, when we make $a\to\infty$, (6).

What seems to me to be the most elegant discussion of the problem is contained in Robert Leslie Ellis's paper "On the Solution of Equations in Finite Differences" (*Cambridge Mathematical Journal*, No. XXII, Vol. IV, 1844; reprinted in *Mathematical and other Writings*, 1863, pp. 203—211). Ellis obtains equations (9·7) and (9·8), but restricts m in the first of these equations to the range $1<m<a+b-1$. (9·7) possesses the same symmetrical character as (9·3), from which we have derived it; Ellis remarks on this symmetry, adding "the result, however, which is the interpretation of this symmetry, may probably be obtained by general considerations."

From (9·8) and (11·1) we have

$${}_nP_b=\frac{1-\left(\dfrac{q}{p}\right)^a}{1-\left(\dfrac{q}{p}\right)^{a+b}}-\sum_{n+1}^{\infty}p_{a,\,x},$$

or

$$(31)\quad {}_nP_b=\frac{1-\left(\dfrac{q}{p}\right)^a}{1-\left(\dfrac{q}{p}\right)^{a+b}}-\frac{(4pq)^{\frac{n+1}{2}}}{a+b}\left(\frac{p}{q}\right)^{\frac{b}{2}}\sum_{r=1}^{a+b-1}\frac{\sin\dfrac{r\pi}{a+b}\sin\dfrac{rb\pi}{a+b}\left(\cos\dfrac{r\pi}{a+b}\right)^n}{1-2\sqrt{pq}\cos\dfrac{r\pi}{a+b}}.$$

The sum of the rth and the $(a+b-r)$th terms in the second member of this equation is

$$\frac{\sin\dfrac{r\pi}{a+b}\sin\dfrac{rb\pi}{a+b}\left(\cos\dfrac{r\pi}{a+b}\right)^n}{1-2\sqrt{pq}\cos\dfrac{r\pi}{a+b}}+\frac{\sin\left(\pi-\dfrac{r\pi}{a+b}\right)\sin\left(b\pi-\dfrac{rb\pi}{a+b}\right)\left\{\cos\left(\pi-\dfrac{r\pi}{a+b}\right)\right\}^n}{1-2\sqrt{pq}\cos\left(\pi-\dfrac{r\pi}{a+b}\right)}$$

$$=\frac{\sin\dfrac{r\pi}{a+b}\sin\dfrac{rb\pi}{a+b}\left(\cos\dfrac{r\pi}{a+b}\right)^n}{p^2-2pq\cos\dfrac{2r\pi}{a+b}+q^2}$$

$$\times\left\{1+2\sqrt{pq}\cos\frac{r\pi}{a+b}+(-1)^{n+b-1}\left(1-2\sqrt{pq}\cos\frac{r\pi}{a+b}\right)\right\}.$$

Thus (31) is equivalent to

$$(31\cdot1)\quad {}_nP_b=\frac{1-\left(\dfrac{q}{p}\right)^a}{1-\left(\dfrac{q}{p}\right)^{a+b}}-\frac{2\,(4pq)^{\frac{n'+1}{2}}}{a+b}\left(\frac{p}{q}\right)^{\frac{b}{2}}\sum_{r=1}^{\left[\frac{a+b-1}{2}\right]}\frac{\sin\dfrac{r\pi}{a+b}\sin\dfrac{rb\pi}{a+b}\left(\cos\dfrac{r\pi}{a+b}\right)^{n'}}{p^2-2pq\cos\dfrac{2r\pi}{a+b}+q^2},$$

where $n'=n+1$ or n, according as $(n+b)$ is even or odd.

If we write $(b+2i)$ for n and $(r+1)$ for r, (31·1) becomes Laplace's equation H.

Let ${}_nQ_a$ be the probability that A will lose his last counter on or before the nth game. From (31) we have, by interchanging a with b, and p with q,

$$ {}_nQ_a = \frac{1-\left(\frac{p}{q}\right)^b}{1-\left(\frac{p}{q}\right)^{a+b}} - \frac{(4pq)^{\frac{n+1}{2}}}{a+b}\left(\frac{q}{p}\right)^{\frac{a}{2}} \sum_{r=1}^{a+b-1} \frac{\sin\frac{r\pi}{a+b}\sin\frac{ra\pi}{a+b}\left(\cos\frac{r\pi}{a+b}\right)^n}{1-2\sqrt{pq}\cos\frac{r\pi}{a+b}}; $$

this equation, with (31), gives, for the probability that the set will end before the $(n+1)$th game,

$$ (32) \quad {}_nP_b + {}_nQ_a = 1 - \frac{(4pq)^{\frac{n+1}{2}}}{a+b}\left(\frac{p}{q}\right)^{\frac{b}{2}} $$

$$ \times \sum_{r=1}^{a+b-1} \frac{\sin\frac{r\pi}{a+b}\left(\cos\frac{r\pi}{a+b}\right)^n}{1-2\sqrt{pq}\cos\frac{r\pi}{a+b}} \left\{\sin\frac{rb\pi}{a+b} + \left(\frac{q}{p}\right)^{\frac{a+b}{2}}\sin\frac{ra\pi}{a+b}\right\}. $$

Since $\sin\frac{ra\pi}{a+b} = (-)^{r-1}\sin\frac{rb\pi}{a+b}$, (32) reduces, when the necessary changes in notation are made, to Lagrange's second solution to his Problème VI (*loc. cit.* pp. 265—269).

If we make $b=a$, we get, from (31·1),

$$ (32·1) \quad {}_nP_a + {}_nQ_a = 1 - \left\{\left(\frac{p}{q}\right)^{\frac{a}{2}} + \left(\frac{q}{p}\right)^{\frac{a}{2}}\right\} \frac{(4pq)^{\frac{n'+1}{2}}}{2a} \sum_{r=1}^{a-1} \frac{\sin\frac{r\pi}{a}\sin\frac{r\pi}{2}\left(\cos\frac{r\pi}{2a}\right)^{n'-1}}{p^2-2pq\cos\frac{r\pi}{a}+q^2}, $$

so that the probability that the set will not end in n games is

$$ (32·2) \quad 1-{}_nP_a-{}_nQ_a = \frac{p^a+q^a}{(pq)^{\frac{a}{2}-1}} \frac{(4pq)^{\frac{n'-1}{2}}}{a} \sum_{s=0}^{\left[\frac{a-2}{2}\right]} \frac{(-1)^s\sin\frac{2s+1}{a}\pi\left(\cos\frac{2s+1}{2a}\pi\right)^{n'-1}}{p^2-2pq\cos\frac{2s+1}{a}\pi+q^2}; $$

if we now make $p=q=\frac{1}{2}$, we get

$$ (32·3) \qquad 1-{}_nP_a-{}_nQ_a = \frac{1}{a}\sum_{s=0}^{\left[\frac{a-2}{2}\right]} \frac{(-1)^s\left(\cos\frac{2s+1}{2a}\pi\right)^{n'}}{\sin\frac{2s+1}{2a}\pi} $$

De Moivre gives the last of these results (*loc. cit.* Problem LXVIII), but instead of (32·2) he gives, if we do not mistake him,

$$ 1-{}_nP_a-{}_nQ_a = \frac{p^n+q^n}{(pq)^{\frac{a}{2}-1}} \frac{(4pq)^{\frac{n}{2}}}{a} \sum_{s=0}^{\left[\frac{a-2}{2}\right]} \frac{(-1)^s\sin\frac{2s+1}{a}\pi\left(\cos\frac{2s+1}{2a}\pi\right)^n}{p^2-2pq\cos\frac{2s+1}{a}\pi+q^2} $$

(*loc. cit.* Problem LXIX).

Except for those mentioned above, I believe the results of this paper to be new; in the case of the others, the methods by which I have established them seem to me to be simpler than those previously used, and to link together results that until now have appeared somewhat disconnected. The use of the Incomplete Beta-Function, in particular, renders almost intuitive the transition from *B*'s chance of losing all his possessions in a given number of games, to his chance of doing so ultimately, a transition previously effected only in the case $a = \infty$, and by Ampère's very elaborate algebra.

This paper originated from some remarks in Professor Pearson's lectures on Laplace; I wish to thank him for his suggestions and advice.

ON THE NATURE OF THE RELATIONSHIP BETWEEN TWO OF "STUDENT'S" VARIATES (z_1 AND z_2) WHEN SAMPLES ARE TAKEN FROM A BIVARIATE NORMAL POPULATION.

BY KARL PEARSON, F.R.S.

(1) IT is well known that "Student" first introduced the study of the variate

$$z = \frac{\text{Mean of Sample} - \text{Mean of Parent Population}}{\text{Standard Deviation of Sample}}$$

as a method of testing whether a small sample has been drawn from a parent population of which the mean has been ascertained*. The method has been developed by several writers since the appearance of "Student's" original memoir. The value of z will certainly determine whether it be exceedingly improbable that the sample was drawn from the supposed parent population, but in my opinion it does not justify us in asserting it probably has been, if we find the value of z has a high degree of probability. The numerator and denominator of z are—at any rate in the proof provided by "Student," i.e. selection from a normal parent population—independent variables. There is nothing to check their variations being in the same direction, and their ratio z may take a very probable value, although individually they might be highly improbable as selections from the given parent population. Further, if the mean of the parent population be so well known that it can be safely used in the numerator, then it would appear that the standard deviation can also be safely determined, and we have two variates instead of a single one to compare with those of the sample. Such a comparison has always seemed to me safer than arguing from a single ratio. But "Student" uses his formula to compare two samples from two populations. Let these variates be given by x and y with standard deviations and means $\sigma_1, \sigma_2, \bar{x}$ and $\bar{y}$ for the samples, and $\Sigma_1, \Sigma_2, m_1, m_2$ for the corresponding parent populations. x and y may be correlated with correlation coefficient in the parent population $=\rho$, and in the samples $=r$. Now if x and y both follow a normal distribution, so will the difference of their differences from their means. In other words the ratio will be

$$z = \frac{(\bar{x} - m_1) - (\bar{y} - m_2)}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2r\sigma_1\sigma_2}}.$$

If now we ask whether x and y are *independent* samples from the *same* population, then we may suppose $m_1 = m_2$ and $r = 0$ to get our result. If they are not independent samples, we may put $m_1 = m_2$ but are not justified in putting $r=0$. The two cases

$$z = \frac{\bar{x} - \bar{y}}{\sqrt{\sigma_1^2 + \sigma_2^2}} \quad \text{and} \quad z' = \frac{\bar{x} - \bar{y}}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2r\sigma_1\sigma_2}}$$

* *Biometrika*, Vol. VI. pp. 7—8.

may lead us to very different conclusions. Both z and z' get rid of a possibly unknown m of the parent population, but the second does not really get rid of the unknown r by the simple process of finding the standard deviation of $x-y$. The wide range of the coefficient of correlation r in *small* samples from a population of correlation ρ is well known, and appears only to be screened in taking the standard deviation of the difference. It seems necessary therefore to be sure that our two samples are wholly independent before using z. If they turn out not to be, i.e. if m_1 is very improbably equal to m_2, then we certainly are not justified when dealing with correlated samples in using z' where m_1 is put equal to m_2.

We may illustrate this in the following manner by asking whether the older generation is of less stature than the succeeding generation. We take a sample of fathers and a sample of sons, not sons of those fathers, and find z is sufficiently small for it to be probable that $m_1=m_2$. We now take the fathers and sons to be correlated individuals, and find owing to the term in r, that z' is so large that it is unreasonable to suppose that m_1 may be put equal to m_2 in the case of sons of the same fathers. It will I think be clear that z' cannot determine what will happen in the case of z. For example, if we test for the relative effectiveness of two drugs or two methods of factory production on the *same* groups of individuals and find a significant difference, we have not obtained evidence that there would be a significant difference had the same drugs or same methods of production been tested on different groups of individuals*.

Notwithstanding the need for caution in the use of z, and the undesirability of exaggerating the efficiency of z tests, it seemed to me worth while to inquire into the relationship of two variates measured by "Student's" ratios.

(2) *Correlation Surface of z_1 and z_2.*

Using the same notation as in the preceding section we take

$$z_1=(\bar{x}-m_1)/\sigma_1,\quad z_2=(\bar{y}-m_2)/\sigma_2,$$

and we suppose the normal parent population defined by m_1, m_2, Σ_1, Σ_2 and correlation ρ. For brevity we may write $s_1=\Sigma_1\sqrt{1-\rho^2}/\sqrt{n}$, $s_2=\Sigma_2\sqrt{1-\rho^2}/\sqrt{n}$, where n is the size of the sample; r will represent the correlation in a particular sample. Z will denote the ordinate of any frequency surface or curve and Z_0 a constant independent of the variates of the particular sample. The constants of the square brackets following a frequency curve denote the appropriate element of volume. The correlation surface for the five variables $\bar{x}$, $\bar{y}$, σ_1, σ_2 and r is

$$Z=Z_0 e^{-\frac{1}{2}\frac{n}{1-\rho^2}\left\{\frac{(\bar{x}-m_1)^2}{\Sigma_1^2}-\frac{2\rho(\bar{x}-m_1)(\bar{y}-m_2)}{\Sigma_1\Sigma_2}+\frac{(\bar{y}-m_2)^2}{\Sigma_2^2}\right\}}$$
$$\times e^{-\frac{1}{2}\frac{n}{1-\rho^2}\left\{\frac{\sigma_1^2}{\Sigma_1^2}-\frac{2r\rho\sigma_1\sigma_2}{\Sigma_1\Sigma_2}+\frac{\sigma_2^2}{\Sigma_2^2}\right\}}(\sigma_1\sigma_2)^{n-2}(1-r^2)^{\frac{n-4}{2}}\,[d\bar{x}\,d\bar{y}\,d\sigma_1\,d\sigma_2\,dr]$$
$$\ldots\ldots\text{(i)}.$$

* For further illustration see *Biometrika*, Vol. xxii. pp. 268—270.

We will first integrate this expression with regard to r between the limits -1 and $+1$. The result is given as Equation (v) of *Biometrika*, Vol. XVII. p. 177, and substituting for $\bar{x}-m_1$ and $\bar{y}-m_2$, we have*

$$Z = Z_0' e^{-\frac{1}{2}\left(\frac{(1+z_1^2)\sigma_1^2}{s_1^2} - \frac{2\rho z_1 z_2 \sigma_1\sigma_2}{s_1 s_2} + \frac{(1+z_2^2)\sigma_2^2}{s_2^2}\right)} (\sigma_1\sigma_2)^{n-1}$$
$$\times\left(1 + \frac{\rho^2}{1!}\frac{\sigma_1^2\sigma_2^2}{(2n-2)s_1^2 s_2^2} + \frac{\rho^4}{2!}\frac{\sigma_1^4\sigma_2^4}{(2n-2)(2n+2)s_1^4 s_2^4} + \dots\right.$$
$$\left.+ \frac{\rho^{2p}}{p!}\frac{\sigma_1^{2p}\sigma_2^{2p}}{(2n-2)(2n+2)\dots(2n+4p-6)s_1^{2p}s_2^{2p}} + \dots\right)[dz_1 dz_2 d\sigma_1 d\sigma_2] \quad \dots\text{(ii)}.$$

To obtain the surface of frequency of z_1, z_2 we need to integrate this for σ_1 and σ_2, from 0 to ∞ in both cases. It is necessary first to expand the exponential term $e^{\frac{\rho z_1 z_2}{s_1 s_2}\sigma_1\sigma_2}$, and we have

$$Z = Z_0' e^{-\frac{1}{2}\frac{1+z_1^2}{s_1^2}\sigma_1^2} e^{-\frac{1}{2}\frac{1+z_2^2}{s_2^2}\sigma_2^2} \times \underset{p'=0}{\overset{\infty}{S}} \frac{1}{p'!}\left(\rho z_1 z_2 \frac{\sigma_1\sigma_2}{s_1 s_2}\right)^{p'} \times \left(\frac{\sigma_1}{s_1}\frac{\sigma_2}{s_2}\right)^{n-1}(s_1 s_2)^n$$
$$\times \underset{p=0}{\overset{p=\infty}{S}} \frac{\rho^{2p}}{p!}\frac{\sigma_1^{2p}\sigma_2^{2p}}{s_1^{2p}s_2^{2p}}\frac{1}{(2n-2)(2n+2)\dots(2n+4p-6)}\left[d\left(\frac{\sigma_1}{s_1}\right)d\left(\frac{\sigma_2}{s_2}\right)\right]$$

Take $v_1 = \frac{1}{2}\frac{1+z_1^2}{s_1^2}\sigma_1^2$, $v_2 = \frac{1}{2}\frac{1+z_2^2}{s_2^2}\sigma_2^2$, then $dv_1 = \sigma_1 d\sigma_1 \frac{1+z_1^2}{s_1^2} = \lambda_1 \frac{\sigma_1}{s_1}\frac{d\sigma_1}{s_1}$, say, and $dv_2 = \sigma_2 d\sigma_2 \frac{1+z_2^2}{s_2^2} = \lambda_2\frac{\sigma_2}{s_2}\frac{d\sigma_2}{s_2}$, say,

$$Z = Z_0'(s_1 s_2)^n e^{-v_1-v_2}\underset{p'=0}{\overset{\infty}{S}}\underset{p=0}{\overset{\infty}{S}} 2^{p'+2p+n-2}(z_1 z_2)^{p'}\frac{\rho^{p'+2p}}{p'!\,p!}$$
$$\times\frac{1}{(2n-2)(2n+2)\dots(2n+4p-6)}\times\frac{v_1^{\frac{1}{2}(p'+2p+n-2)}v_2^{\frac{1}{2}(p'+2p+n-2)}}{\lambda_1^{\frac{1}{2}(p'+2p+n)}\lambda_2^{\frac{1}{2}(p'+2p+n)}}[dv_1 dv_2].$$

Integrate for v_1 and v_2 from 0 to ∞, and we have for the surface of frequency of z_1 and z_2

$$Z = Z_0'(s_1 s_2)^n \underset{p'=0}{\overset{\infty}{S}}\underset{p=0}{\overset{\infty}{S}} 2^{p'+2p+n-2}\frac{\rho^{p'+2p}(z_1 z_2)^{p'}}{p'!\,p!}$$
$$\times\frac{1}{(2n-2)(2n+2)\dots(2n+4p-6)}\times\frac{\Gamma^2(\frac{1}{2}(p'+2p+n))}{(\lambda_1\lambda_2)^{\frac{1}{2}(p'+2p+n)}}.$$

We may write this in a somewhat different form, namely:

$$Z = Z_0''\frac{1}{(\lambda_1\lambda_2)^{\frac{1}{2}n}}\underset{p'=0}{\overset{\infty}{S}}\frac{2^{p'}\rho^{p'}}{p'!}\left(\frac{z_1 z_2}{\sqrt{\lambda_1\lambda_2}}\right)^{p'}\Gamma^2(\tfrac{1}{2}(p'+n))$$
$$\times\left(1 + \frac{2^2\rho^2(\frac{1}{2}(p'+n))^2}{1!\,\lambda_1\lambda_2(2n-2)} + \frac{2^4\rho^4(\frac{1}{2}(p'+n))^2(\frac{1}{2}(p'+n)+1)^2}{2!(\lambda_1\lambda_2)^2(2n-2)(2n+2)} + \dots\right)$$
$$= Z_0''\frac{1}{(\lambda_1\lambda_2)^{\frac{1}{2}n}}\underset{p'=0}{\overset{\infty}{S}}\frac{2^{p'}\rho^{p'}}{p'!}\left(\frac{z_1 z_2}{\sqrt{\lambda_1\lambda_2}}\right)^{p'}\Gamma^2(\tfrac{1}{2}(p'+n))$$
$$\times F\left(\tfrac{1}{2}(p'+n), \tfrac{1}{2}(p'+n), \tfrac{1}{2}(n-1), \frac{\rho^2}{\lambda_1\lambda_2}\right).$$

* $Z_0' = Z_0 B(\frac{1}{2}, \frac{1}{2}(n-2))$. We shall however pay no attention to the changes in the constant Z_0.

But by Euler's transformation of the hypergeometrical function

$$F\left(\tfrac{1}{2}(p'+n), \tfrac{1}{2}(p'+n), \tfrac{1}{2}(n-1), \frac{\rho^2}{\lambda_1\lambda_2}\right) = \left(1-\frac{\rho^2}{\lambda_1\lambda_2}\right)^{-(\frac{1}{2}(n+1)+p')}$$
$$\times F\left(-\tfrac{1}{2}(p'+1), -\tfrac{1}{2}(p'+1), \tfrac{1}{2}(n-1), \frac{\rho^2}{\lambda_1\lambda_2}\right),$$

and accordingly

$$Z = Z_0'' \frac{\sqrt{\lambda_1\lambda_2}}{(\lambda_1\lambda_2-\rho^2)^{\frac{1}{2}(n+1)}} \mathop{S}_{p'=0}^{\infty} \frac{\Gamma^2(\frac{1}{2}(p'+n))}{p'!} \left(\frac{2\rho z_1 z_2 \sqrt{\lambda_1\lambda_2}}{\lambda_1\lambda_2-\rho^2}\right)^{p'}$$
$$\times F\left(-\tfrac{1}{2}(p'+1), -\tfrac{1}{2}(p'+1), \tfrac{1}{2}(n-1), \frac{\rho^2}{\lambda_1\lambda_2}\right),$$

or, substituting for the λ's,

$$Z = Z_0'' \sqrt{\frac{(1+z_1^2)(1+z_2^2)}{((1+z_1^2)(1+z_2^2)-\rho^2)^{n+1}}} \mathop{S}_{p'=0}^{\infty} \frac{\Gamma^2(\frac{1}{2}(p'+n))}{p'!} \left(\frac{2\rho z_1 z_2\sqrt{(1+z_1^2)(1+z_2^2)}}{(1+z_1^2)(1+z_2^2)-\rho^2}\right)^{p'}$$
$$\times F\left(-\tfrac{1}{2}(p'+1), -\tfrac{1}{2}(p'+1), \tfrac{1}{2}(n-1), \frac{\rho^2}{(1+z_1^2)(1+z_2^2)}\right) \quad \ldots(iii).$$

When $\rho = 0$, this reduces, as it should do, to

$$Z = Z_0'' \frac{1}{(1+z_1^2)^{\frac{1}{2}n}} \times \frac{1}{(1+z_2^2)^{\frac{1}{2}n}},$$

for the case of z_1 and z_2 independent.

I have not succeeded in reducing (iii) in the general case to any more concise form, and it appears too complicated for any numerical reduction for particular values of n and ρ. I leave it in the hopes that a stronger algebraist may possibly achieve something with it. We need not, however, despair of learning something about the nature of the z_1, z_2 frequency surface, for we can calculate its principal characters. I shall now proceed to determine (*a*) the coefficient of correlation of z_1 and z_2, (*b*) the regression curve of z_1 on z_2, and (*c*) the scedasticity of the arrays of z_1 for a given z_2.

(3) *Determination of the Correlation between z_1 and z_2.*

We have seen that the frequency surface of r, σ_1, σ_2, $\bar{x}$, $\bar{y}$ is given by

$$Z = Z_0 e^{-\frac{1}{2}\frac{n}{1-\rho^2}\left\{\frac{(\bar{x}-m_1)^2}{\Sigma_1^2} - \frac{2\rho(\bar{x}-m_1)(\bar{y}-m_2)}{\Sigma_1\Sigma_2} + \frac{(\bar{y}-m_2)^2}{\Sigma_2^2}\right\}}$$
$$\times e^{-\frac{1}{2}\frac{n}{1-\rho^2}\left\{\frac{\sigma_1^2}{\Sigma_1^2} - 2r\rho\frac{\sigma_1\sigma_2}{\Sigma_1\Sigma_2} + \frac{\sigma_2^2}{\Sigma_2^2}\right\}} (\sigma_1\sigma_2)^{n-2}(1-r^2)^{\frac{n-4}{2}},$$

with the element dV of "volume" $= d\bar{x}\,d\bar{y}\,d\sigma_1\,d\sigma_2\,dr$. Now to obtain the correlation of z_1 and z_2 we need, if N = number of samples,

$$P_{z_1z_2} = \frac{1}{N}\int Z\left(\frac{\bar{x}-m_1}{\sigma_1}\right)\left(\frac{\bar{y}-m_2}{\sigma_2}\right)dV,$$

or the product moment of z_1, z_2, the integral being taken over the whole of space. But

$$P_{s_1 s_2} = \frac{Z_0}{N}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\frac{n}{1-\rho^2}\left\{\frac{(\bar{x}-m_1)^2}{\Sigma_1^2} - \frac{2\rho(\bar{x}-m_1)(\bar{y}-m_2)}{\Sigma_1\Sigma_2} + \frac{(\bar{y}-m_2)^2}{\Sigma_2^2}\right\}} \times d\bar{x}\,d\bar{y}\,(\bar{x}-m_1)(\bar{y}-m_2)$$
$$\times \int_0^\infty\int_0^\infty\int_{-1}^{+1} e^{-\frac{1}{2}\frac{n}{1-\rho^2}\left\{\frac{\sigma_1^2}{\Sigma_1^2} - 2r\rho\frac{\sigma_1\sigma_2}{\Sigma_1\Sigma_2} + \frac{\sigma_2^2}{\Sigma_2^2}\right\}} (\sigma_1\sigma_2)^{n-3}(1-r^2)^{\frac{n-4}{2}}\,d\sigma_1 d\sigma_2 dr.$$

Now the integral with regard to $d\bar{x}\,d\bar{y}$ can be taken at once. It is

$$2\pi\sqrt{1-\rho^2}\,\frac{\Sigma_1\Sigma_2}{n}\times\rho\,\frac{\Sigma_1\Sigma_2}{n}.$$

Hence

$$P_{s_1 s_2} = \frac{Z_0}{N}2\pi\sqrt{1-\rho^2}\,\rho\,\frac{\Sigma_1^2\Sigma_2^2}{n^2}$$
$$\times \int_0^\infty\int_0^\infty\int_{-1}^{+1} e^{-\frac{1}{2}\frac{n}{1-\rho^2}\left\{\frac{\sigma_1^2}{\Sigma_1^2} - 2r\rho\frac{\sigma_1\sigma_2}{\Sigma_1\Sigma_2} + \frac{\sigma_2^2}{\Sigma_2^2}\right\}} (\sigma_1\sigma_2)^{n-3}(1-r^2)^{\frac{n-4}{2}}\,d\sigma_1 d\sigma_2 dr.$$

But if $\gamma = \frac{n\rho}{1-\rho^2}\frac{\sigma_1\sigma_2}{\Sigma_1\Sigma_2}$, then the $\int_{-1}^{+1} e^{\gamma r}(1-r^2)^{\frac{n-4}{2}}dr$ is known to be*

$$B\left(\tfrac{1}{2}, \frac{n-2}{2}\right)\left(1 + \frac{\gamma^2}{1!}\frac{1}{2n-2} + \frac{\gamma^4}{2!}\frac{1}{(2n-2)(2n+2)} + \dots + \frac{\gamma^{2p}}{p!}\frac{1}{(2n-2)(2n+2)\dots(2n+4p-6)} + \dots\right).$$

Hence, writing as before, $s_1 = \sqrt{1-\rho^2}\,\Sigma_1/\sqrt{n}$, $s_2 = \sqrt{1-\rho^2}\,\Sigma_2/\sqrt{n}$, the second integral, I_2, in the value of $P_{s_1 s_2}$ reduces to

$$B\left(\tfrac{1}{2}, \frac{n-2}{2}\right)\int_0^\infty\int_0^\infty e^{-\frac{1}{2}\left(\frac{\sigma_1^2}{s_1^2}+\frac{\sigma_2^2}{s_2^2}\right)}\left(\frac{\sigma_1\sigma_2}{s_1 s_2}\right)^{n-3}(s_1 s_2)^{n-2}$$
$$\times\left(1 + \frac{\rho^2}{1!}\left(\frac{\sigma_1\sigma_2}{s_1 s_2}\right)^2\frac{1}{2n-2} + \frac{\rho^4}{2!}\left(\frac{\sigma_1\sigma_2}{s_1 s_2}\right)^4\frac{1}{(2n-2)(2n+2)} + \dots\right.$$
$$\left. + \frac{\rho^{2p}}{p!}\left(\frac{\sigma_1\sigma_2}{s_1 s_2}\right)^{2p}\frac{1}{(2n-2)(2n+2)\dots(2n+4p-6)} + \dots\right)\frac{d\sigma_1}{s_1}\frac{d\sigma_2}{s_2}.$$

Now put
$$\tfrac{1}{2}\frac{\sigma_1^2}{s_1^2} = v_1, \quad \tfrac{1}{2}\frac{\sigma_2^2}{s_2^2} = v_2,$$

and we have

$$I_2 = B\left(\tfrac{1}{2}, \frac{n-2}{2}\right)(s_1 s_2)^{n-2}\int_0^\infty\int_0^\infty 2^{n-4}e^{-v_1-v_2}(v_1 v_2)^{\frac{n-4}{2}}$$
$$\times\left(1 + \frac{(2\rho)^2}{1!}\frac{v_1 v_2}{2n-2} + \frac{(2\rho)^4}{2!}\frac{(v_1 v_2)^2}{(2n-2)(2n+2)} + \dots\right.$$
$$\left. + \frac{(2\rho)^{2p}}{p!}(v_1 v_2)^p\frac{1}{(2n-2)(2n+2)\dots(2n+4p-6)} + \dots\right)dv_1 dv_2$$

* *Biometrika*, Vol. XVII. p. 177.

$$= 2^{n-4} B\left(\tfrac{1}{2}, \frac{n-2}{2}\right)(s_1 s_2)^{n-2}\left(\Gamma^2\left(\frac{n-2}{2}\right) + \frac{(2\rho)^2}{1\,!}\Gamma^2\left(\frac{n}{2}\right)\frac{1}{2n-2}\right.$$
$$+ \frac{(2\rho)^4}{2\,!}\Gamma^2\left(\frac{n+2}{2}\right)\frac{1}{(2n-2)(2n+2)} + \dots$$
$$\left. + \frac{(2\rho)^{2p}}{p\,!}\,\frac{\Gamma^2\left(\frac{n-2+2p}{2}\right)}{(2n-2)(2n+2)\dots(2n+4p-6)} + \dots\right)$$
$$= 2^{n-4} B\left(\tfrac{1}{2}, \frac{n-2}{2}\right)(s_1 s_2)^{n-2}\Gamma^2\left(\frac{n-2}{2}\right)$$
$$\times\left(1 + \frac{(2\rho)^2}{1\,!}\left(\frac{n-2}{2}\right)^2\frac{1}{2n-2} + \frac{(2\rho)^4}{2\,!}\left(\frac{n-2}{2}\right)^2\left(\frac{n}{2}\right)^2\frac{1}{(2n-2)(2n+2)}\right.$$
$$\left. + \left(\frac{n-2}{2}\right)^2\left(\frac{n}{2}\right)^2\left(\frac{n+1}{2}\right)^2\frac{(2\rho)^6}{3\,!}\,\frac{1}{(2n-2)(2n+2)(2n+6)} + \dots\right)$$
$$= 2^{n-4} B\left(\tfrac{1}{2}, \frac{n-2}{2}\right)(s_1 s_2)^{n-2}\Gamma^2\left(\frac{n-2}{2}\right) F\left(\frac{n-2}{2}, \frac{n-2}{2}, \frac{n-1}{2}, \rho^2\right).$$

But by Euler's Theorem

$$F(\alpha, \beta, \gamma, x) = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma, \rho^2).$$

Accordingly

$$I_2 = 2^{n-4} B\left(\tfrac{1}{2}, \frac{n-2}{2}\right)(s_1 s_2)^{n-2}\Gamma^2\left(\frac{n-2}{2}\right)(1-\rho^2)^{-\frac{n-3}{2}} F\left(\tfrac{1}{2}, \tfrac{1}{2}, \frac{n-1}{2}, \rho^2\right).$$

Thus we have

$$P_{s_1 s_2} = \frac{Z_0}{N}\,\frac{2\pi}{(1-\rho^2)^{\frac{1}{2}}}\,\rho\,(s_1 s_2)^n\, 2^{n-4} B\left(\tfrac{1}{2}, \frac{n-2}{2}\right)\Gamma^2\left(\frac{n-2}{2}\right)$$
$$\times (1-\rho^2)^{-\frac{n-3}{2}} F\left(\tfrac{1}{2}, \tfrac{1}{2}, \frac{n-1}{2}, \rho^2\right) \quad \dots\text{(iv)}.$$

We must now find Z_0 from the relation

$$N = \int z\,dV$$
$$= Z_0 2\pi\sqrt{1-\rho^2}\,\frac{\Sigma_1\Sigma_2}{n}\times\int_0^\infty\int_0^\infty\int_{-1}^{+1}(\sigma_1\sigma_2)^{n-2}(1-r^2)^{\frac{n-4}{2}}$$
$$\times e^{-\frac{1}{2}\frac{n}{1-\rho^2}\left(\frac{\sigma_1^2}{\Sigma_1^2} - \frac{2r\rho\sigma_1\sigma_2}{\Sigma_1\Sigma_2} + \frac{\sigma_2^2}{\Sigma_2^2}\right)}\,d\sigma_1\,d\sigma_2\,dr.$$

The integration with regard to r gives the same result as before, and the sole difference is the term $(\sigma_1\sigma_2)^{n-2}$ instead of $(\sigma_1\sigma_2)^{n-3}$. Accordingly

$$N = Z_0 2\pi\sqrt{1-\rho^2}\,\frac{\Sigma_1\Sigma_2}{n}(s_1 s_2)^{n-1} B\left(\tfrac{1}{2}, \frac{n-2}{2}\right)$$
$$\times\int_0^\infty\int_0^\infty e^{-\frac{1}{2}\left(\frac{\sigma_1^2}{s_1^2}+\frac{\sigma_2^2}{s_2^2}\right)}\left(\frac{\sigma_1\sigma_2}{s_1 s_2}\right)^{n-3}\left(1 + \frac{\rho^2}{1\,!}\left(\frac{\sigma_1\sigma_2}{s_1 s_2}\right)^2\frac{1}{2n-2}\right.$$
$$\left. + \frac{\rho^4}{2\,!}\left(\frac{\sigma_1\sigma_2}{s_1 s_2}\right)^4\frac{1}{(2n-2)(2n+2)} + \dots\right)\frac{\sigma_1}{s_1}\,d\left(\frac{\sigma_1}{s_1}\right)\frac{\sigma_2}{s_2}\,d\left(\frac{\sigma_2}{s_2}\right)$$

Or, making the same transformation as before,

$$N = Z_0 2\pi \sqrt{1-\rho^2}\, \frac{\Sigma_1 \Sigma_2}{n} (s_1 s_2)^{n-1} B\left(\tfrac{1}{2}, \frac{n-2}{2}\right) 2^{n-3}$$

$$\times \int_0^\infty \int_0^\infty e^{-v_1-v_2} (v_1 v_2)^{\frac{n-3}{2}} \left(1 + \frac{(2\rho)^2}{1!} \frac{v_1 v_2}{2n-2} + \frac{(2\rho)^4}{2!} \frac{(v_1 v_2)^2}{(2n-2)(2n+2)} + \dots \right.$$

$$\left. + \frac{(2\rho)^{2p}}{p!} (v_1 v_2)^p \frac{1}{(2n-2)(2n+2)\dots(2n+4p-6)} + \dots \right) dv_1 dv_2$$

$$= Z_0 \frac{2\pi}{\sqrt{1-\rho^2}} (s_1 s_2)^n B\left(\tfrac{1}{2}, \frac{n-2}{2}\right) 2^{n-3}$$

$$\times \left(\Gamma^2\left(\frac{n-1}{2}\right) + \frac{(2\rho)^2}{1!} \frac{\Gamma^2\left(\frac{n+1}{2}\right)}{2n-2} + \frac{(2\rho)^4}{2!} \frac{\Gamma^2\left(\frac{n+3}{2}\right)}{(2n-2)(2n+2)} \right.$$

$$\left. + \frac{(2\rho)^6}{3!} \frac{\Gamma^2\left(\frac{n+5}{2}\right)}{(2n-2)(2n+2)(2n+6)} + \dots \right)$$

$$= Z_0 \frac{2\pi}{\sqrt{1-\rho^2}} (s_1 s_2)^n B\left(\tfrac{1}{2}, \frac{n-2}{2}\right) 2^{n-3} \Gamma^2\left(\frac{n-1}{2}\right)$$

$$\times \left(1 + \frac{n-1}{2} \frac{\rho^2}{1!} + \frac{n-1}{2} \frac{n+1}{2} \frac{\rho^4}{2!} + \frac{n-1}{2} \frac{n+1}{2} \frac{n+3}{2} \frac{\rho^6}{3!} + \dots \right).$$

Thus
$$N = \frac{Z_0 2\pi (s_1 s_2)^n B\left(\tfrac{1}{2}, \frac{n-2}{2}\right) 2^{n-3} \Gamma^2\left(\frac{n-1}{2}\right)}{(1-\rho^2)^{n/2}}$$

Or
$$Z_0 = \frac{N (1-\rho^2)^{n/2}}{2\pi (s_1 s_2)^n B\left(\tfrac{1}{2}, \frac{n-2}{2}\right) 2^{n-3} \Gamma^2\left(\frac{n-1}{2}\right)} \quad \dots\dots\dots\dots\dots\dots\text{(v)}.$$

Returning to Equation (iv) and substituting we have

$$P_{z_1 z_2} = \tfrac{1}{2} \rho F\left(\tfrac{1}{2}, \tfrac{1}{2}, \frac{n-1}{2}, \rho^2\right) \Gamma^2\left(\frac{n-2}{2}\right) \Big/ \Gamma^2\left(\frac{n-1}{2}\right).$$

But we need to divide by $\sigma_{z_1} \times \sigma_{z_2}$ to find $r_{z_1 z_2}$. Now*

$$\sigma_{z_1} = \sigma_{z_2} = \frac{1}{\sqrt{n-3}}.$$

Accordingly we have

$$r_{z_1 z_2} = \frac{n-3}{2} \frac{\Gamma^2\left(\frac{n-2}{2}\right)}{\Gamma^2\left(\frac{n-1}{2}\right)} \rho F\left(\tfrac{1}{2}, \tfrac{1}{2}, \frac{n-1}{2}, \rho^2\right)$$

$$= \frac{\Gamma^2\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right) \Gamma\left(\frac{n-3}{2}\right)} \rho F\left(\tfrac{1}{2}, \tfrac{1}{2}, \frac{n-1}{2}, \rho^2\right) \dots\dots\dots\dots\dots\text{(v)}.$$

* See *Biometrika*, Vol. VI. p. 12.

TABLE I.

Correlation of z_1 and z_2 for various Values of ρ and n.

Samples of n.

$n=$	4	5	6	7	8	9	10	11	12
$\rho=0·0$	·0000	·0000	·0000	·0000	·0000	·0000	·0000	·0000	·0000
0·1	·0638	·0786	·0850	·0884	·0906	·0921	·0932	·0940	·0946
0·2	·1282	·1579	·1706	·1773	·1816	·1845	·1867	·1883	·1896
0·3	·1940	·2384	·2570	·2671	·2734	·2777	·2808	·2832	·2850
0·4	·2620	·3209	·3453	·3584	·3665	·3720	·3759	·3789	·3813
0·5	·3333	·4063	·4360	·4517	·4614	·4678	·4725	·4760	·4787
0·6	·4097	·4960	·5302	·5480	·5587	·5658	·5709	·5747	·5776
0·7	·4936	·5919	·6293	·6482	·6594	·6667	·6718	·6756	·6785
0·8	·5933	·6970	·7357	·7541	·7646	·7713	·7760	·7793	·7819
0·9	·7129	·8204	·8540	·8687	·8766	·8814	·8847	·8860	·8887
1·0	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000

$n=$	13	14	15	16	17	18	19	20	21
$\rho=0·0$	·0000	·0000	·0000	·0000	·0000	·0000	·0000	·0000	·0000
0·1	·0952	·0956	·0960	·0963	·0965	·0968	·0970	·0971	·0973
0·2	·1906	·1914	·1921	·1927	·1932	·1937	·1941	·1944	·1947
0·3	·2865	·2877	·2887	·2896	·2903	·2909	·2915	·2920	·2924
0·4	·3831	·3847	·3859	·3870	·3879	·3888	·3895	·3901	·3906
0·5	·4808	·4826	·4841	·4853	·4864	·4873	·4881	·4888	·4894
0·6	·5799	·5818	·5834	·5847	·5858	·5868	·5876	·5884	·5890
0·7	·6807	·6826	·6841	·6854	·6865	·6875	·6883	·6890	·6896
0·8	·7839	·7855	·7868	·7879	·7888	·7896	·7903	·7909	·7915
0·9	·8900	·8910	·8919	·8926	·8932	·8937	·8941	·8945	·8948
1·0	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000

$n=$	22	23	24	25	26	27	28	29	30
$\rho=0·0$	·0000	·0000	·0000	·0000	·0000	·0000	·0000	·0000	·0000
0·1	·0974	·0976	·0977	·0978	·0979	·0980	·0980	·0981	·0982
0·2	·1950	·1952	·1955	·1957	·1959	·1960	·1962	·1963	·1965
0·3	·2928	·2932	·2935	·2938	·2941	·2943	·2946	·2948	·2950
0·4	·3911	·3916	·3920	·3923	·3927	·3930	·3933	·3935	·3938
0·5	·4900	·4905	·4910	·4914	·4918	·4921	·4924	·4927	·4930
0·6	·5896	·5902	·5906	·5911	·5915	·5918	·5922	·5925	·5927
0·7	·6902	·6907	·6912	·6916	·6920	·6923	·6926	·6929	·6932
0·8	·7919	·7924	·7928	·7931	·7934	·7937	·7940	·7942	·7944
0·9	·8951	·8954	·8956	·8958	·8960	·8962	·8964	·8965	·8966
1·0	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000

$n=$	50	52	100	102	400	402	1000	∞	
$\rho=0·0$	·0000	·0000	·0000	·0000	·0000	·0000	·0000	·0000	
0·1	·0990	·0990	·0995	·0995	·0999	·0999	·1000	·1000	
0·2	·1980	·1980	·1990	·1990	·1998	·1998	·1999	·2000	
0·3	·2971	·2972	·2986	·2986	·2997	·2997	·2999	·3000	
0·4	·3964	·3966	·3983	·3983	·3996	·3996	·3998	·4000	
0·5	·4960	·4962	·4981	·4981	·4995	·4995	·4998	·5000	
0·6	·5959	·5960	·5980	·5980	·5995	·5995	·5998	·6000	
0·7	·6961	·6963	·6981	·6982	·6995	·6996	·6998	·7000	
0·8	·7969	·7970	·7985	·7985	·7996	·7996	·7999	·8000	
0·9	·8981	·8982	·8991	·8991	·8998	·8998	·8999	·9000	
1·0	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	

But by *Biometrika*, Vol. XI. p. 336, the *mean* value of a correlation coefficient in samples of n is given by

$$\bar{r}_n = \frac{\Gamma^2\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}\rho F\left(\tfrac{1}{2}, \tfrac{1}{2}, \frac{n+1}{2}, \rho^2\right) \quad \text{.........(vi).}$$

Change n to $(n-2)$ and we have the result that: *The mean value of the correlation coefficient in samples of* $(n-2)$ *from a parent population of correlation* ρ *is equal to the correlation of* z_1 *and* z_2 *in samples of size* n *from the same parent population.*

This is a somewhat remarkable theorem; it enables us at once to provide values of $r_{z_1z_2}$ from those already calculated for $\bar{r}_n$. These are given in Table I.

(4) *Determination of the Regression Equation.*

A knowledge of the correlation coefficient of z_1 with z_2 is, however, of small service, if, the regression being non-linear, we have not some measure of its approach to linearity. We will therefore find the true regression of z_1 on z_2.

Let $\bar{z}_1$ be the mean value of the array of z_1 for a given value z_2 of z_2. We have, if n_{z_2} denote the frequency of the array of z_1's for the given z_2,

$$n_{z_2} \times \bar{z}_1 = \int_{-\infty}^{+\infty}\int_0^{\infty}\int_0^{\infty} Z z_1 d\sigma_1 d\sigma_2 dz_1,$$

where Z is the ordinate of the frequency surface. Hence

$$n_{z_2} \times \bar{z}_1 = Z_0\int_0^{\infty} d\sigma_1\int_0^{\infty} d\sigma_2\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(z_1\frac{\sigma_1}{s_1} - \rho z_2\frac{\sigma_2}{s_2}\right)^2 - \frac{1}{2}\frac{\sigma_2^2}{s_2^2}(1+z_2^2(1-\rho^2))}$$
$$\times e^{-\frac{1}{2}\frac{\sigma_1^2}{s_1^2}}\frac{s_1}{\sigma_1}\left(\left\{z_1\frac{\sigma_1}{s_1} - \rho z_2\frac{\sigma_2}{s_2}\right\} + \rho z_2\frac{\sigma_2}{s_2}\right)(\sigma_1\sigma_2)^{n-1} Q\, dz,$$

where Q is the series

$$1 + \frac{\rho^2\sigma_1^2\sigma_2^2}{1!(2n-2)s_1^2s_2^2} + \frac{\rho^4\sigma_1^4\sigma_2^4}{2!(2n-2)(2n+2)s_1^4s_2^4} + \dots$$
$$+ \frac{\rho^{2p}\sigma_1^{2p}\sigma_2^{2p}}{p!(2n-2)(2n+2)\dots(2n+4p-6)s_1^{2p}s_2^{2p}} + \dots.$$

Writing $\kappa^2 = 1 + z_2^2(1-\rho^2)$ and $z_1\frac{\sigma_1}{s_1} - \rho z_2\frac{\sigma_2}{s_2} = u$, we have

$$n_{z_2} \times \bar{z}_1 = Z_0\int_0^{\infty} d\sigma_1\int_0^{\infty} d\sigma_2\int_{-\infty}^{+\infty} e^{-\frac{1}{2}u^2}\frac{s_1^2}{\sigma_1^2}\left(u + \rho\frac{z_2\sigma_2}{s_2}\right) du\,(\sigma_1\sigma_2)^{n-1} Q e^{-\frac{1}{2}\left(\frac{\sigma_1^2}{s_1^2} + \frac{\sigma_2^2\kappa^2}{s_2^2}\right)}.$$

But $\int_{-\infty}^{+\infty} e^{-\frac{u^2}{2}} u\,du = 0$ and $\int_{-\infty}^{+\infty} e^{-\frac{1}{2}u^2} du = \sqrt{2\pi}$. Thus

$$n_{z_2} \times \bar{z}_1 = Z_0\int_0^{\infty} d\sigma_1\int_0^{\infty} d\sigma_2\sqrt{2\pi}\,\rho z_2\frac{s_1^2}{\sigma_1^2}\frac{\sigma_2}{s_2}(\sigma_1\sigma_2)^{n-1} Q e^{-\frac{1}{2}\frac{\sigma_1^2}{s_1^2}} e^{-\frac{1}{2}\frac{\sigma_2^2\kappa^2}{s_2^2}}$$

Now put $\frac{1}{2}\frac{\sigma_1^2}{s_1^2}=v_1$ and $\frac{1}{2}\frac{\sigma_2^2\kappa^2}{s_2^2}=v_2$ and we find

$$n_{z_2}\times\bar{z}_1=Z_0\sqrt{2\pi}\,\frac{s_1^n s_2^n}{\kappa^{n+1}}\,2^{n-\frac{5}{2}}\,\rho s_2\int_0^\infty\int_0^\infty e^{-v_1}e^{-v_2}$$
$$\times\mathop{S}_{p=0}^{p=\infty}\left(\frac{2\rho}{\kappa}\right)^{2p}\frac{1}{p!}\frac{v_1^{\frac{n-4+2p}{2}}\,v_2^{\frac{n-1+2p}{2}}\,dv_1dv_2}{(2n-2)(2n+2)\dots(2n+4p-6)}$$
$$=Z_0\sqrt{2\pi}\,\frac{s_1^n s_2^n}{\kappa^{n+1}}\,2^{n-\frac{5}{2}}\,\rho s_2\mathop{S}_{p=0}^{p=\infty}\left(\frac{2\rho}{\kappa}\right)^{2p}\frac{1}{p!}\frac{\Gamma\left(\frac{n-2}{2}+p\right)\Gamma\left(\frac{n+1}{2}+p\right)}{(2n-2)(2n+2)\dots(2n+4p-6)}$$
$$=Z_0\sqrt{2\pi}\,\frac{s_1^n s_2^n}{\kappa^{n+1}}\,2^{n-\frac{5}{2}}\,\rho s_2\,\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{n+1}{2}\right)F\left(\frac{n-2}{2},\frac{n+1}{2},\frac{n-1}{2},\frac{\rho^2}{\kappa^2}\right)$$
$$=Z_0\sqrt{2\pi}\,\frac{s_1^n s_2^n}{\kappa^{n+1}}\,2^{n-\frac{5}{2}}\,\rho s_2\,\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{n+1}{2}\right)\left(1-\frac{\rho^2}{\kappa^2}\right)^{-\frac{1}{2}n}$$
$$\times F\left(\tfrac{1}{2},-1,\frac{n-1}{2},\frac{\rho^2}{\kappa^2}\right),$$

by Euler's transformation,

$$=Z_0\sqrt{2\pi}\,\frac{s_1^n s_2^n}{\kappa^{n+1}}\,2^{n-\frac{5}{2}}\,\rho s_2\,\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{n+1}{2}\right)\left(1-\frac{\rho^2}{\kappa^2}\right)^{-\frac{1}{2}n}$$
$$\times\left(1-\frac{1}{n-1}\frac{\rho^2}{\kappa^2}\right)\quad\dots\dots\dots\dots\text{(vii)}.$$

It remains now to find n_{z_2}. We have

$$n_{z_2}=\int_0^\infty\int_0^\infty\int_{-\infty}^{+\infty}Z\,d\sigma_1 d\sigma_2 dz_1$$
$$=Z_0\int_0^\infty\int_0^\infty\int_{-\infty}^{+\infty}e^{-\frac{1}{2}\left(\frac{z_1\sigma_1}{s_1}-\rho\frac{z_2\sigma_2}{s_2}\right)^2}dz_1\,e^{-\frac{1}{2}\frac{\sigma_1^2}{s_1^2}}e^{-\frac{1}{2}\frac{\sigma_2^2}{s_2^2}\kappa^2}\,Q(\sigma_1\sigma_2)^{n-1}d\sigma_1 d\sigma_2.$$

Integrating out for dz_1, there results

$$n_{z_2}=Z_0\sqrt{2\pi}\int_0^\infty\int_0^\infty\frac{s_1}{\sigma_1}e^{-\frac{1}{2}\frac{\sigma_1^2}{s_1^2}}e^{-\frac{1}{2}\frac{\sigma_2^2}{s_2^2}\kappa^2}\,Q(\sigma_1\sigma_2)^{n-1}d\sigma_1\,d\sigma_2.$$

Changing again to v_1 and v_2 we find

$$n_{z_2}=Z_0\sqrt{2\pi}\,2^{n-\frac{5}{2}}\frac{s_1^n s_2^n}{\kappa^n}\int_0^\infty\int_0^\infty e^{-v_1}e^{-v_2}\mathop{S}_0^\infty\left(\frac{2\rho}{\kappa}\right)^{2p}\frac{1}{p!}\frac{v_1^{\frac{n-3}{2}+p}\,v_2^{\frac{n-2}{2}+p}\,dv_1dv_2}{(2n-2)(2n+2)\dots(2n+4p-6)}$$
$$=Z_0\sqrt{2\pi}\,2^{n-\frac{5}{2}}\frac{s_1^n s_2^n}{\kappa^n}\,S\left(\frac{2\rho}{\kappa}\right)^{2p}\frac{1}{p!}\frac{\Gamma\left(\frac{n-1}{2}+p\right)\Gamma\left(\frac{n}{2}+p\right)}{(2n-2)(2n+2)\dots(2n+4p-6)}$$
$$=Z_0\sqrt{2\pi}\,2^{n-\frac{5}{2}}\frac{s_1^n s_2^n}{\kappa^n}\,\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}\right)$$
$$\times\left(1+\frac{\frac{n-1}{2}\,\frac{n}{2}}{1!\,(2n-2)}\left(\frac{2\rho}{\kappa}\right)^2+\frac{\frac{n-1}{2}\,\frac{n+1}{2}\,\frac{n}{2}\left(\frac{n}{2}+1\right)}{2!\,(2n-2)(2n+2)}\left(\frac{2\rho}{\kappa}\right)^4+\dots\right).$$

The series reduces to

$$1+\frac{n}{2}\frac{\rho^2}{\kappa^2}+\frac{\frac{n}{2}\left(\frac{n}{2}+1\right)}{1.2}\frac{\rho^4}{\kappa^4}+\ldots=\left(1-\frac{\rho^2}{\kappa^2}\right)^{-\frac{1}{2}n}$$

Thus

$$n_{z_2}=Z_0\sqrt{2\pi}\,2^{n-\frac{1}{2}}\frac{s_1{}^n s_2{}^n}{\kappa^n}\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}\right)\left(1-\frac{\rho^2}{\kappa^2}\right)^{-\frac{1}{2}n} \quad \text{.........(viii).}$$

Dividing (vii) by (viii) we reach, on substituting for κ^2,

$$\tilde{z}_1=\frac{\rho z_2}{\sqrt{1+(1-\rho^2)z_2^2}}\left(\frac{n-1}{n-2}\right)\left(1-\frac{1}{n-1}\frac{\rho^2}{1+(1-\rho^2)z_2^2}\right) \quad \text{.........(ix).}$$

This may be put into the simple form

$$\tilde{z}_1=\frac{\rho z_2}{\sqrt{1+(1-\rho^2)z_2^2}}\left(1+\frac{1}{n-2}\frac{(1-\rho^2)(1+z_2^2)}{1+(1-\rho^2)z_2^2}\right) \quad \text{.........(ix)}^{bis}.$$

This is the regression equation of z_1 on z_2.

We see that if n be finite it is by no means linear. As n grows indefinitely large, it tends to

$$\tilde{z}_1=\frac{\rho z_2}{\sqrt{1+(1-\rho^2)z_2^2}},$$

but, if we remember that the standard deviation of z_2 is $1/\sqrt{n-3}$, z_2^2 will be negligible before z_2, and accordingly we have

$$n\to\infty,\quad \tilde{z}_1\to\rho z_2.$$

The form of the curve is algebraically somewhat complicated. It has a point of inflexion at the origin and for asymptotes has the horizontal lines

$$z_1=\pm\frac{n-1}{n-2}\frac{\rho}{\sqrt{1-\rho^2}}.$$

There are further points of inflexion given by

$$z_2^2=\frac{3\rho^2-(n-1)}{(n-1+2\rho^2)(1-\rho^2)},$$

but these will be imaginary if $n-1>3\rho^2$, which it will be if $n=4$, and for practical purposes it is hard to conceive a problem where correlations could be based on samples of 2 or 3. The tangent at the origin, i.e. that at the point of inflexion, is

$$z_1=\frac{n-1-\rho^2}{n-2}\rho z_2 \quad \text{..............................(x).}$$

If $n-1>3\rho^2$ the tangent (x) does not meet again the curve (ix). The general form of (ix) is given diagrammatically in Fig. 1 on p. 416.

Clearly linearity increases more and more as n approaches nearer to infinity. Our figure of the regression line is drawn for the case when n is $=$ or >4, and accordingly the two other points of inflexion vanish.

The accompanying Table II, prepared by Mr E. C. Fieller, indicates how, for various values of ρ, the ordinate z_2 and the size of the sample n, the mean value of $\tilde{z}_1$, differs (i) from the z_1 of the tangent at the point of inflexion, and (ii) from the

line $z_1 = \rho z_2$, the regression straight line. Three values of z_2 are taken respectively equal to once, twice and thrice the standard deviation, $\frac{1}{\sqrt{n-3}}$, of z_2; these will cover the really important part of the regression curve. It will be seen that the deviation from linear regression can be fairly considerable even for a sample of 50.

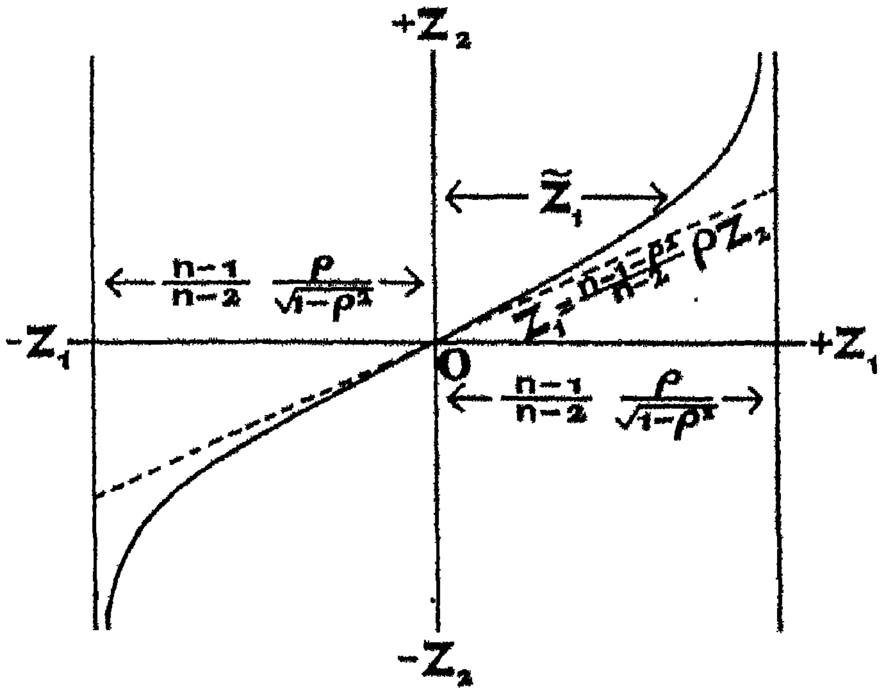


Fig. 1.

(5) *Scedasticity of Arrays.*

We shall now show that the arrays of z_1 for a given value of z_2 are heteroscedastic. In order to obtain the variance of an array we require first to determine

$$n_{z_2} \times {}_{z_2}\mu'_{2,z_1} = \int_{-\infty}^{+\infty}\int_0^\infty\int_0^\infty Z z_1^2\, dz_1\, d\sigma_1\, d\sigma_2$$

$$= Z_0 \int_0^\infty d\sigma_1 \int_0^\infty d\sigma_2 \int_{-\infty}^{+\infty} e^{-\frac{1}{2}u^2}\left(u + \rho z_2 \frac{\sigma_2}{s_2}\right)^2 du\, \frac{s_1^3}{\sigma_1^3}\, Q$$

$$\times (\sigma_1\sigma_2)^{n-1} e^{-\frac{1}{2}\left(\frac{\sigma_1^2}{s_1^2} + \frac{\sigma_2^2}{s_2^2}\kappa^2\right)},$$

the symbols having the same significance as on p. 413. Hence

$$n_{z_2} \times {}_{z_2}\mu'_{2,z_1} = Z_0 \int_0^\infty d\sigma_1 \int_0^\infty d\sigma_2 \sqrt{2\pi}\left(1 + \rho^2 z_2^2 \frac{\sigma_2^2}{s_2^2}\right) \frac{s_1^3}{\sigma_1^3} \times (\sigma_1\sigma_2)^{n-1}$$

$$\times \mathop{S}_{p=0}^{p=\infty} \frac{\rho^{2p}\sigma_1^{2p}\sigma_2^{2p}}{p!\,(2n-2)(2n+2)\dots(2n+4p-6)\, s_1^{2p} s_2^{2p}}\, e^{-\frac{1}{2}\left(\frac{\sigma_1^2}{s_1^2} + \frac{\sigma_2^2}{s_2^2}\kappa^2\right)}$$

$$= \frac{\sqrt{2\pi}\, Z_0 s_1^n s_2^n}{\kappa^n}\, 2^{n-\frac{1}{2}} \int_0^\infty dv_1 \int_0^\infty dv_2\, e^{-v_1-v_2}\left(1 + \frac{2\rho^2 z_2^2}{\kappa^2} v_2\right)$$

$$\times S\left(\frac{2^2\rho^2}{\kappa^2}\right)^p \frac{1}{p!}\, v_1^{\frac{n-5}{2}+p}\, v_2^{\frac{n-2}{2}+p}\, \frac{1}{(2n-2)(2n+2)\dots(2n+4p-6)}.$$

TABLE II.

Regression Values of $\bar{x}_1$ on z_2.

z_2	$\rho=\cdot2$: $\bar{x}_1$	$\rho=\cdot2$: $\frac{n-1-\rho^2}{n-2}\rho x_2$	$\rho=\cdot2$: ρx_2	$\rho=\cdot4$: $\bar{x}_1$	$\rho=\cdot4$: $\frac{n-1-\rho^2}{n-2}\rho x_2$	$\rho=\cdot4$: ρx_2	$\rho=\cdot6$: $\bar{x}_1$	$\rho=\cdot6$: $\frac{n-1-\rho^2}{n-2}\rho x_2$	$\rho=\cdot6$: ρx_2	$\rho=\cdot8$: $\bar{x}_1$	$\rho=\cdot8$: $\frac{n-1-\rho^2}{n-2}\rho x_2$	$\rho=\cdot8$: ρx_2
$n=10$												
$1/\sqrt{n-3}$	·079448	·084684	·075593	·158163	·167060	·151186	·235256	·244921	·226779	·309308	·315978	·302372
$2/\sqrt{n-3}$	·136285+	·169328	·151186	·276258	·334121	·302372	·423833	·489842	·453557	·583043	·631957	·604743
$3/\sqrt{n-3}$	·170342	·253992	·226779	·350772	·501181	·453557	·554452	·734763	·680336	·802734	·947935−	·907115−
$n=50$												
$1/\sqrt{n-3}$	·029458	·029756	·029173	·058847	·059367	·058346	·087519	·088686	·087519	·117134	·117567	·116692
$2/\sqrt{n-3}$	·057225−	·059513	·058346	·114730	·118734	·116692	·172796	·177372	·175038	·231704	·235134	·233384
$3/\sqrt{n-3}$	·082056	·089269	·087519	·165377	·178101	·175038	·251319	·266058	·262557	·341430	·352702	·350076
$n=100$												
$1/\sqrt{n-3}$	·020405	·020506	·020307	·040786	·040962	·040614	·061119	·061319	·060921	·081377	·081526	·081228
$2/\sqrt{n-3}$	·040612	·041012	·040614	·080545+	·081924	·081228	·121062	·122637	·121842	·161871	·163052	·162455
$3/\sqrt{n-3}$	·060625	·061518	·060921	·118374	·122886	·121842	·178762	·183956	·182762	·240644	·244578	·243683
$n=500$												
$1/\sqrt{n-3}$	·008980	·008989	·008971	·017958	·017973	·017942	·026931	·026948	·026914	·035898	·035911	·035885−
$2/\sqrt{n-3}$	·017908	·017977	·017942	·035825−	·035945+	·035885−	·053758	·053897	·053827	·071718	·071822	·071770
$3/\sqrt{n-3}$	·026734	·026966	·026914	·053513	·053918	·053827	·080381	·080845−	·080741	·107384	·107733	·107655−

Now
$$\frac{2^{2p}}{(2n-2)(2n+2)\dots(2n+4p-6)}=\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-1}{2}+p\right)};$$

hence

$$n_{z_1}\times{}_{z_2}\mu'_{2,z_1}=\frac{\sqrt{2\pi}\,Z_0 s_1{}^n s_2{}^n}{\kappa^n}\,2^{n-\frac{3}{2}}\,\Gamma\left(\frac{n-1}{2}\right)$$
$$\times\sum_0^\infty\left[\frac{\Gamma\left(\frac{n-3}{2}+p\right)}{\Gamma\left(\frac{n-1}{2}+p\right)}\Gamma\left(\frac{n}{2}+p\right)\left(\frac{\rho^2}{\kappa^2}\right)^p\frac{1}{p!}\right.$$
$$\left.+\frac{2\rho^2 z_2{}^2}{\kappa^2}\,\frac{\Gamma\left(\frac{n-3}{2}+p\right)\Gamma\left(\frac{n}{2}+p+1\right)}{\Gamma\left(\frac{n-1}{2}+p\right)}\left(\frac{\rho^2}{\kappa^2}\right)^p\frac{1}{p!}\right]$$
$$=\sqrt{2\pi}\,Z_0\frac{s_1{}^n s_2{}^n}{\kappa^n}\,2^{n-\frac{3}{2}}\,\Gamma\left(\frac{n-1}{2}\right)$$
$$\times\sum_0^\infty\frac{1}{\frac{n-3}{2}+p}\,\Gamma\left(\frac{n}{2}+p\right)\left(\frac{\rho^2}{\kappa^2}\right)^p\frac{1}{p!}\left(1+\frac{(n+2p)\,\rho^2 z_2{}^2}{\kappa^2}\right)\quad\dots\dots(\text{xi}).$$

But
$$n_{z_1}=\sqrt{2\pi}\,Z_0\frac{s_1{}^n s_2{}^n}{\kappa^n}\,2^{n-\frac{3}{2}}\,\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}\right)\left(1-\frac{\rho^2}{\kappa^2}\right)^{-n},$$

thus
$${}_{z_2}\mu'_{2,z_1}=\frac{\left(1-\frac{\rho^2}{\kappa^2}\right)^n}{\Gamma\left(\frac{n}{2}\right)}\sum_0^\infty\frac{\Gamma\left(\frac{n}{2}+p\right)}{(n-3+2p)}\left(\frac{\rho^2}{\kappa^2}\right)^p\frac{1}{p!}\left(1+\frac{(n+2p)\,\rho^2 z_2{}^2}{\kappa^2}\right)\dots\dots(\text{xii}).$$

We have two series to consider, namely

$$\sum_0^\infty\frac{\Gamma\left(\frac{n}{2}+p\right)}{\Gamma\left(\frac{n}{2}\right)(n-3+2p)}\left(\frac{\rho^2}{\kappa^2}\right)^p\frac{1}{p!},$$

and
$$\sum_0^\infty\frac{n+2p}{(n-3+2p)}\,\frac{\Gamma\left(\frac{n}{2}+p\right)}{\Gamma\left(\frac{n}{2}\right)}\left(\frac{\rho^2}{\kappa^2}\right)^p\frac{1}{p!}.$$

The latter

$$=\sum_1^\infty\frac{\Gamma\left(\frac{n}{2}+p\right)}{\Gamma\left(\frac{n}{2}\right)}\left(\frac{\rho^2}{\kappa^2}\right)^p\frac{1}{p!}+\sum_0^\infty\frac{3}{(n-3+2p)}\,\frac{\Gamma\left(\frac{n}{2}+p\right)}{\Gamma\left(\frac{n}{2}\right)}\left(\frac{\rho^2}{\kappa^2}\right)^p\frac{1}{p!}$$
$$=\left(1-\frac{\rho^2}{\kappa^2}\right)^{-\frac{n}{2}}+3\sum_0^\infty\frac{1}{(n-3+2p)}\,\frac{\Gamma\left(\frac{n}{2}+p\right)}{\Gamma\left(\frac{n}{2}\right)}\left(\frac{\rho^2}{\kappa^2}\right)^p\frac{1}{p!}.$$

Thus we have, if we write the first series Σ,

$$_{z_2}\mu'_{2,z_1} = \frac{\rho^2 z_2^2}{\kappa^2} + \left(1 - \frac{\rho^2}{\kappa^2}\right)^{\frac{n}{2}} \left(1 + \frac{3\rho^2 z_2^2}{\kappa^2}\right) \Sigma \quad \text{.................(xiii).}$$

Now

$$\Sigma = \frac{1}{n-3} + \frac{\frac{1}{2}n}{(n-1)1!}\frac{\rho^2}{\kappa^2} + \frac{\frac{1}{2}n(\frac{1}{2}n+1)}{(n+1)2!}\frac{\rho^4}{\kappa^4} + \dots$$

$$= \int_0^1 u^{n-4}\left(1 - \frac{\rho^2}{\kappa^2}u^2\right)^{-\frac{1}{2}n} du \quad \text{..................................(xiv).}$$

The series for Σ converges, since if t_p be the pth term

$$t_{p+1} = \left(t + \frac{n}{2(p+1)}\right)\left(\frac{1}{1 + \dfrac{2}{n+p}}\right)\frac{\rho^2}{\kappa^2}t_p,$$

and the first factor can be made to approach as near to unity as we please by indefinitely increasing p, the second and third factors are always less than unity; thus the series approaches a geometrical series of radix ρ^2/κ^2. It converges, however, far too slowly to be of service for computing. We need to transform the integral into a more rapidly converging series. We put $\frac{\rho^2}{\kappa^2}u^2 = \frac{v}{1+v}$, and find if v_0 be given by $\frac{\rho^2}{\kappa^2} = \frac{v_0}{1+v_0}$ that

$$\Sigma = \int_0^1 u^{n-4}\left(1 - \frac{\rho^2}{\kappa^2}u^2\right)^{-\frac{1}{2}n} du$$

$$= \frac{1}{2}\left(\frac{\kappa}{\rho}\right)^{n-3}\int_0^{v_0} v^{\frac{n-5}{2}}(1+v)^{\frac{1}{2}} dv.$$

Integrating by parts, raising the power of v, we find

$$\Sigma = \frac{1}{n-3}\left(\frac{\kappa}{\rho}\right)^{n-3}(1+v_0)^{\frac{1}{2}} v_0^{\frac{n-3}{2}}\left(1 - \frac{1}{n-1}\frac{v_0}{1+v_0} - \frac{1}{(n-1)(n+1)}\left(\frac{v_0}{1+v_0}\right)^2\right.$$
$$\left. - \frac{1.3}{(n-1)(n+1)(n+3)}\left(\frac{v_0}{1+v_0}\right)^3 - \frac{1.3.5}{(n-1)(n+1)(n+3)(n+5)}\left(\frac{v_0}{1+v_0}\right)^4 - \text{etc.}\right),$$

a sufficiently converging series for practical purposes.

Substituting for v_0 we have

$$\Sigma = \frac{1}{n-3}\frac{1}{\left(1 - \frac{\rho^2}{\kappa^2}\right)^{\frac{n-3}{2}}}\left(1 - \frac{1}{n-1}\frac{\rho^2}{\kappa^2} - \frac{1}{(n-1)(n+1)}\frac{\rho^4}{\kappa^4}\right.$$
$$\left. - \frac{1.3}{(n-1)(n+1)(n+3)}\frac{\rho^6}{\kappa^6} - \frac{1.3.5}{(n-1)(n+1)(n+3)(n+5)}\frac{\rho^8}{\kappa^8} - \text{etc.}\right) \quad \text{......(xv)}$$

$$= \frac{1}{n-3}\frac{1}{\left(1 - \frac{\rho^2}{\kappa^2}\right)^{\frac{n-3}{2}}} \times \Sigma', \text{ say.}$$

Accordingly we have from (xiii)

$$_{z_2}\mu'_{2,z_1} = \frac{\rho^2 z_2^2}{\kappa^2}\left(1 + \frac{3}{n-3}\left(1 - \frac{\rho^2}{\kappa^2}\right)\Sigma'\right) + \frac{1}{n-3}\left(1 - \frac{\rho^2}{\kappa^2}\right)\Sigma' \quad \text{......(xvi),}$$

TABLE III.

Measurement of the Scedasticity of z_1 for a given z_2. Exact Value and Approximations.

	$\rho=\cdot 2$				$\rho=\cdot 4$			
z_2	True $\sigma_{z_1 . z_2}$	Formula (xxi)	$\sqrt{\frac{1-r^2 z_1 z_2}{n-3}}$	$\sqrt{\frac{1-\rho^2}{n-3}}$	True $\sigma_{z_1 . z_2}$	Formula (xxi)	$\sqrt{\frac{1-r^2 z_1 z_2}{n-3}}$	$\sqrt{\frac{1-\rho^2}{n-3}}$
$n=10$								
$1/\sqrt{n-3}$	·371586	·371645+	·37132	·370328	·351067	·351273	·35024	·346410
$2/\sqrt{n-3}$	·375630	·375679	"	"	·366949	·367142	"	"
$3/\sqrt{n-3}$	·379065+	·379106	"	"	·381290	·381670	"	"
$n=50$								
$1/\sqrt{n-3}$	·142981	·142981	·14298	·142918	·133923	·133923	·13392	·133687
$2/\sqrt{n-3}$	·143318	·143318	"	"	·135195−	·135195+	"	"
$3/\sqrt{n-3}$	·143802^5	·143802^5	"	"	·137055−	·137055+	"	"
$n=100$								
$1/\sqrt{n-3}$	·099504	·099504	·09950+	·099483	·093136	·093136	·09314	·093058
$2/\sqrt{n-3}$	·099622	·099622	"	"	·093581	·093581	"	"
$3/\sqrt{n-3}$	·099805+	·099805+	"	"	·094274	·094274	"	"
$n=500$								
$1/\sqrt{n-3}$	·043952	·043952	·04395−	·043950−	·041118	·041118	·04112	·041111
$2/\sqrt{n-3}$	·043962	·043962	"	"	·041157	·041157	"	"
$3/\sqrt{n-3}$	·043979	·043979	"	"	·041222	·041222	"	"
	$\rho=\cdot 6$				$\rho=\cdot 8$			
$n=10$								
$1/\sqrt{n-3}$	·311304	·311647	·31032	·302372	·238098	·238426	·23839	·226779
$2/\sqrt{n-3}$	·345172	·345559	"	"	·288839	·289299	"	"
$3/\sqrt{n-3}$	·379422	·379827	"	"	·350873	·351495−	"	"
$n=50$								
$1/\sqrt{n-3}$	·117150−	·117150+	·11714	·116692	·088122	·088122	·08812	·087519
$2/\sqrt{n-3}$	·119685−	·119685+	"	"	·091570	·091571	"	"
$3/\sqrt{n-3}$	·123502	·123503	"	"	·096991^5	·096992	"	"
$n=100$								
$1/\sqrt{n-3}$	·081380	·081380	·08138	·081228	·061123	·061123	·06112	·060921
$2/\sqrt{n-3}$	·082260	·082260	"	"	·062307	·062307	"	"
$3/\sqrt{n-3}$	·083653	·083653	"	"	·064225+	·064225+	"	"
$n=500$								
$1/\sqrt{n-3}$	·035897	·035897	·03590	·035885−	·026931	·026931	·02693	·026914
$2/\sqrt{n-3}$	·035976	·035975+	"	"	·027035−	·027035−	"	"
$3/\sqrt{n-3}$	·036103	·036103	"	"	·027206	·027206	"	"

and for the variance of the array of z_1's for a given z_2 by (ix)

$$\sigma^2_{z_1 . z_2} = \frac{\rho^2 z_2^2}{\kappa^2}\left\{1 + \frac{3}{n-3}\left(1 - \frac{\rho^2}{\kappa^2}\right)\Sigma' - \left(\frac{n-1}{n-2}\right)^2\left(1 - \frac{1}{n-1}\frac{\rho^2}{\kappa^2}\right)^2\right\} + \frac{1}{n-3}\left(1 - \frac{\rho^2}{\kappa^2}\right)\Sigma' \quad \text{.........(xvii).}$$

It is now possible to compute $\sigma_{z_1 . z_2}$ for given values of ρ, z_2 and n, remembering that Σ' is given by

$$\Sigma' = 1 - \frac{1}{n-1}\frac{\rho^2}{\kappa^2} - \frac{1}{(n-1)(n+1)}\frac{\rho^4}{\kappa^4} - \frac{1 . 3}{(n-1)(n+1)(n+3)}\frac{\rho^6}{\kappa^6} - \frac{1 . 3 . 5}{(n-1)(n+1)(n+3)(n+5)}\frac{\rho^8}{\kappa^8} - \text{etc.} \quad \text{......(xviii),}$$

and that

$$\kappa^2 = 1 + (1 - \rho^2) z_2^2.$$

It is clear, however, that $\sigma^2_{z_1 . z_2}$ is a fairly involved function of z_2, or, in other words, the system is far from homoscedastic. Table III gives the true values of $\sigma_{z_1 . z_2}$ for selected values of ρ, z_2 and n in the first column of each section corresponding to a given ρ; in the third column is given the homoscedastic value, i.e.

$$\sigma'_{z_1 . z_2} = \frac{1}{\sqrt{n-3}}\sqrt{1 - r^2_{z_1 z_2}},$$

where $r_{z_1 z_2}$ is the correlation of z_1 and z_2, and in the fourth column the homoscedastic value

$$\sigma''_{z_1 . z_2} = \frac{1}{\sqrt{n-3}}\sqrt{1 - \rho^2},$$

where ρ is the correlation in the parent population. Finally in the second column is given the approximation to $\sigma_{z_1 . z_2}$ now to be found, where we assume that terms of the order $\frac{1}{(n-3)^4}$ may be neglected.

To find an Approximation for $\sigma^2_{z_1 . z_2}$ in terms of $1/(n-3)$.

We first find $\bar{z}_1^2$,

$$\bar{z}_1^2 = \left(\frac{\rho}{\kappa}\right)^2 z_2^2 \left(\frac{n-1}{n-2}\right)^2 \left(1 - \frac{1}{n-1}\left(\frac{\rho}{\kappa}\right)^2\right)^2.$$

Now

$$\left(\frac{n-1}{n-2}\right)^2 = \left(\frac{1 + \dfrac{2}{n-3}}{1 + \dfrac{1}{n-3}}\right)^2 = 1 + \frac{2}{n-3} - \frac{1}{(n-3)^2} + \frac{0}{(n-3)^3},$$

and

$$\left(1 - \frac{1}{n-1}\frac{\rho^2}{\kappa^2}\right)^2 = 1 - \frac{2}{n-1}\frac{\rho^2}{\kappa^2} + \frac{1}{(n-1)^2}\frac{\rho^4}{\kappa^4}$$
$$= 1 - \frac{2}{n-3}\frac{\rho^2}{\kappa^2} + \frac{1}{(n-3)^2}\frac{\rho^2}{\kappa^2}\left(4 + \frac{\rho^2}{\kappa^2}\right) - \frac{4}{(n-3)^3}\frac{\rho^2}{\kappa^2}\left(2 + \frac{\rho^2}{\kappa^2}\right).$$

Combining these we have

$$\bar{z}_1^2 = \frac{\rho^2 z_2^2}{\kappa^2}\left(1+\frac{2}{n-3}\left(1-\frac{\rho^2}{\kappa^2}\right)-\frac{1}{(n-3)^2}\left(1-\frac{\rho^4}{\kappa^4}\right)+\frac{2}{(n-3)^3}\frac{\rho^2}{\kappa^2}\left(1-\frac{\rho^2}{\kappa^2}\right)\right) \quad \text{(xix)*}.$$

In the same manner we find

$$\frac{1}{n-3}\left(1-\frac{\rho^2}{\kappa^2}\right)\Sigma' = \frac{1}{n-3}\left(1-\frac{\rho^2}{\kappa^2}\right)\left(1-\frac{1}{n-3}\frac{\rho^2}{\kappa^2}+\frac{1}{(n-3)^2}\frac{\rho^2}{\kappa^2}\left(2-\frac{\rho^2}{\kappa^2}\right)\right) \quad \text{(xx)}.$$

Substituting (xix) and (xx) in (xvii) we reach, after some reductions,

$$\sigma^2_{z_1.z_2} = \frac{1}{n-3}\frac{(1-\rho^2)(1+z_2^2)^2}{(1+(1-\rho^2)z_2^2)^2}$$
$$\times\left\{1-\frac{1}{n-3}\rho^2\frac{1-z_2^2(1-\rho^2)}{1+(1-\rho^2)z_2^2}+\frac{1}{(n-3)^2}\frac{\rho^2(\rho^2+2(1-\rho^2)(1+(1+\rho^2)z_2^2))}{(1+(1-\rho^2)z_2^2)^2}\right\}$$
$$\ldots\ldots\text{(xxi)}.$$

This is the expression of $\sigma^2_{z_1.z_2}$ up to the third order terms in $\frac{1}{n-3}$.

It will be remarked on examination of Mr Fieller's table, Table III, that after $n=50$, the approximate formula (xxi) agrees with the true value of $\sigma_{z_1.z_2}$ practically to a unit in the sixth decimal place. For statistical practice it is really efficient down to $n=25$, i.e. it will only differ in the fifth decimal place. For lower values of n it will be needful to evaluate the full series Σ' of formula (xviii). We note further that the distribution is not adequately homoscedastic even for $n=500$, the distribution of z_1 for a given z_2 continues to increase in variability as we increase z_2. Of the two suggested formulae for homoscedastic values that for which we use the correlation of z_1 and z_2 gives a better result than that for which we use the correlation of the parent population.

Clearly the non-linearity of the regression and the heteroscedasticity of the arrays are not in favour of using z_1 and z_2 as variates in samples drawn from a parent population with correlated variables. The investigation of the probability that an observed z_1 should be associated with an observed z_2, if the parental population had known means and a given correlation, would require much arithmetical labour.

I have to thank heartily my colleague Mr E. C. Fieller for his help in the preparation of this paper.

* This Equation for samples of 25 and over will give very accurately the mean value of $\bar{z}_1$ for a given z_2.

MISCELLANEA.

I. Note on Tests for Normality.

In an earlier part of the present volume of *Biometrika* I have given certain approximate tables of the 5 % and 1 % points for $\sqrt{\beta_1}$ and β_2 in sampling from a normal population. The results were based on expansions in series of inverse powers of n, the sample size, which it had been possible to derive as far as the terms in n^{-3} *. Since the publication of this paper Dr R. A. Fisher has been able to obtain exact expressions for the moment-coefficients of the sampling distribution of these two constants in the case of a normal population †. The quantities with which he deals are

$$\gamma = k_3 k_2^{-\frac{3}{2}} = \frac{\sqrt{n(n-1)}}{n-2}\sqrt{\beta_1} \text{ and } \delta = k_4 k_2^{-2} = \frac{n^2-1}{(n-2)(n-3)}\left\{\beta_2 - \frac{3(n-1)}{n+1}\right\} \quad \text{......(1),}$$

and from these relations, and making use of the value of $\kappa(4^4)$ given above by Wishart ‡, it is possible to obtain the following results:

Distribution of $\sqrt{\beta_1}$.

$$\sigma^2_{\sqrt{\beta_1}} = \frac{6(n-2)}{(n+1)(n+3)} \quad \text{......(2),}$$

$$B_2(\sqrt{\beta_1}) = 3 + \frac{36(n-7)(n^2+2n-5)}{(n-2)(n+5)(n+7)(n+9)} \quad \text{......(3),}$$

$$B_4(\sqrt{\beta_1}) = 15 + \frac{540\{n^7+60n^6-131n^5-2798n^4-3629n^3+21352n^2+32943n-70070\}}{(n-2)^2(n+5)(n+7)(n+9)(n+11)(n+13)(n+15)} \quad \text{...(4).}$$

Distribution of β_2.

$$\text{Mean } \beta_2 = \frac{3(n-1)}{n+1} \quad \text{......(5),}$$

$$\sigma^2_{\beta_2} = \frac{24n(n-2)(n-3)}{(n+1)^2(n+3)(n+5)} \quad \text{......(6),}$$

$$B_1(\beta_2) = \frac{216}{n}\,\frac{(n+3)(n+5)(n^2-5n+2)^2}{(n-3)(n-2)(n+7)^2(n+9)^2} \quad \text{......(7),}$$

$$B_2(\beta_2) = 3 + \frac{36(15n^6-36n^5-628n^4+982n^3+5777n^2-6402n+900)}{n(n-3)(n-2)(n+7)(n+9)(n+11)(n+13)} \quad \text{......(8).}$$

As a check on the previous work it is satisfactory to find that the expressions (2), (3), (6), (7) and (8) give on expansion as far as the terms in n^{-3} exactly the same coefficients as those contained in equations (10), (11), (21), (22) and (23) of my earlier paper. For the standard errors of $\sqrt{\beta_1}$ and β_2 the approximate expressions that I had used at $n=50$ and $n=100$ respectively are identical with the true values to four places of decimals. Further the values of $B_2(\sqrt{\beta_1})$ are as follows:

	$n=50$	$n=75$	$n=100$
True B_2	3·452	3·351	3·284
Value used in making tables	3·46	3·35	3·28

* The results were based on the investigations of R. A. Fisher, *Proc. Lond. Math. Soc.* (2), Vol. xxx. (1929), pp. 199—238, and J. Wishart, *Biometrika*, Vol. xxii. pp. 224—238.

† *Proceedings of the Royal Society*, Series A, Vol. 130, No. A 812, pp. 16—28.

‡ P. 238, Equation (15).

And for the distribution of β_2:

	$n=100$	$n=150$		$n=100$	$n=150$
True B_1	1·631	1·192	True B_2	6·774	5·826
Value used	1·63	1·19	Value used	6·85	5·84

Although I have not refitted the curve, I have little doubt that the error in B_2 (β_2) for $n=100$ could only affect the values tabled for the 5 °/ₒ and 1 °/ₒ points at the most by two units in the second decimal place, and probably only by one unit. Any approximation which may be found to exist will therefore not be due to the use of the first four terms in the series instead of the true values of the moment-coefficients, but to the assumption that Type VII and Type IV curves may be used to find the two probability limits. This point cannot be completely solved until the actual frequency laws for $\sqrt{\beta_1}$ and β_2 have been found.

EGON S. PEARSON.

II. Some recent Researches in the Theory of Statistics and Actuarial Science. By J. F. STEFFENSEN. Cambridge: published for the Institute of Actuaries, at the University Press, 1930. Price 5*s.* net.

THIS little volume gives, in a somewhat extended form, the substance of the three lectures delivered by Professor Steffensen for the University of London in the spring of 1930, and will enable them to reach, as they deserve, a much wider audience. Its modest bulk of fifty-two pages is packed with matter.

Professor Steffensen took as the general subject of his lectures some of the efforts he had made "to introduce more rigour into certain questions of theoretical statistics and actuarial science." In mathematics we have a science which investigates the relations between numbers. Observations may contradict each other, but mathematical relations are not allowed to contain contradictions: theory must be presented in such a form that the theoretical relations or assumptions contain no contradictions. The first lecture is devoted to showing how we may be led astray by neglect of this principle. The opening sections make a critical examination of the notion of *Biometric Functions* (Life Table functions). A number of interesting inequalities are obtained, and some common but loose modes of statement or assumption, e.g. the assumption of an "oldest possible" age at which the l_x column abruptly terminates, come in for useful discussion. The author then turns to the thorny question of "presumptive values" of frequency constants, taking as an illustration presumptive values of the moment-coefficients. Here we are brought up rather sharply by an apparent paradox. The mathematical expectation of the second moment-coefficient *about the mean* in a sample of n is $(n-1)/n$ times the second moment-coefficient in the universe sampled: hence the not-infrequently used formula $n/(n-1)$ times the sample value for the "presumptive value" in the universe. But the mathematical expectation of a moment-coefficient of any order *about a fixed origin* is identical with the value in the universe; and the value about the mean is expressible in terms of the values about a fixed origin. The presumptive values are therefore identical with those in the sample. Professor Steffensen concludes (p. 20): "It appears thus that neither of the two systems of presumptive values of frequency-constants...is free from contradictions, and that a strong case can be made even against the time-honoured Gaussian formula $\overline{m}_2=\frac{n}{n-1}m_2$. If, on the other hand, we use the uncorrected [sample moments] as the best available approximations to [the moments in the universe] we are at least sure that no contradictions can ever be met with." The conclusion is comforting to one who has always worked with the sample values. But in this case it looks as if precisely the same assumptions led to contradictory conclusions, and Professor Steffensen does not seem to show how they do so. What is the source of the discrepancy?

In the second lecture the author considers problems of approximation and interpolation. Two objects of interpolation are distinguished. When the problem is to find the value of a *defined* function for a certain value of the argument, the function being tabulated for certain other arguments, the problem is one of *approximation*. When the problem is to fill up in a reasonable way a gap in a series of values of an *undefined* function, the process is more analogous to graduation and, it is suggested, might be termed *intercalation*. We are here on different ground, and it is a question of "plausibility" rather than "accuracy" of the result. Illustrations are drawn from the life-table and again some useful inequalities are obtained.

Mathematics are also used for describing facts of observation. Formulae used for the purpose may be empirical, or may be based on theoretical considerations, and theoretical reasons may be given for believing that one formula is likely to fit the facts better than another. Theoretical foundations are therefore of importance, and frequency-functions are chosen as the subject of the third lecture. Professor Karl Pearson's methods and those associated mainly with the names of Thiele, Charlier and Bruns are discussed and contrasted. One of the earliest pupils of Professor Pearson may perhaps be allowed the pleasure of citing from the fifth paragraph of the lecture (p. 35): "It is the lasting merit of Professor Karl Pearson of the University of London to have pointed out convincingly that the *natural source* of practically useful types of frequency-functions is the elementary calculus of probabilities. What nature does in producing a new individual, practically comes to the same thing as drawing from various urns and mixing up the results. This explains the great success of Pearson's types." And again from the conclusion (p. 48): "We are therefore inclined to think that the apparent generality of (28) [the general series] is rather a disadvantage than otherwise, and that Pearson's types are as a rule preferable."

Every statistician who is interested in theory should possess the volume.

G. UDNY YULE.

III. A Problem in Probability.

POSITIVE quantities $x_1, x_2, \ldots x_n$ are taken at random subject to the conditions

$$\text{I} \qquad a_1x_1 \geqslant a_2x_2 \geqslant a_3x_3 \geqslant \ldots \geqslant a_nx_n \qquad (a_i>0),$$

$$\text{II} \qquad \alpha_1a_1x_1 + \alpha_2a_2x_2 + \alpha_3a_3x_3 + \ldots + \alpha_na_nx_n = 1 \qquad (\alpha_i>0).$$

What are the mean values of $x_1, x_2, \ldots x_n$?

Represent the set of quantities $x_1, x_2, \ldots x_n$ by the point $P(x_1, x_2, \ldots x_n)$ in S_n. All points corresponding to sets satisfying the given conditions lie in Γ, the $(n-1)$-dimensional simplex cut out by the primes

$$(1) \qquad \left\{\begin{array}{l} a_1x_1 = a_2x_2 \\ a_2x_2 = a_3x_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n-1}x_{n-1} = a_nx_n \\ a_nx_n = 0, \end{array}\right.$$

from the prime

$$(2) \qquad \alpha_1a_1x_1 + \alpha_2a_2x_2 + \ldots + \alpha_na_nx_n = 1.$$

If we omit the rth of equations (1) and solve the remainder with (2), we get for the coordinates of P_r, the rth vertex of Γ,

$$(3) \qquad \begin{aligned} x_i &= 1 \Big/ a_i \sum_{p=1}^{r} \alpha_p \qquad &(i=1,..r), \\ &= 0 \qquad &(i>r). \end{aligned}$$

Subject to conditions involved in I, $x_1, x_2, \ldots x_{n-1}$ may be taken at random, and x_n is then determined by II. The chance that x_1 should lie between x_1 and x_1+dx_1, x_2 between x_2 and

$x_2+dx_2, \ldots x_{n-1}$ between x_{n-1} and $x_{n-1}+dx_{n-1}$ is proportional to $dx_1 dx_2 \ldots dx_{n-1}$, which is the content of the hyper-rectangle in which the projection of P on $x_n=0$ must lie. But the content of this hyper-rectangle is proportional to the content of the portion of Γ of which it is the orthogonal projection.

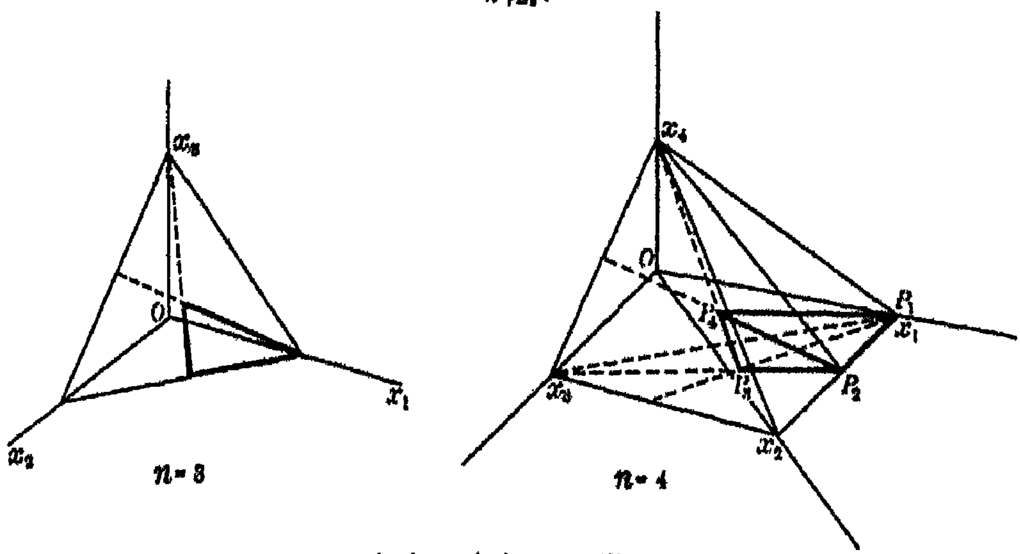
It follows that all positions of P in Γ are equally likely, so that the mean values $\bar{x}_1, \bar{x}_2, \ldots \bar{x}_n$ of $x_1, x_2, \ldots x_n$ are the coordinates of the centroid of Γ; thus

$$\bar{x}_s = \frac{1}{n} \sum_{r=s}^{n} \left(1 \Big/ a_r \sum_{p=1}^{r} \alpha_p\right)$$

Corollary. If we put $a_1 = a_2 = \ldots = a_n = 1 = \alpha_1 = \alpha_2 = \ldots = \alpha_n$, we get, for the mean values of positive quantities $x_1, x_2, \ldots x_n$ chosen subject to the conditions

$$\begin{cases} x_1 \geqslant x_2 \geqslant x_3 \ldots \geqslant x_n \\ x_1+x_2+x_3+\ldots+x_n = 1, \end{cases}$$

$$\bar{x}_s = \frac{1}{n} \sum_{r=s}^{n} \frac{1}{r}.$$



$n=3$ $n=4$

Γ is shown in heavy outline.

This is the result that Laplace uses for his theory of voting, when he says*: "Donnons à chaque votant, une urne qui renferme un nombre infini de boules; et supposons qu'il les distribue sur les diverses propositions, en raison des probabilités respectives qu'il leur attribue.... Le problème se réduit donc à déterminer les combinaisons dans lesquelles les boules seront réparties, de manière qu'il y en ait plus sur la première proposition du billet, que sur la seconde; plus sur la seconde que sur la troisième, etc.; à faire les sommes de tous les nombres de boules, relatifs à chaque proposition dans ces diverses combinaisons; et à diviser cette somme, par le nombre des combinaisons: les quotiens seront les nombres de boules, que l'on doit attribuer aux propositions sur un billet quelconque. On trouve par l'analyse, qu'en partant de la dernière proposition, pour remonter à la première; ces quotiens sont entre eux, comme les quantités suivantes: 1° l'unité divisée par le nombre des propositions; 2° la quantité précédente augmentée de l'unité divisée par le nombre des propositions moins une; 3° cette seconde quantité augmentée de l'unité divisée par le nombre des propositions moins deux; et ainsi du reste."

It is worth noticing that Laplace's wording, as it stands, is somewhat ambiguous; moreover, as Professor Pearson points out, the assumption of equal probability for all modes of division is hardly likely to be justified in practice.

* *Essai philosophique sur les probabilités.*

Laplace insists on his voters' dividing up all their balls between the various proposals, but the result holds without this condition. An argument similar to that used above shows that the mean values $\bar{x}_s$ of quantities x_s chosen at random subject to the conditions

I $$a_1 x_1 \geqslant a_2 x_2 \geqslant a_3 x_3 \geqslant \dots \geqslant a_n x_n \qquad (a_i > 0),$$

and II′ $$\alpha_1 a_1 x_1 + \alpha_2 a_2 x_2 + \alpha_3 a_3 x_3 + \dots + \alpha_n a_n x_n \leqslant 1 \qquad (\alpha_i > 0),$$

are the coordinates of the centroid of the n-dimensional simplex $OP_1P_2 \dots P_r \dots P_n$. Thus

$$\bar{x}_s = \frac{1}{n+1} \sum_{r=s}^{n} \left(1 \Big/ a_s \sum_{p=1}^{r} \alpha_p\right),$$

so that the generalization of condition II does not alter the ratios of the mean values.

E. C. Fieller.